Tensor model for 3d gravity

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Based on Alex Belin, Jan de Boer, DLJ, Pranjal Nayak, Julian Sonner
WIP with Liza Rozenberg and Gabriel Wong

- Introduction
- Approximate CFTs
- Random tensor model
- 3d gravity

Ensembles for pseudo-random microdata

- Features of high energy microstates of chaotic quantum systems are pseudo-random.
- One can encode their statistical properties in an ensemble, such that typical elements preserve EFT observables.
- For quantum mechanics, this gives a random matrix model for the matrix elements of simple operators between the high energy microstates, for example $H_{ij}, \mathcal{O}_{ij}$.
- For CFTs, there are also the three point functions of high dimension operators, so it is a random tensor model, C_{ijk} .

Holographic matrix ensembles

- The Saad Shenker Stanford matrix model for Jackiw-Teitelboim gravity reinterprets a minimal string theory matrix model as the Wigner ensemble for the holographically dual Hamiltonian. The 't Hooft expansion of $< \mathrm{Tr}(e^{-\beta_1 H})\ \mathrm{Tr}(e^{-\beta_2 H}).. >$ gives the sum over 2d topologies, as dense Feynman diagrams fill out the surfaces in a double scaled limit.

- The lack of euclidean factorization is puzzling from the perspective of exact 2d holography.

- How does it work in higher dimensional versions of this?

[Mertens Turiaci; Collier Maloney Maxfield Tsiares; Cotler Jensen; Belin de Boer Liska; Belin de Boer Nayak Sonner; Anous Belin de Boer Liska; Chandra Collier Hartman Maloney …]

CFTs

- The microdata of a CFT consists of the dilation operator acting on the Hilbert space graded by spin, and the structure constants, C. These give a matrix and a tensor.

- The operators best characterized by an ensemble are in the dense part of the spectrum. For simplicity, could consider a large c 2d CFT, with no operators below the black hole threshold beside the identity Virasoro module.

- If there were light operators, these would define matrices C_{LHH}

- At small c, the discussion applies to the high dimension region of the spectrum.

OPE statistics from 3d gravity

- The Cardy formula gives the asymptotics of the spectral density

$$\rho \approx \exp\left[2\pi(\frac{c}{6}(h - \frac{c}{24}))\right] \exp\left[2\pi(\frac{c}{6}(\bar{h} - \frac{c}{24}))\right]$$

- The leading order variance of the structure constants is given by $|C_{ijk}|^2 \approx C_0(h_i, h_j, h_k)C_0(\bar{h}_i, \bar{h}_j, \bar{h}_k)$, where C_0 is the Liouville three point function given by the DOZZ formula, normalized by the spectral density.

- The associated Gaussian ensemble reproduces certain handlebody and euclidean wormhole topologies in 3d gravity.

[Chandra Collier Hartman Maloney]

Higher d has more constraints

- An important point is that unlike QM, where any Hamiltonian is allowed, CFTs have many consistency constraints. 4 point crossing encodes locality, the vanishing of operator commutators outside the lightcone. In 2d, there is also torus 1 point function modular invariance.

- Therefore, we do not expect there to exist ensembles with e^S parameters over exact CFT data, in contrast to QM Hamiltonians. No disordered exact duals are possible for higher dimensional gravity.

- We will define a notion of approximate CFT, which in an appropriate limit would localize to a true CFT.

Which gravity EFTs have UV completions?

- In AdS, this turns into a problem of classifying CFTs: is there an exact CFT whose light operators behave like a given EFT's correlation functions in AdS?

- Around the AdS vacuum, perturbative solutions to the bootstrap exist for any EFT. There are some sharp "naturalness" bounds on the Wilson coefficients from dispersion relations.

  [Heemskerk Penedones Polchinski Sully; Caron-Huot Mazac Rastelli Simmons-Duffin]

- For black hole backgrounds, corresponding to thermal correlators, there are consistent ensembles for any EFT.

  [DLJ Kolchmeyer Mukhametzhanov Sonner]

- Major constraints must come from mutually locality of black hole microstate operators among themselves.

Approximate CFTs

- Consists of a full collection of CFT data, subject to the conditions that crossing is obeyed up to a tolerance T, subject to bounds on
  - Total number of insertions
  - Dimension of external operators
  - Genus
  - Avoidance of extreme Lorentzian kinematics (Regge regime)

- These allow e^S moduli of deformation away from a true CFT.

From EFT to microstate

- One can show that this implies approximate crossing for almost all operators, including heavy operators, above the previous cutoff. This is done by performing an appropriate inversion transform on a higher genus or higher point function to pick out contributions of (bands of) heavy operators in some channel.

- It follows from Moore-Seiberg that it is equivalent to having 4 point crossing and torus 1 point modular invariance approximately satisfied for almost all operators.

- The CFT data purely for light operators can be taken to be fixed, or approximately fixed.

Variance of crossing

- Although the expectation value of the crossing equation vanishes in the gaussian ensemble, individual instantiations strongly violate it and are not approximate CFTs. This is captured by a large variance, which must be cancelled by higher moments.

From genus 2

$$G_{1122}(x)|_{\mathrm{s}} = \sum_k |C_{12k}|^2 |\mathcal{F}_{1221}(O_k|x)|^2, \quad G_{1122}(x)|_{\mathrm{t}} = \sum_{k'} C_{11k'}C_{22k'}|\mathcal{F}_{1122}(O_{k'}|1-x)|^2$$

$$\overline{s-t} = 0, \text{ but } \overline{(s-t)^2} = \overline{s^2} + \overline{t^2} - 2\overline{st} \neq 0$$

- The s t cross term vanishes in the Gaussian ensemble. Therefore there must be a quartic moment to cancel the variance.

$$\overline{C_{ijk}C_{iml}C_{njl}C_{nmk}}\Big|_c = \left\{ \begin{matrix} \mathcal{O}_k & \mathcal{O}_j & \mathcal{O}_i \\ \mathcal{O}_l & \mathcal{O}_m & \mathcal{O}_n \end{matrix} \right\} = \left| \mathbf{F}_{kl}\left[\begin{matrix} n & j \\ m & i \end{matrix} \right] C_0(h_i,h_j,h_k)C_0(h_k,h_n,h_m)\rho_0(h_l)^{-1} \right|^2$$

"Local" version of crossing

- The Ponsot-Teschner crossing kernel F is exactly the object which transforms s-channel to t-channel blocks.

$$\mathcal{F}_{ijmn}(O_k|x) = \int \mathrm{d}[O_l]\, \mathbf{F}_{kl}\left[\begin{array}{cc} n & j \\ m & i \end{array}\right] \mathcal{F}_{imjn}(O_l|1-x)$$

- Expand the crossing equation in s-channel principal series

$$\sum_q \left(C_{i_1 i_2 q} C_{i_3 i_4 q} \delta^{(2)}(P_s - P_q) - C_{i_1 i_4 q} C_{i_2 i_3 q} \left| \mathbf{F}_{P_q P_s}\left[\begin{array}{cc} P_3 & P_4 \\ P_2 & P_1 \end{array}\right] \right|^2 \right) = 0\,,$$

- For Virasoro, the principal series are the above threshold physical weights. For other operators (including Id), the δ involves contour manipulation. In approximate CFTs, the restriction on kinematics implies the δ is smeared.

$$P = \sqrt{h - (c-1)/24}$$

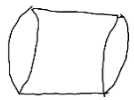
Tensor model

- Instead of specifying moments, write an explicit ensemble:

$$\mathcal{Z} = \int D[L_0, \bar{L}_0]\, D[C]\, e^{-\frac{1}{\hbar} V[L_0, \bar{L}_0, C]}$$

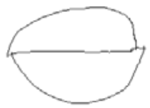
- The maximum ignorance ensemble consistent with crossing has V given by the sum of squares of the constraints.

$$V_4 = 2 \sum_{i_1 \cdots i_4}{}' \sum_{p,q} \left(\frac{C_{i_1 i_2 p} C_{i_3 i_4 p} C_{i_1 i_2 q} C_{i_3 i_4 q}}{|\rho_0(p) C_0(12p) C_0(34p)|^2} \delta^{(2)}(P_p - P_q) - \frac{C_{i_1 i_2 p} C_{i_3 i_4 p} C_{i_1 i_4 q} C_{i_2 i_3 q}}{|C_0(12p) C_0(34p) C_0(23q) C_0(14q)|^2} \begin{Bmatrix} \mathcal{O}_q & \mathcal{O}_4 & \mathcal{O}_1 \\ \mathcal{O}_p & \mathcal{O}_2 & \mathcal{O}_3 \end{Bmatrix} \right)$$



- p=Id, 1=2, 3=4 gives the propagator.

$$V_2 = \sum_{ijk} \frac{C_{ijk}^2}{|C_0(ijk)|^2}$$



Scaling limit

- The random tensor model is defined by truncating to a finite number of primaries, and then taking a triple scaled limit, in which the rank is taken to infinity, while h and the width of the smeared delta functions are taken to zero.

- In principle, in the strict limit, crossing is imposed exactly and one either finds a specific exact CFT (or collection thereof) or no solutions. In the latter case $Z \sim e^{-N/h}$.

- It is strongly coupled as a tensor model (and C=0 is not a minimum), so it might seem impossible to analyze.

3d topological expansion

- We made a very particular choice for the square of the crossing equation, which has amazing integrability properties. Diagrams fall into classes labelled by 3d-manifolds, and are equal in each class up to overall factors of h.

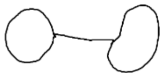
- What is interesting is to study the theory in the 't Hooft matrix + triple line tensor Feynman expansion. The expansion parameter is e^{-c}. Find a finite h →0 limit term by term.

- The ultimate fate of the strict limit at fixed c is a "doubly non-perturbative" question - sum over all topologies.

Modular invariance

- Similarly, modular invariance of torus 1 point functions can be written in terms of the modular inversion kernel, S, and we take its square as part of the potential.

$$V_{\mathrm{S},\neq\mathbf{1}} = 2\sum_{i,j,k}{}' \left|\rho_0(P_i) C_0(P_i,P_i,P_j)\right|^2 C_{iij} C_{kkj} \left(\delta^{(2)}\left(P_i - P_k\right) - \left|\mathbf{S}\left[\mathcal{O}_j\right]_{P_i P_k}\right|^2\right)$$



- This is another quadratic term. With the Id inserted, one gets

$$V_{\mathrm{S},\mathbf{1}} = \sum_{i,j}{}' \left(\delta^{(2)}(P_i - P_j) - \left|\cos(4\pi P_i P_j)\right|^2\right) - \sum_{i}{}' \left|\sinh(2\pi b P_i)\sinh\left(2\pi \frac{P_i}{b}\right)\right|^2$$

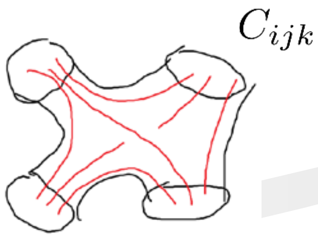
- T invariance imposes integrality of spin

$$V_T = \sum_{i,j} \delta^{(2)}(P_i - P_j)\sin^2(\pi(h_i - \bar{h}_i))$$

3d gravity

- Conjecture: in the case with no fixed primaries aside from Id (thus a gap to the black hole threshold), then in e^{-c} expansion, the triple scaled limit of the tensor model is exactly pure 3d gravity, including the sum over all hyperbolic 3-manifolds!

- Building block of the tensor part is the 4 boundary wormhole associated to the 6j vertex. The index loops are filled in with 't Hooft diagrams of the matrices – leading disk corresponds to BTZ filling of solid torus.



[Collier Eberhardt Zhang]

2 and 4 point resummed

- However, the bare propagator of the tensor model has a factor of h. This is similar to imposing $\langle x^2 \rangle = v$ by an integral

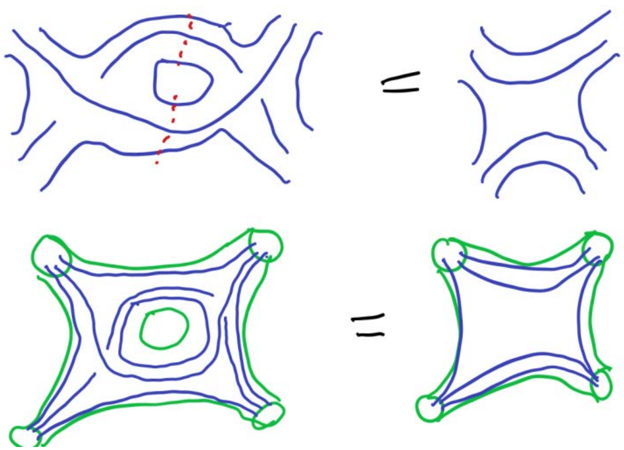
$$\int dx\ e^{-\frac{1}{h}(x^2/v-1)^2} \sim \int dx\ e^{-\frac{1}{h}(-2x^2/v+x^4/v^2)}$$

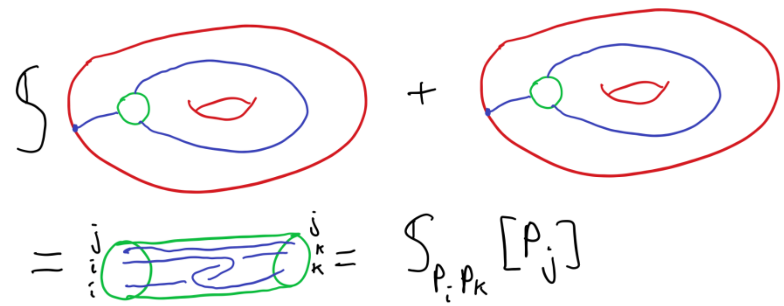
- Need to resum infinitely many diagrams. One can check Schwinger-Dyson equations

- Very similar to simplicial 3d gravity of Regge, …, Boulatov also involving 6j symbol vertices. Here matrix part also plays important role. But there is a sum over many diagrams for each 3-manifold. Integrability implies that they are all proportional! So also like Turaev-Viro, Collier-Eberhardt-Zhang.

Relation to 3d gravity

- Examples of tensor diagrams evaluating to 3d gravity partition functions, dressed with Wilson lines. Closed Wilson lines are integrated over the BTZ spectrum, corresponding to S transform toroidal surgery.



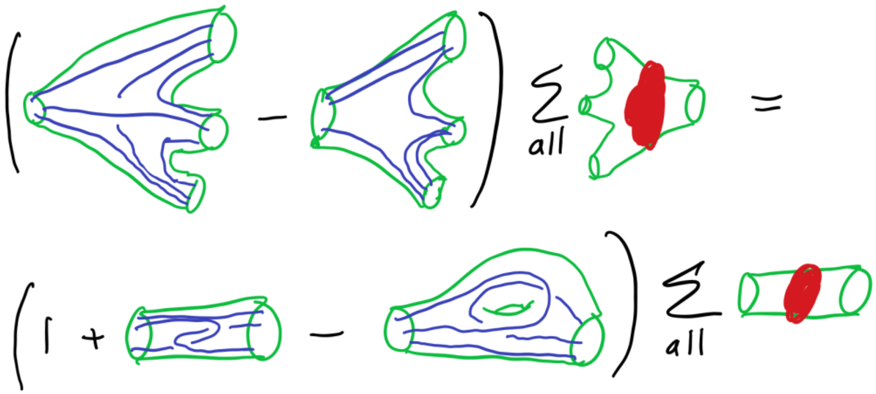


$$\int dP\rho_0(P)\frac{\begin{Bmatrix}\mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_3\\ \mathcal{O}_4 & \mathcal{O}_5 & \mathcal{O}_p\end{Bmatrix}\begin{Bmatrix}\mathcal{O}_1 & \mathcal{O}_4 & \mathcal{O}_p\\ \mathcal{O}_5 & \mathcal{O}_2 & \mathcal{O}_6\end{Bmatrix}}{C_0(14p)C_0(25p)} = \frac{C_0(123)C_0(345)}{\rho_0(3)}\delta(P_3-P_6)$$

Summing the series in h

- Conjecture is that the coefficients of each topology becomes 1 (up to symmetry factors) as h → 0, so one obtains exactly the 3d gravity partition function. It's clear that all 3-manifolds are produced since one can obtain surgeries on arbitrary links.

- If true, this implies infinitely many Schwinger-Dyson equations that are topological/combinatoric 3d analogues of 2d topological recursion.



Connection to simplicial gravity

- Also involved tensor models with 6j symbols. Didn't work, in the sense that in the double scaled limit, the diagrams didn't correspond to dense tetrahedral decompositions of smooth manifolds. Instead, melonic dominance.

- Here, we also have the "pillow" vertex, so there is a delicate balance. Moreover, the basic BTZ saddle comes from the matrix part of the theory.

- Mathematical results (used often in Chern-Simons theory) relate toroidal surgeries on similar building blocks to simplicial decomposition, but we're still working out the exact story here.

Doubly non-perturbative theory

- The e^{-c} expansion is an asymptotic series, as the number of manifolds grows faster than exponentially in the volume (on-shell 3d gravity action). There are also accumulation points, but these always correspond to hyperbolic cusps in an SL(2,Z) sum over solid torii in some piece of the manifold, which are usually regulated by zeta functions.

- However, since the action is the square of a constraint, one expects a very different completion than for 2d gravity – a specific CFT2 (probably with some (many?) operators pushed off the above threshold contour).

Relation to M5 branes

- An intriguing fact is that the reduction of the 2 M5 brane theory on S^3 in the context of $\mathcal{N}$=2 sphere partition functions is exactly the integration cycle for SL(2,C) Chern-Simons theory that gives TTQFT on the 3-manifold.

  [Dimofte Gaiotto Gukov; Cordova DLJ; Mikhaylov]

- There is probably a connection of the tensor model to the M5 brane similar to the matrix model/minimal string.

- A distinction again is that here the sum over all of the complicated 3-manifold topologies ends up imposing exact crossing, rigidifying the model.

Higher dimensions

- Higher d CFTs always have light operators, at least from TT fusion. Thus there are many matrices, and the crossing squared potential depends on the light structure constants. Same in 3d gravity with matter.

- Still only tensors with 3 indices – topologies with 3d skeletons.

- Moreover, the analog of modular invariance involves higher d operators. Perhaps their data together with constraints give more general 4d and higher manifold topologies.

Summary

- Defined a notion of approximate CFT, which is CFT data that obeys crossing up to small corrections away from extreme kinematics. One can define ensembles of such data, which must be strongly non-Gaussian.

- Leads to a tensor model that is completely determined by conformal symmetries – maximum ignorance ensemble. Its expansion is related to 3d gravity, connecting simplicial gravity and VTQFT.

- The sum over all of the hyperbolic 3-manifolds is successively imposing exact CFT crossing!