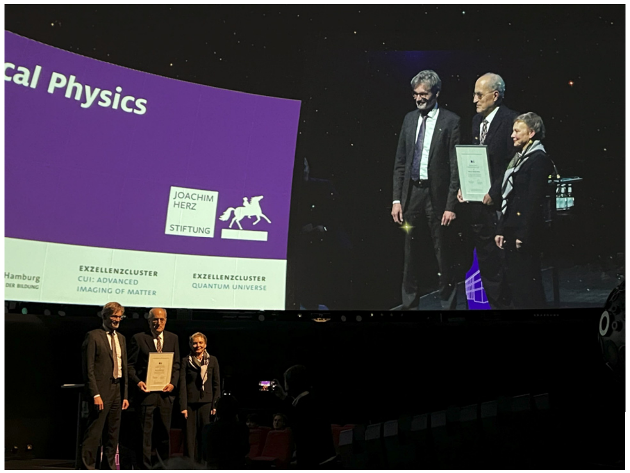
Symmetry Resolution in Conformal Field Theory

Hirosi Ooguri

Theoretical Physics Symposium 2023

8 – 10 November 2023 , Wolfgang Pauli Centre, Hamburg

Congratulations, Edward!

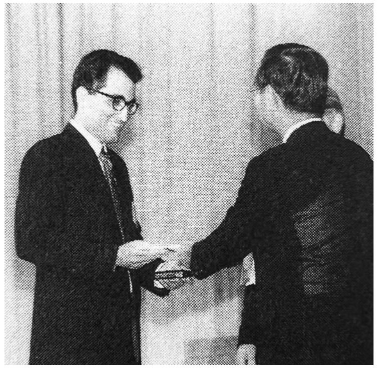


JOACHIM HERZ STIFTUNG

DESY.

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Since 1990, Edward has visited Japan at least five times:

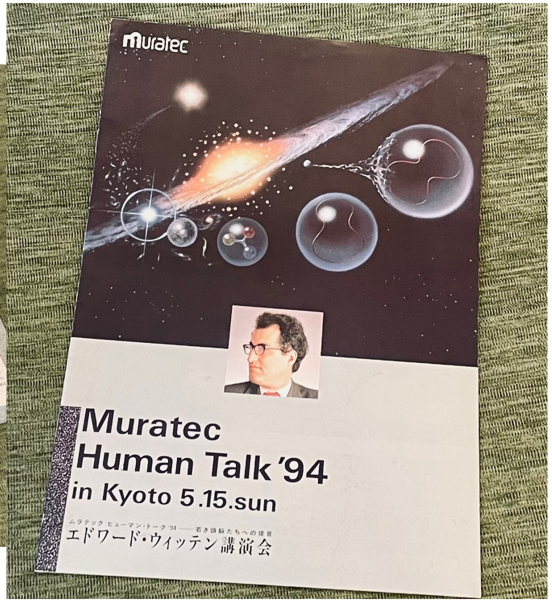


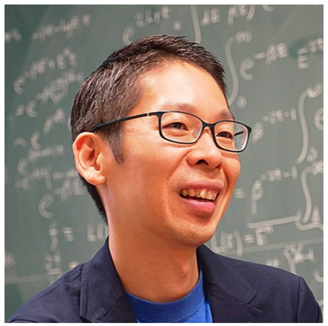
ICM-90
International
Congress of
Mathematicians
KYOTO

1990 Fields Medal

- ICM **1990**
- **1994** Muratech Public Lecture
- Strings **2003**
- **2014** Kyoto Prize
- Strings **2018**

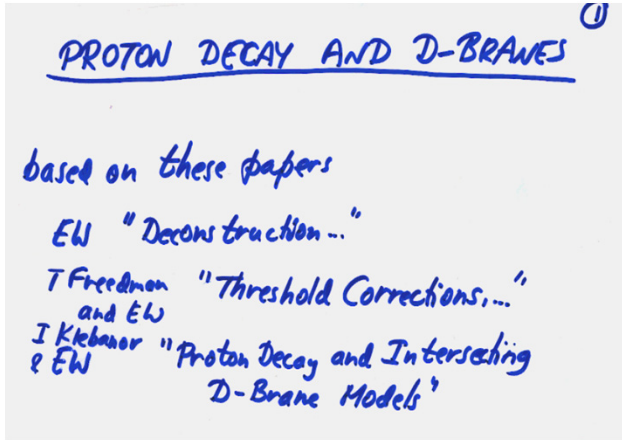
Muratech Public Lecture (Kyoto, 1994)





inspired high school students
such as **Koji Hashimoto** and
Yuji Tachikawa to study physics.

Strings 2003 in Kyoto



Public Lectures

2014 Kyoto Prize

with Princess Takamado

Visit to Kavli IPMU

in Tokyo

Interview with Edward Witten

Hirosi Ooguri

This is a slightly edited version of an interview with Edward Witten that appeared in the December 2014 issue of *Kavli IPMU News*, the news publication of the Kavli Institute for the Physics and Mathematics of the Universe (IPMU). An abridged Japanese translation has also appeared in the April and May 2015 issues of *Sugaku Seminar*, the popular mathematics magazine. The interview is published here with permission of the Kavli IPMU.

The interview took place in November 2014 on the occasion of Witten's receipt of the 2014 Kyoto Prize in Basic Sciences. Witten is Charles Simonyi Professor in the School of Natural Sciences at the Institute for Advanced Study in Princeton. Yukinobu Toda and Masahito Yamazaki, two junior faculty members of the Kavli IPMU, also participated in the interview.

Ooguri: *First I would like to congratulate Edward on his Kyoto Prize. Every four years, the Kyoto Prize in Basic Sciences is given in the field of mathematical sciences, and this is the first prize in this category awarded to a physicist.*

Witten: Well, I can tell you I'm deeply honored to have this prize.

Ooguri: *It is wonderful that your work in the area at the interface of mathematics and physics has been recognized as some of the most important progress in mathematics as well as in physics. For those of us working in this area, this is also very gratifying.*

Witten: Actually, in my acceptance speech a couple of days ago, I remarked that I regard it also as a recognition of the field, not just of me.

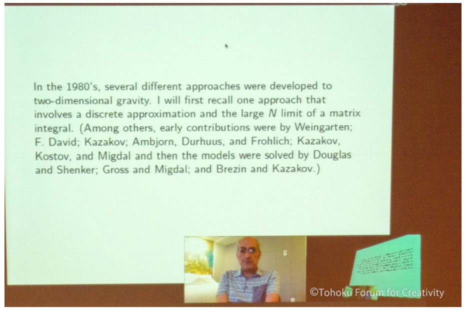
Generalizing Chern-Simons

Ooguri: *This conversation will appear as an article as well as in the Kavli IPMU News. You have already given two interviews for Sugaku Seminar. In 1990, at the International Congress of Mathematicians in Kyoto, you received the Fields Medal. On that occasion, Tohru Eguchi had an interview with you. You also had a discussion with Vaughan Jones, another Fields Medalist at the congress, and I remember you expressed interest in generalizing your work in the Chern-Simons theory with a spectral parameter, which is very natural from*

Edward Witten.

Notices of the American Mathematical Society (May 2015)

Strings 2018 in Okinawa



String-Math 2018
Tohoku University

Monotonicity of Relative Entropy In Quantum Field Theory

Edward Witten

Strings 2018

Symmetry Resolution in Conformal Field Theory

Based on:

- with Harlow [2109.03838]
- with Kang, Lee [2206.14814]
- with Benjamin, Lee, Simmons-Duffin [2306.08031]
- with Kusuki, Murchiano, Pal [2309.03287]

For any unitary conformal field theory on $\mathbb{R} \times S^{d-1}$, denote the subspace of the Hilbert space with conformal weights $\in [\Delta, \Delta + \delta]$ by $\mathcal{H}_{[\Delta,\Delta+\delta]}$.

For $1 \ll \delta \ll \Delta$, a simple dimensional analysis shows,

$$\dim \mathcal{H}_{[\Delta,\Delta+\delta]} \approx \exp\left[\left(f \Delta^{d-1}\right)^{1/d}\right] \text{ for some } f > 0.$$

(For $d = 2$, the coefficient $f = \frac{4\pi^2}{3} c$, the Cardy formula.)

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If the conformal field theory has a **global symmetry $\boldsymbol{G}$**, we can decompose the Hilbert space into irreducible representations of G:

$$\mathcal{H}_{[\Delta,\Delta+\delta]} = \bigoplus_R \left(\mathcal{H}^{(R)}_{[\Delta,\Delta+\delta]} \otimes R\right).$$

If G is a finite group and $1 \ll \delta \ll \Delta$,

$$\dim \mathcal{H}^{(\boldsymbol{R})}_{[\Delta,\Delta+\delta]} \approx \frac{\dim R}{|G|} \dim \mathcal{H}_{[\Delta,\Delta+\delta]}$$

If G is a compact Lie group and if we take $1 \ll \delta \ll \Delta$ with R fixed,

$$\dim \mathcal{H}^{(\boldsymbol{R})}_{[\Delta,\Delta+\delta]} \approx \frac{\dim R}{\Lambda} \dim \mathcal{H}_{[\Delta,\Delta+\delta]},$$

where $\Lambda = \left(k\, \Delta^{(d-1)/d}/4\pi\right)^{\dim G/2}$ for some $k > 0$.

For a large representation R, we can calculate corrections systematically as,

$$\dim \mathcal{H}^{(\boldsymbol{R})}_{[\Delta,\Delta+\delta]} \approx \frac{\dim R}{\Lambda} \exp\left[-\frac{c_2(R)}{k\, \Delta^{(d-1)/d}} + \cdots\right] \dim \mathcal{H}_{[\Delta,\Delta+\delta]},$$

The main part of the formular, dim $\mathcal{H}^{(\boldsymbol{R})}_{[\Delta,\Delta+\delta]} \propto \dim R$, would follow from,

$$\mathrm{Tr}\left[\, U(g)\, e^{-\beta H} \,\right] \ll \mathrm{Tr}\left[\, e^{-\beta H} \,\right]$$

for $g \neq e$ and $\beta = 1/T \to 0.$

If G is a finite group,

$$\log_{\beta\to 0} \frac{\mathrm{Tr}\left[\, U(g)\, e^{-\beta H} \,\right]}{\mathrm{Tr}\left[\, e^{-\beta H} \,\right]} = \delta(g, e) = \sum_R \frac{\dim R}{|G|} \chi_R(g) \,.$$

Comparing this with,

$$\mathcal{H}_{[\Delta,\Delta+\delta]} = \bigoplus_R \left(\mathcal{H}^{(R)}_{[\Delta,\Delta+\delta]} \otimes R \right),$$

we see,

$$\dim \mathcal{H}^{(\boldsymbol{R})}_{[\Delta,\Delta+\delta]} \approx \frac{\dim R}{|G|} \dim \mathcal{H}_{[\Delta,\Delta+\delta]} \,.$$

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If G is a compact Lie group, the correction factor, $e^{-\frac{c_2(R)}{k\,\Delta^{(d-1)/d}}+\cdots}$, follows from,

$$\mathrm{Tr}\big[\, U\big(e^{i\omega}\big)\, e^{-\beta H} \,\big] \approx e^{-\frac{k}{4} T^{d-1} \langle\omega,\omega\rangle + \cdots} \times \mathrm{Tr}\big[\, e^{-\beta H} \,\big].$$

Why

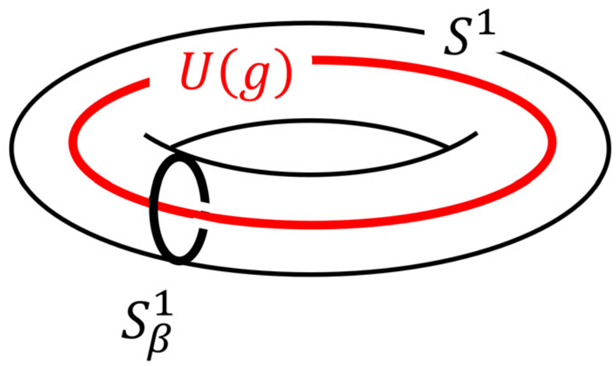
$$\mathrm{Tr}\Big[\, U(g)\, e^{-\beta H} \,\Big] \ll \mathrm{Tr}\Big[\, e^{-\beta H} \,\Big]$$

for $g \neq e$ and $\beta = 1/T \to 0$?

When $\boldsymbol{d} = \mathbf{2}$ and on $S^1_\beta \times S^1$,

$$\mathrm{Tr}_{\mathcal{H}}\left[\, U(g)\, e^{-\beta H}\, \right] = \mathrm{Tr}_{\mathcal{H}_g}\left[\, e^{-4\pi^2\beta^{-1}H}\, \right]$$

by **modular invariance**, where $\mathcal{H}$ is the original Hilbert space on S^1 and $\mathcal{H}_g$ is twisted by g. In the limit $\beta \to 0$, the left-hand side becomes $e^{-\frac{4\pi^2}{\beta}\left(\Delta_g - \frac{c}{12}\right)}$, where Δ_g is the lowest conformal weight in $\mathcal{H}_g$.



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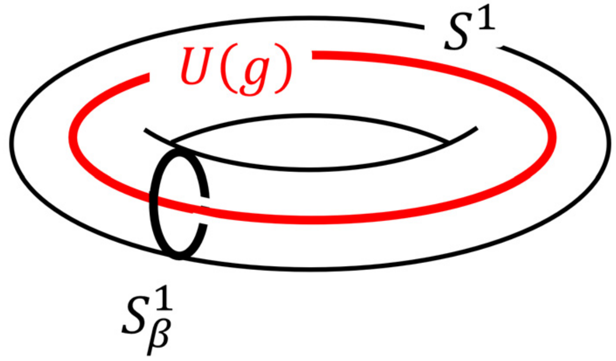
We must have $\boldsymbol{\Delta_g > 0}$, otherwise the twist operator becomes topological and contradicts with the assumption that ***G* acts nontrivially** on the conformal field theory. Therefore,



$$\mathrm{Tr}_{\mathcal{H}}\left[\, U(g)\, e^{-\beta H}\,\right]$$

$$\approx e^{-\frac{4\pi^2}{\beta}\left(\Delta_g - \frac{c}{12}\right)} \ll e^{\frac{\pi^2 c}{3}} \approx \mathrm{Tr}_{\mathcal{H}}\left[\, e^{-\beta H}\,\right]$$

for $g \neq e$ in the limit $\beta \to 0$.

When $\boldsymbol{d} = \mathbf{2}$ and on $S^1_\beta \times S^1$,

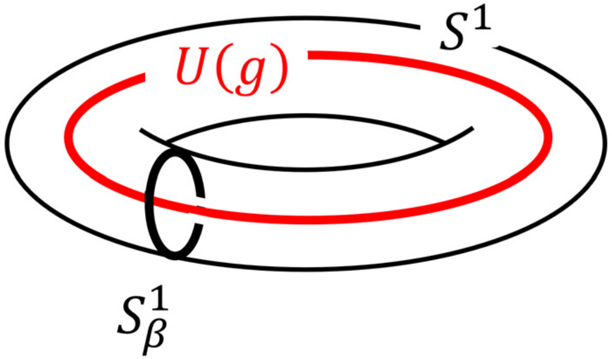
$$\mathrm{Tr}_{\mathcal{H}}\left[\, U(g)\, e^{-\beta H}\, \right] = \mathrm{Tr}_{\mathcal{H}_g}\left[\, e^{-4\pi^2\beta^{-1}H}\, \right].$$

If $\boldsymbol{G}$ **is a compact Lie group,** we can express the Noether currents in terms of free bosons φ and parafermions (Gepner, 1987). In particular, the Noether currents J along the maximum torus is given by $J = i\sqrt{k/2}\,\partial\varphi$, where $\boldsymbol{k}$ **is the level of the Kac-Moody algebra.**

Using this, we can show

$$\mathrm{Tr}\left[\, U\!\left(e^{i\omega}\right) e^{-\beta H}\, \right]$$

$$\approx e^{-\frac{k}{4\beta}\langle\omega,\omega\rangle+\cdots} \times\ \mathrm{Tr}\left[\, e^{-\beta H}\, \right] .$$



When $\boldsymbol{d} > \mathbf{2}$ and on $S^1_\beta \times S^{d-1}$, there are no S^1 to exchange with S^1_β.

Alternatively, we can consider a more general space $S^1_\beta \times \Sigma_{d-1}$ with Σ_{d-1} containing S^1, but how can we relate its partition function to the spectrum of the conformal weights Δ ?

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David Simmons-Duffin

Why don't you use **thermal effective field theory**?

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Alternatively, we can consider a more general space $S^1_\beta \times \Sigma_{d-1}$ with Σ_{d-1} containing S^1, but how can we relate its partition function to the spectrum of the conformal weights Δ ?

David Simmons-Duffin

Why don't you use **thermal effective field theory**?

For $\beta \to 0$, we can write down a $(d-1)$-dimensional effective action by dimensionally reducing on S^1_β.

$$S_{\text{eff}} = \int_{\boldsymbol{\Sigma_{d-1}}} \sqrt{h}\, dx^{d-1} \left(T^{d-1} V(g) + \cdots \right)$$

where $g \in G$ is the holonomy around S^1_β.

By diffeomorphism invariance, the potential $\boldsymbol{V(g)}$ at high temperature **is independent of the geometry of $\boldsymbol{\Sigma_{d-1}}$.**

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By diffeomorphism invariance, the potential $V(g)$ at high temperature is independent of the geometry of Σ_{d-1}.

We can interpret $\boldsymbol{V(g)}$ as the **lowest energy in the $\boldsymbol{g}$-twisted sector around the thermal circle** S^1_β.

Therefore, $\boldsymbol{V(g) > V(e)}$ when $g \neq e$, and

$$\mathrm{Tr}\big[\, U(g)\, e^{-\beta H} \,\big] \ll \mathrm{Tr}\big[\, e^{-\beta H} \,\big] \quad \text{for } \beta = 1/T \to 0.$$

To summarize:

If G is a finite group and $1 \ll \delta \ll \Delta$,

$$\dim \mathcal{H}^{(\boldsymbol{R})}_{[\Delta,\Delta+\delta]} \approx \frac{\dim R}{|G|} \dim \mathcal{H}_{[\Delta,\Delta+\delta]}$$

If G is a compact Lie group,

$$\dim \mathcal{H}^{(\boldsymbol{R})}_{[\Delta,\Delta+\delta]} \approx \frac{\dim R}{\Lambda} \exp\left[-\frac{c_2(R)}{k\,\Delta^{(d-1)/d}} + \cdots\right] \dim \mathcal{H}_{[\Delta,\Delta+\delta]},$$

where $\Lambda = \left(k\,\Delta^{(d-1)/d}/4\pi\right)^{\dim G/2}$ for some $b > 0$.

By the AdS/CFT correspondence, these formulae also teach us about charged black holes. For example, **if the theory contains fermions, a half of large black hole microstates are fermionic.**

Asymptotic Density of States in High Energy **resolved with respect to Spacetime Symmetry**

$$\mathrm{Tr}\left[\, e^{-\beta(H+i\vec{\Omega}\cdot\vec{L})} \,\right] = \int_0^\infty \sum_{\ell=0}^\infty \rho_d(\Delta, \ell)\, e^{-\beta(\Delta+\varepsilon_d)+i\beta\vec{\Omega}\cdot\vec{\ell}}\, d\Delta,$$

where H and $\vec{L}$ are the Hamiltonian and the angular momentum on S^{d-1},
$\rho_d(\Delta, \ell)$ is the density of local operators with scaling dimension Δ and spin $\vec{\ell}$,

and ε_d is the Casimir energy on S^{d-1} given by,

Euler density

$$\varepsilon_d = \frac{(d-1)!!}{(-2)^{d/2}} a_d \quad \text{for } d\text{: even}, \quad \text{and} \quad \langle T_\mu^\mu \rangle = \frac{a_d}{(-4\pi)^{d/2}} E_d + \cdots$$

$$\varepsilon_d = 0 \quad \text{for } d\text{: odd}.$$

For example, $\varepsilon_{d=2} = -\dfrac{c}{12}$

Asymptotic Density of States in High Energy **resolved with respect to Spacetime Symmetry**

$$ds^2_{\text{cylinder}} = \beta d\tau + d\Omega^2_{S^{d-1}}, \quad d\Omega^2_{S^{d-1}} = \sum_{a=1}^{\lceil d/2 \rceil} dr_a^2 + \sum_{a=1}^{\lfloor d/2 \rfloor} r_a^2 d\theta_a^2$$

$$\mathrm{Tr}\left[\, e^{-\beta(H+i\vec{\Omega}\cdot\vec{L})} \,\right] \Leftrightarrow (\tau, \theta_a) \sim (\tau+1, \theta_a - \beta\Omega_a)$$ [Omega background]

$$\Leftrightarrow \; ds^2_{\text{Kaluza-Klein}} = h_{ij}(x) dx^i dx^j + e^{2\phi(x)} (d\tau + A_i(x))^2,$$

where $x \in S^{d-1}$.

- $e^{2\phi(x)} = \beta^2 \left(1 + \sum_a r_a^2 \Omega_a^2\right),$
- $A_i(x) dx^i = \frac{\sum_a r_a^2 \Omega_a d\theta_a}{\beta\left(1+\sum_a r_a^2 \Omega_a^2\right)},$
- $h_{ij}(x)\, dx^i dx^j = \sum_{a=1}^{\lceil d/2 \rceil} dr_a^2 + \sum_{a,b=1}^{\lfloor d/2 \rfloor} \left(r_a^2 \delta_{ab} + \frac{r_a^2 r_b^2 \Omega_a \Omega_b}{1+\sum_c r_c^2 \Omega_c^2} \right) d\theta_a d\theta_b.$

Asymptotic Density of States in High Energy **resolved with respect to Spacetime Symmetry**

Universal behavior with respect to the angular momentum $\vec{M}$

$$\mathrm{Tr}\left[\, e^{-\beta(H+i\vec{\Omega}\cdot\vec{L})} \,\right] \approx \exp\left(\frac{G(T,\vec{\Omega})}{\prod_a (1+\Omega_a^2)} + \cdots \right)$$

When $\Delta, \ell \gg 1$ and $|\,\Delta - \ell\,| \sim \sqrt{f\Delta}$, we can use the saddle-point approximation to invert the Laplace transform,

$$\int_0^\infty d\Delta \sum_{J=0}^\infty \rho_d\left(\Delta, \vec{\ell}\right) e^{-\beta(\Delta+\varepsilon_d)+i\beta\vec{\Omega}\cdot\vec{\ell}} = \mathrm{Tr}\left[\, e^{-\beta(H+i\vec{\Omega}\cdot\vec{L})} \,\right],$$

to calculate $\rho_d\left(\Delta, \vec{\ell}\right)$.

Asymptotic Density of States in High Energy **resolved with respect to Spacetime Symmetry**

For $\Delta, \ell \gg 1$ and $|\,\Delta - \ell\,| \sim \sqrt{f\Delta}$,

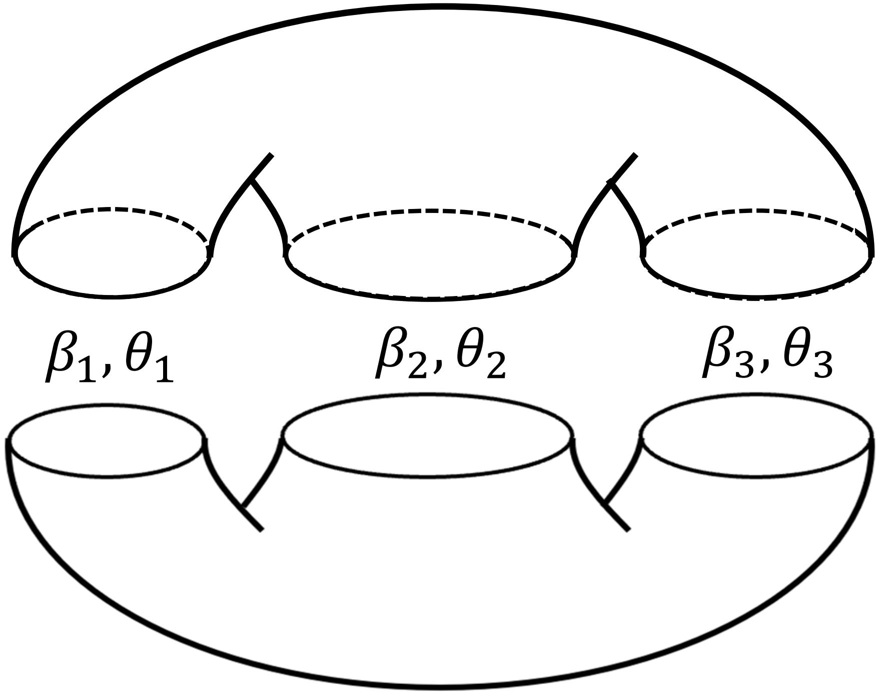
$$\rho_d(\Delta,\ell) \approx \exp\left[\frac{d}{d-1}(\Delta+\varepsilon_d)^{\frac{d-1}{d}}\left(\frac{f(d-1)\mathrm{vol}\,S^{d-1}}{2}\left(1+\sqrt{1+\frac{(d-3)(d-1)\,\ell^2}{(\Delta+\varepsilon_d)^2}}\right)\right)^{1/d}\right.$$
$$\left.\times\left(\frac{d-2}{d-3}-\frac{1}{d-3}\sqrt{1+\frac{(d-3)(d-1)\ell^2}{(\Delta+\varepsilon_d)^2}}\right)^{1-2/d}\right].$$

When $d=2$, this becomes,

$$\rho_{d=2}(\Delta,\ell) \approx \exp\left[\sqrt{\frac{2c}{3}}\pi\left(\sqrt{\frac{\Delta+\ell-c/12}{2}}+\sqrt{\frac{\Delta-\ell-c/12}{2}}\right)\right],$$

reproducing the Cardy formula

Three-Point Function at Large Conformal Dimensions



$d = 2$:

A genus-2 surface can be constructed by connecting a pair of spheres by three cylinders (Whittaker/Schottky parametrization).

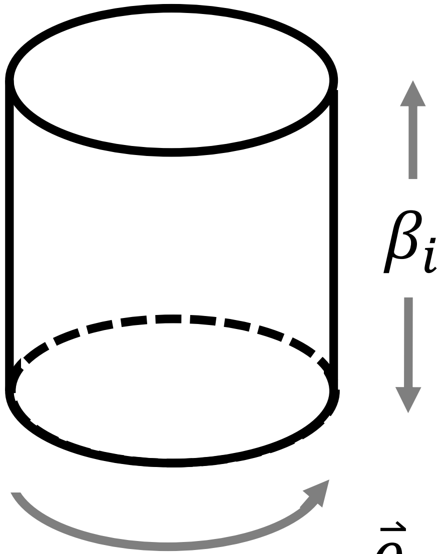
In the limit where the three cylinders become thick, the Laplace transformation with respect to the lengths of the cylinders can be used to calculate:

$$\left|C_{\Delta_1 \Delta_2 \Delta_3}\right|^2 \rho(\Delta_1)\rho(\Delta_2)\rho(\Delta_3)$$

Cardy, Malony, Maxfield: 1705.05855
Collier, Maloney, Maxifield, Tsiares: 1912.00222

Three-Point Function at Large Conformal Dimensions

Benjamin, Lee, Simmons-Duffin + H.O.: 2306.08031

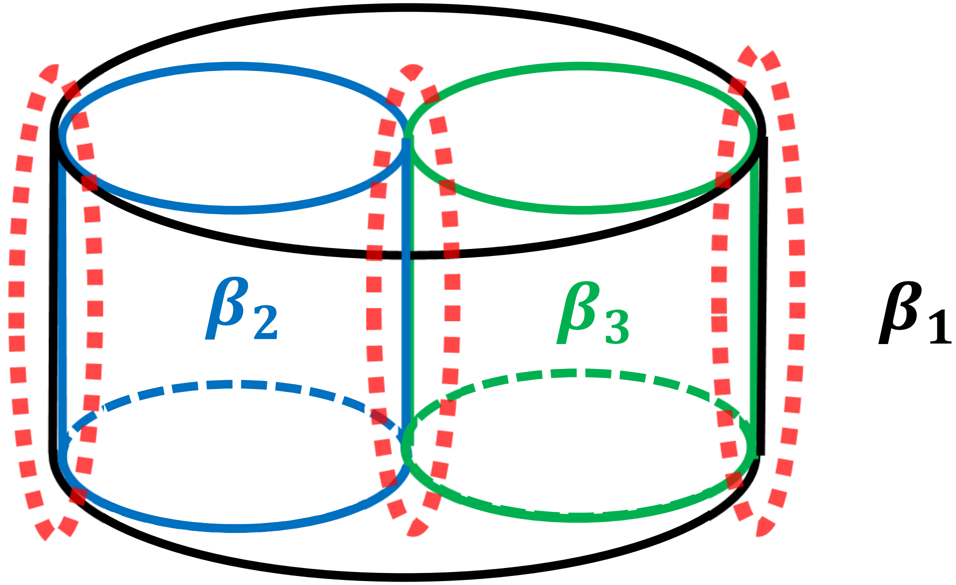


$$\vec{\theta}_i = \beta_i \, \vec{\Omega}_i$$

We can generalize the Whittaker/Schottky parametrization to $d > 2$ by connecting two $(d-1)$-spheres by three cylinders.

There are 3-cycles whose periods vanish in the limit of $\beta_i \longrightarrow 0$.

We call them **HOT SPOTs**.



Three-Point Function at Large Conformal Dimensions

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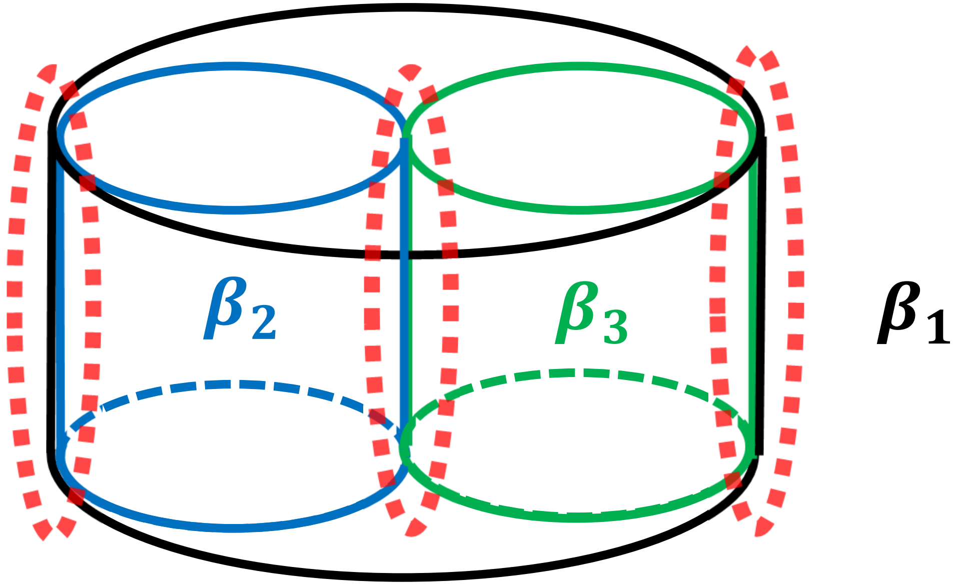

- The whole “genus-2” geometry is NOT a circle fibration.
- We assume that the leading contribution in the limit of $\beta_i \rightarrow 0$ can be estimated by the thermal effective action applied to any region where the **geometry locally looks like a circle fibration with a large local temperature**.

Three-Point Function at Large Conformal Dimensions

- The whole “genus-2” geometry is NOT a circle fibration.
- We assume that the leading contribution in the limit of $\beta_i \to 0$ can be estimated by the thermal effective action applied to any region where the **geometry locally looks like a circle fibration with a large local temperature**.

$$Z(\text{"genus 2"}) \approx \exp\left(\frac{f\,\mathrm{vol}(S^{d-1})}{\boldsymbol{\beta}_{\mathbf{12}}^{d-1}\prod_a\left(1 + \boldsymbol{\Omega}_{\mathbf{12},a}^2\right)} + [23] + [31]\cdots\right)$$

$\boldsymbol{\beta}_{\mathbf{12}}$ **is a local temperature** at one of the three hot spots, and $\vec{\boldsymbol{\Omega}}_{\mathbf{12}}$ is the twist there. They are **determined explicitly in terms of $\boldsymbol{\beta}_{\boldsymbol{i}}$ and $\vec{\boldsymbol{\Omega}}_{\boldsymbol{i}}$** $(i = 1,2,3)$ for the three cylinders.

Three-Point Function at Large Conformal Dimensions

We need to decompose $Z(\text{"genus 2"})$ into **conformal blocks**.

Conformal blocks are eigenfunctions of the conformal Casimir operators and satisfying appropriate boundary conditions.

There is another class of functions, **conformal partial waves**, which satisfy the same Casimir equations, but obey different boundary conditions.

Conformal partial waves are easier to build in terms of inner products of the principal series representations with $\Delta = d/2 + ir$, $(r \in \mathbb{R})$:

$$\Psi^{ss'}(g_1, g_2, g_3) = (V^s|g_1 g_2 g_3|V^{s'})$$

$V^s, V^{s'}$: Clebsch-Gordan coefficients for 3 principal series representations.

Three-Point Function at Large Conformal Dimensions

“genus-2” Conformal Blocks

Conformal partial waves $\Psi^{ss'}(g_1, g_2, g_3)$

$\Downarrow$

Conformal blocks

Monodromy projection by Karateev, Kravchuk, Simmons-Duffin: 1809.05111.

- Each channel of a conformal partial wave has $\Delta = d/2 + ir$ and its **shadow** $\Delta = d/2 - ir$.
- The space of solutions to the conformal Casimir equations is $8 = 2^3$ dimensions.
- Conformal partial waves and conformal blocks are different basis of this 8-dimensional space.

Three-Point Function at Large Conformal Dimensions

After constructing the conformal blocks, we still need to:

- Construct the measure so that the blocks are orthogonal.
- Calculate the overlap of the genus-2 partition function with a conformal block in an appropriate limit.

$$Z(\text{"genus 2"}) \approx \exp\left(\frac{f\ \mathrm{vol}(S^{d-1})}{\beta_{12}^{d-1}\prod_a\left(1 + \Omega_{12,a}^2\right)} + [23] + [31]\ \cdots\right)$$

$$\Longrightarrow \qquad c_{123}^{s}\ \overline{c_{123}^{s\prime}}\ \rho_1\ \rho_2\ \rho_3$$

Three-Point Function at Large Conformal Dimensions

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \sum c_{123}^{s}\ V^{s}(x_1, x_2, x_3),$$

$d = 2$

$$c_{123}\ \sqrt{\rho(\Delta,\ell_1)\rho(\Delta,\ell_2)\rho(\Delta,\ell_3)} \approx \left(\frac{3}{2}\right)^{3\Delta} \frac{2^{13/2} f e^{\frac{9}{4}\left(\pi^2 f^2 \Delta\right)^{1/3}}}{3^{9/2}\pi^{1/2}\Delta^{5/2}}$$

$d = 3$

$$c_{123}^{s}\ \sqrt{\rho(\Delta,\ell_1)\rho(\Delta,\ell_2)\rho(\Delta,\ell_3)} \approx \left(\frac{3}{2}\right)^{3\Delta} \frac{2^{49/8} f^{9/8} e^{\frac{3}{2}(2\pi f \Delta)^{1/2}}}{3^{19/4}\pi^{1/4}\Delta^{31/8}}$$

$$\times \prod_{i=1}^{3} (2\ell_i + 1)\binom{2\ell_i}{\ell_i + q_I}$$

$s = [q_1, q_2, q_3]$: conformal block parameters

Three-Point Function at Large Conformal Dimensions

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \sum c^{s}_{123}\ V^{s}(x_1, x_2, x_3),$$

$d = 2$

$$c_{123}\ \sqrt{\rho(\Delta,\ell_1)\rho(\Delta,\ell_2)\rho(\Delta,\ell_3)} \approx \left(\frac{3}{2}\right)^{3\Delta} \frac{2^{13/2} f e^{\frac{9}{4}(\pi^2 f^2 \Delta)^{1/3}}}{3^{9/2}\pi^{1/2}\Delta^{5/2}}$$

$d = 3$

$$c^{s}_{123}\ \sqrt{\rho(\Delta,\ell_1)\rho(\Delta,\ell_2)\rho(\Delta,\ell_3)} \approx \left(\frac{3}{2}\right)^{3\Delta} \frac{2^{49/8} f^{9/8} e^{\frac{3}{2}(2\pi f \Delta)^{1/2}}}{3^{19/4}\pi^{1/4}\Delta^{31/8}}$$

$$\times \prod_{i=1}^{3} (2\ell_i + 1) \binom{2\ell_i}{\ell_i + q_I}$$

$s = [q_1, q_2, q_3]$: conformal block parameters

The leading behavior is independent of spacetime dimensions.

Three-Point Function at Large Conformal Dimensions

Questions:

- These results are valid for $\Delta, \ell \gg 1, | \Delta - \ell | \sim \sqrt{f\Delta}$. Can we interpolate them to the **light-cone bootstrap** results for $| \Delta - \ell | \ll \sqrt{f\Delta}$?
- Effective actions parametrize our ignorance. For the 3d $O(N)$ model with $N \gg 1$, Sachdev (1993) calculated f. How about **higher order coefficients**? Can we bound them?
- In 2d, C_{HHH}, C_{HHL}, and C_{HLL} are related to the **DOZZ formula of the Liouville theory**. Does it generalize to $d > 2$?
- In 2d, the three-point function for global blocks $(3/2)^{3\Delta}$ is much larger than the one for Virasoro blocks $(27/16)^{3\Delta/2}$. This means that **states with boundary gravitons have much larger OPE's** than those for pure black holes. Why?

Symmetry-Resolved Density of States

- $\rho(\Delta, R) \approx \dfrac{\dim R}{|G|} \rho(\Delta)$, when G is a **finite group**.

Harlow + H.O.: 2111.04725

- $$\rho(\Delta, R) \approx \dim R \left(\frac{4\pi}{k\,\Delta^{(d-1)/d}}\right)^{\dim G/2} \exp\left[-\frac{c_2(R)}{k\,\Delta^{(d-1)/d}}\right] \rho(\Delta) ,$$

when G is a **compact Lie group**. Kang, Lee + H.O.: 2206.14814

- $$\rho_d(\Delta, \ell) \approx \exp\left[\frac{d}{d-1}(\Delta+\varepsilon_d)^{\frac{d-1}{d}}\left(\frac{f(d-1)\mathrm{vol}\,S^{d-1}}{2}\left(1+\sqrt{1+\frac{(d-3)(d-1)\,\ell^2}{(\Delta+\varepsilon_d)^2}}\right)\right)^{1/d} \times\left(\frac{d-2}{d-3}-\frac{1}{d-3}\sqrt{1+\frac{(d-3)(d-1)\ell^2}{(\Delta+\varepsilon_d)^2}}\right)^{1-2/d}\right].$$

for **spacetime spin** ℓ. Shaghoulian: 1512.06855
Benjamin, Lee, Simmons-Duffin + H.O., 2306.08031

Three-Point Function at Large Conformal Dimensions

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \sum c_{123}^{s}\; V^{s}(x_1, x_2, x_3),$$

$d = 2$

$$c_{123}\sqrt{\rho(\Delta,\ell_1)\rho(\Delta,\ell_2)\rho(\Delta,\ell_3)} \approx \left(\frac{3}{2}\right)^{3\Delta}\frac{2^{13/2} f e^{\frac{9}{4}(\pi^2 f^2 \Delta)^{1/3}}}{3^{9/2}\pi^{1/2}\Delta^{5/2}}$$

$d = 3$

$$c_{123}^{s}\sqrt{\rho(\Delta,\ell_1)\rho(\Delta,\ell_2)\rho(\Delta,\ell_3)} \approx \left(\frac{3}{2}\right)^{3\Delta}\frac{2^{49/8} f^{9/8} e^{\frac{3}{2}(2\pi f\Delta)^{1/2}}}{3^{19/4}\pi^{1/4}\Delta^{31/8}} \times \prod_{i=1}^{3}(2\ell_i+1)\left(\frac{2\ell_i}{\ell_i+q_I}\right)$$

$s = [q_1, q_2, q_3]$: conformal block parameters

Benjamin, Lee, Simmons-Duffin + H.O., 2306.08031

Congratulations, Edward!