Quantum field theory
and
the nature of space-time

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Very roughly, this means giving vector subpaces $\mathcal{O}_x$ of A, indexed by the points of spacetime, which generate $\mathcal{A}$ *multiplicatively*.

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The early attempts to make this precise took space-time as *given*, and a QFT gave no information about space-time.

The **Wightman axioms** are formulated only for standard Minkowski space $\mathbb{M}$.

Organizing principle: $\mathcal{A}$ is generated by an *operator-valued distribution* $x \mapsto \psi(x)$ on $\mathbb{M}$.

All other properties
— domains of definition, local commutativity, positivity of energy, unitarity —
are put in by hand.

Positivity of energy is ensured by a version of Wick rotation:

the distributions

$$(x_1, \dots, x_k) \;\mapsto\; \psi(x_1)\dots\psi(x_k)$$

are boundary-values of **holomorphic** operator-valued **functions** defined in an open subset $\mathcal{U}_k \subset (\mathbb{M}_\mathbb{C})^k$.

$\mathcal{U}_k$ contains the configuration-space $C_k(\mathbb{E})$ of the Euclidean subspace $\mathbb{E}$ of $\mathbb{M}_\mathbb{C}$, so we do have actual operators parametrized by $C_k(\mathbb{E})$.

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Example

Kac-Moody algebra $\mathcal{A}$ — defined combinatorially

$$\text{Out}(\mathcal{A}) \quad = \quad \text{Diff}(S^1) \quad \neq \quad \text{Homeo}(S^1)$$

$\rightarrow$ loop groups and positive energy representations

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modularity of characters $\leftrightarrow$ character associated to *torus*

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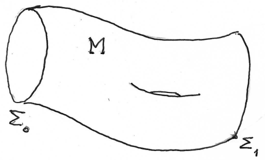
The **cobordism functor** definition attempts to model the path-integral idea. (See arXiv:2105.10161 for details.)

Closed $(d-1)$-manifold $\Sigma \quad \mapsto \quad$ topological vector space $\mathcal{H}_\Sigma$

Space-time manifold M bounded in the past and the future by hypersurfaces Σ_0 and Σ_1



$\mapsto$ *trace-class linear map* $U_M : \mathcal{H}_{\Sigma_0} \to \mathcal{H}_{\Sigma_1}$

The manifolds M and hypersurfaces Σ have *allowable complex metrics* given by symmetric tensors g_{ij} with complex components. These metrics form a contractible domain, with the Lorentzian metrics on its boundary.

Axioms:

(a) U_M depends holomorphically on the metric of M.

(b) The functor takes disjoint unions to tensor products.

(c) $*$-property.

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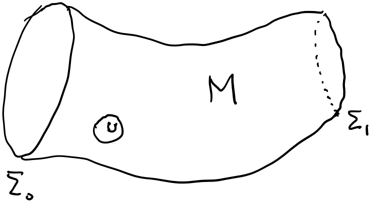
For any $x \in M$ we define $\mathcal{O}_x$ as the set of compatible families $\psi_x = \{\psi_U \in \mathcal{H}_{\partial U}\}$ for all neighbourhoods $x \in U \subset M$.

Thus ψ_x is an operator $\psi_x : \mathcal{H}_{\Sigma_0} \to \mathcal{H}_{\Sigma_1}$.



$$\Sigma_0 \sqcup \partial U \rightsquigarrow \Sigma_1$$

$$\mathcal{H}_{\Sigma_0} \otimes \mathcal{H}_{\partial U} \longrightarrow \mathcal{H}_{\Sigma_1}$$

Much of the usual structure of QFT appears automatically.

Positivity of energy and unitarity come from the use of complex metrics, using the

Theorem If the metric of M tends to a Lorentzian metric (on the boundary of the domain of allowable metrics) which is globally hyperbolic, then U_M tends to a unitary operator.

Cf. The unitary group U_n lies on the boundary of its holomorphic hull, the contraction operators in $\mathbb{C}^n$.

Some essential features of the path-integral picture are **not** captured by the cobordism formulation.

We cannot deduce that

- a deformation of the theory is defined by a local field
- the energy-momentum tensor is an element of $\mathcal{O}_x$.

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• In flat Minkowski space, are the field operators *distributions*?

• Can we construct an **'extended'** theory? We would like to have:

a *linear tensor category* $\mathcal{C}_Z$ for each closed $(d-2)$-manifold Z

with an object $\mathcal{H}_\Sigma$ for each Σ^{d-1} with boundary Z,

such that

if $\hat{\Sigma}$ is the closed manifold $\bar{\Sigma}_1 \sqcup_Z \Sigma_2$ then

$$\mathcal{H}_{\hat{\Sigma}} = \mathrm{Hom}_{\mathcal{C}_Z}(\mathcal{H}_{\Sigma_1}; \mathcal{H}_{\Sigma_2}).$$

$$= \mathcal{H}_{\bar{\Sigma}_1} \otimes_{\mathcal{C}_Z} \mathcal{H}_{\Sigma_2}.$$

I conjecture that all four of these statements follow from the assumption of **asymptotic conformality** at short distances.

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This requires the spaces $\mathcal{H}_{\partial U}$ associated to the neighbourhoods U of a point $x \in M$ to *stabilize* as U shrinks to the point x.

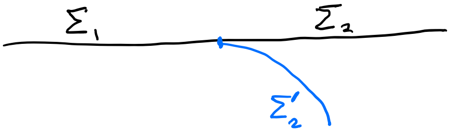
It allows us to introduce the concept of the **scaling dimension** of a field operator, i.e. an *increasing filtration*

$$\mathcal{O}_x^{(0)} \subset \mathcal{O}_x^{(1)} \subset \dots \subset \mathcal{O}_x,$$

and to set up the usual structure of *operator product expansions* in terms of it.

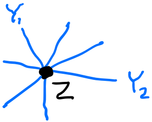
Let $\mathcal{A}_{\Sigma_1}$ denote the algebra of operators on $\mathcal{H}_{\Sigma_1 \cup \Sigma_2}$ generated by the elements of $\mathcal{O}_x$ with $x \in \Sigma_1$, and let $\mathcal{A}^o_{\Sigma_1}$ denote its commuting algebra.

Tomita-Takesaki theory



$$\mathcal{A}^{\circ}_{\Sigma_1} \cong \begin{cases} \hat{\mathcal{A}}_{\Sigma_2} \\ \hat{\mathcal{A}}_{\Sigma_2} \end{cases}$$

Category $\mathcal{C}_Z$ for a $(d-2)$-manifold Z



$\mathcal{A}_Y$ = alg of ops assoc to Y (Type III)

$\mathcal{H}_{Y_1 \cup_Z Y_2}$ is an $\mathcal{A}_Y - \mathcal{A}_Y$ bimodule (realizing the modular flow)

For any Σ^{d-1} with $\partial\Sigma = Z$ we have a family of $\mathcal{A}_Y$ modules $\{\mathcal{H}_{\Sigma \cup Y}\}$ with modular flows between them.