Signal communication and modular theory

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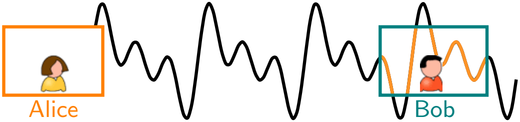
Signal communication

Suppose Alice sends a signal to Bob that is codified by a function of time f. Bob can measure the value f only within *a certain time interval*; moreover, the frequency of f is filtered by the signal device within a *certain interval in the spectrum amplitude*



Say both intervals are equal to $B = (-1, 1)$. As is well known, if a function f and its Fourier transform $\hat{f}$ are both supported in bounded intervals, then f is the zero function. So one is faced with the problem of *simultaneously maximizing* the portions of energy and amplitude spectrum within the intervals

$$||f||_{2,B}, \quad ||\hat{f}||_{2,B}\,,$$

$||f||_2 = ||\hat{f}||_2 = 1$, the concentration problem.

The concentration problem

The problem of best approximating, with support concentration, a function and its Fourier transform is a classical problem; in particular, it lies behind Heisenberg uncertainty relations in Quantum Mechanics and is studied in Quantum Field Theory too (Jaffe, etc.)

In the '60ies, this problem was studied in seminal works by Slepian, Pollak and Landau. Denote by $\mathcal{F} : f \mapsto \hat{f}$ the Fourier transform and by $\mathcal{F}_B$ the truncated Fourier transform

$$\mathcal{F}_B = E_B \mathcal{F} E_B$$

$E_B = \chi_B$,

$$(\mathcal{F}_B f)(p) = \frac{\chi_B(p)}{\sqrt{2\pi}} \int_B f(x) e^{-ixp} dx$$

as an operator on $L^2(B)$.

The functions that best maximize the concentration problem are eigenfunctions of $\mathcal{F}_B$ with the highest eigenvalues.

Since $||\mathcal{F}_B^* \mathcal{F}_B|| = ||\mathcal{F}_B||^2$, one can equivalently consider the angle operator

$$T_B \equiv \mathcal{F}_B^* \mathcal{F}_B = E_B \hat{E}_B E_B$$

with $\hat{E}_B = \mathcal{F}^* E_B \mathcal{F}$. This is a $L^2(B)$ Hilbert-Schmidt operator

$$T_B = \int_B k_B(x-y) f(y) dy$$

$$k(x) = \frac{1}{(2\pi)^{1/2}} \frac{\sin x}{x}$$

and one has the eigenvalue problem

$$T_B f = \lambda f$$

The *eigenvalue* λ *measures the level of concentration* of the corresponding eigenfunction f.

Table I—Values of $\lambda_n(c) = L_n(c) \times 10^{-p_n(c)}$

n	$c = 0.5$		$c = 1.0$		$c = 2.0$		$c = 4.0$		$c = 8.0$	
	L	p	L	p	L	p	L	p	L	p
0	3.0969	1	5.7258	1	8.8056	1	9.9589	1	1.0000	0
1	8.5811	3	6.2791	2	3.5564	1	9.1211	1	9.9988	1
2	3.9175	5	1.2375	3	3.5868	2	5.1905	1	9.9700	1
3	7.2114	8	9.2010	6	1.1522	3	1.1021	1	9.6055	1
4	7.2714	11	3.7179	8	1.8882	5	8.8279	3	7.4790	1
5	4.6378	14	9.4914	11	1.9359	7	3.8129	4	3.2028	1
6	2.0413	17	1.6716	13	1.3661	9	1.0951	5	6.0784	2
7	6.5766	21	2.1544	16	7.0489	12	2.2786	7	6.1263	3
8	1.6183	24	2.1207	19	2.7768	14	3.6066	9	4.1825	4

Figure: The first eigenvalues λ_n of T_B

$c =$ product of support lengths

The lucky accident

The spectral analysis of the angle operator is not easily doable a priori.
However, by the lucky accident figured out in by Slepian et al. , this integral operator commutes with a linear differential operator, the prolate operator

$$W = \frac{d}{dx}(1 - x^2)\frac{d}{dx} - x^2 \,,$$

indeed, $\mathcal{F}_B$ commutes with W, so these eigenfunctions were computed.

W is a classical operator, it arises by separating the 3-dimensional scalar wave equation in a prolate spheroidal coordinate system.

Connes and Moscovici recently showed a remarkable relation of the prolate spectrum with the asymptotic distribution of the zeros of the Riemann ζ-function.

Here, I want to understand the role of the prolate operator on a conceptual basis, in relation to the mentioned lucky accident: the prolate operator as an entropy operator.

I will generalize the prolate operator in higher dimensions, guided by QFT

$$W = \nabla(1 - r^2)\nabla^2 - r^2 = (1 - r^2)\nabla^2 - 2r\partial_r - r^2$$

On $S(\mathbb{R}^d)$, W is Hermitian non-selfadjont but admits a natural extension that commutes $\mathcal{F}_B$ (In the one-dimensional case, the extension is selfadjoint (Connes)).

The expectation values of W on $L^2(B)$ will be entropy quantities

Tomita-Takesaki modular theory

$\mathcal{M}$ a von Neumann algebra on $\mathcal{H}$, $\varphi = (\Omega, \cdot\Omega)$ normal faithful state on $\mathcal{M}$. Embed $\mathcal{M}$ into $\mathcal{H}$

$$S_0 : X\Omega \mapsto X^*\Omega, \quad X \in \mathcal{M}$$

$S_{\mathcal{M}} = \bar{S}_0 = J_{\mathcal{M}}\Delta_{\mathcal{M}}^{1/2}$, polar decomposition, $\Delta_{\mathcal{M}}$ and $J_{\mathcal{M}}$ modular operator and conjugation

$$t \in \mathbb{R} \mapsto \sigma_t^{\varphi} \in \mathrm{Aut}(\mathcal{M})$$

$$\sigma_t^{\varphi}(X) = \Delta_{\mathcal{M}}^{it} X \Delta_{\mathcal{M}}^{-it}, \quad X \in \mathcal{M}$$

modular automorphisms **intrinsic evolution associated with φ!**

$$J_{\mathcal{M}}\mathcal{M}J_{\mathcal{M}} = \mathcal{M}' \quad \text{on } \mathcal{H}$$

$\log \Delta_{\mathcal{M}}$ is called the modular Hamiltonian of φ

Araki's relative entropy

An infinite quantum system is described by a von Neumann algebra $\mathcal{M}$ typically not of type I so Tr does not exist; however Araki's relative entropy between two faithful normal states φ and ψ on $\mathcal{M}$ is defined in general by

$$S(\varphi\|\psi) \equiv -(\eta, \log \Delta_{\xi,\eta}\, \eta)$$

where ξ, η are cyclic vector representatives of φ, ψ and $\Delta_{\xi,\eta}$ is the relative modular operator associated with ξ, η.

$$S(\varphi\|\psi) \geq 0$$

positivity of the relative entropy

Standard subspaces

$\mathcal{H}$ complex Hilbert space and $H \subset \mathcal{H}$ a *closed, real linear subspace*. *Symplectic complement*:

$$H' = \{\xi \in \mathcal{H} : \Im(\xi, \eta) = 0 \ \forall \eta \in H\}$$

H is a standard subspace if it is $\mathcal{H}$ *cyclic* if $\overline{H + iH} = \mathcal{H}$ and *separating* $H \cap iH = \{0\}$

$$\textit{H standard subspace} \rightarrow \textit{anti-linear operator } S_H$$

$$S_H : \xi + i\eta \rightarrow \xi - i\eta, \ \ \xi, \eta \in H$$

$S_H^2 = 1|_{D(S_H)}$, $D(S_H) = H + iH$. S_H is closed, densely defined, $S_H^* = S_{H'}$

Modular theory for standard subspaces

Set $S_H = J_H \Delta_H^{1/2}$, polar decomposition. Then J_H is an anti-unitary involution, $\Delta_H > 0$ is non-singular called the modular conjugation and the modular operator; they satisfy $J_H \Delta_H J_H = \Delta_H^{-1}$ and

$$\Delta_H^{it} H = H, \quad J_H H = H'$$

(one particle Tomita-Takesaki theorem).

Example: $\mathcal{M}$ von Neumann algebra on $\mathcal{H}$, Ω cyclic separating vector

$$H = \overline{\mathcal{M}_{\mathrm{s.a.}}\Omega} \text{ is a standard subspace of } \mathcal{H}$$

$$\Delta_H = \Delta_{\mathcal{M}}, \quad J_H = J_{\mathcal{M}}$$

Passivity

$\log \Delta_H$ is characterised by complete passivity, following Pusz and Woronowicz in the von Neumann algebra case

$\mathcal{H}$ a complex Hilbert space, $H \subset \mathcal{H}$ a standard subspace and A a selfadjoint linear operator on $\mathcal{H}$ such that $e^{isA}H = H$, $s \in \mathbb{R}$.

A is passive with respect to H if

$$-(\xi, A\xi) \geq 0\,, \quad \xi \in D(A) \cap H\,.$$

A is completely passive w.r.t. H if the generator of $e^{itA} \otimes e^{itA} \cdots \otimes e^{itA}$ is passive with respect to the n-fold tensor product $H \otimes H \otimes \cdots \otimes H$, all $n \in \mathbb{N}$.

A is completely passive with respect to H iff $\log \Delta_H = \lambda A$ for some $\lambda \geq 0$.

positivity of energy $\leftrightsquigarrow$ comp. passivity of modular Hamiltonian (equivalence in principle)

Entropy of a vector relative to a real linear subspace

Let $\mathcal{H}$ be a complex Hilbert space and $H \subset \mathcal{H}$ a standard subspace

The entropy of a vector $h \in \mathcal{H}$ with respect to $H \subset \mathcal{H}$ is defined by

$$S(h\|H) = -\Im(h, P_H i \log \Delta_H\, h) = \Re(h, iP_H i \log \Delta_H\, h)$$

(in a quadratic form sense), where P_H is the cutting projection

$$P_H : H + H' \to H\,, \quad h + h' \mapsto h$$

We have $P_H^* = -iP_H i$ and the formula

$$P_H = (1 + S_H)(1 - \Delta_H)^{-1} \\ = (1 - \Delta_H)^{-1} + J_H \Delta_H^{1/2}(1 - \Delta_H)^{-1}\,;$$

(P_H is the closure of the right-hand side).

In QFT, the **cutting projection** P_H is geometric.

Properties of the entropy of a vector

Some of the main properties of the entropy of a vector are:

- $S(h\|H) \geq 0$ or $S(h\|H) = +\infty$ positivity
- If $K \subset H$, then $S(h\|K) \leq S(h\|H)$ monotonicity
- If $h_n \to h$, then $S(h\|H) \leq \liminf_n S(h_n|H)$ lower semicontinuity
- If $H_n \subset H$ is an increasing sequence with $\overline{\bigcup_n H_n} = H$, then $S(h\|H_n) \nearrow S(h\|H)$ monotone continuity
- If $h \in D(\log \Delta_H)$ then $S(h\|H) < \infty$ finiteness on smooth vectors
- $S(h\|H) = S(k\|H)$ if $k - h \in H'$ locality

Entropy of coherent sectors

Given $\Phi \in \mathcal{H}$ consider coherent state φ_Φ on Weyl von Neumann algebra $\mathcal{A}(H)$ on the Bose Fock space $e^{\mathcal{H}}$:

The vacuum relative entropy of φ_Φ on $\mathcal{A}(H)$ is given by

$$S(\varphi_\Phi \| \varphi_0) = S(\Phi \| H)$$

Araki's relative entropy

Entropy of vector

(φ_0 vacuum state)

Entropy operator

The entropy operator $\mathcal{E}_H$ is defined by

$$\mathcal{E}_H = iP_H i \log \Delta_H$$

(closure of the right-hand side). We have

$$S(h\|H) = (h, \mathcal{E}_H h)\,, \quad k \in \mathcal{H}\,.$$

real quadratic form sense.

The entropy operator $\mathcal{E}_H$ is *real linear, positive, and selfadjoint* w.r.t. to the real part of the scalar product.

In my view, an *entropy operator* $\mathcal{E}$ *is a real linear operator on a real or complex Hilbert space* $\mathcal{H}$*, such* $\mathcal{E}$ *is positive, selfadjoint and its expectation values* $(f, \mathcal{E}f)$*,* $f \in \mathcal{H}$*, correspond to entropy quantities*

The information in a wave packet

By a wave (or wave packet), we mean a real solution of the wave equation

$$\Box\Phi = 0 \ ,$$

with compactly supported, smooth Cauchy data $\Phi|_{x^0=0}$, $\Phi'|_{x^0=0}$.

Classical field theory describes Φ by the stress-energy tensor $T_{\mu\nu}$, which provides the energy-momentum density of Φ at any time.

But, how to define the information, or entropy, carried by Φ in a given region at a given time?

We give an answer to a classical question by Operator Algebras and Quantum Field Theory

Joint works with F. Ciolli, G. Ruzzi, G. Morsella

Local entropy of a wave packet

The real linear wave's space $\mathcal{T}$ is given in Cauchy data

$$\Phi \leftrightarrow \langle f, g \rangle \in S(\mathbb{R}^d) \times S(\mathbb{R}^d)$$

• The complex structure on $\mathcal{T}$ is then

$$\imath_0 = \begin{bmatrix} 0 & \mu^{-1} \\ -\mu & 0 \end{bmatrix}, \quad \mu = \sqrt{-\nabla^2}$$

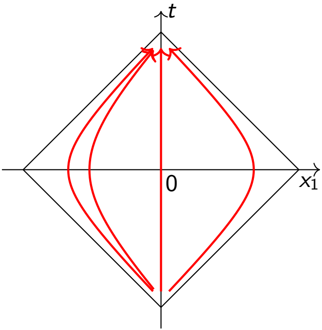
• The scalar product on $\mathcal{T}$ is the unique Poincaré covariant one

• Local structure: Waves with Cauchy data supported in region O (causal envelop of a space region B) form a real linear subspace $H(O) \equiv H(B)$.

• The information $S(\Phi \| O)$ carried by the wave Φ in the region O is the entropy $S(\Phi \| H(O))$ of the vector Φ w.r.t. $H(O)$

Double cone, conformal case

For a bounded region O (double cone, causal envelop of a space ball B), in the conformal case the modular group is given by the geometric transformation (Hislop, L. '81)



local modular trajectories

$$(u, v) \mapsto \big((Z(u, s), Z(v, s)\big)$$

$$Z(z, s) = \frac{(1+z) + e^{-s}(1-z)}{(1+z) - e^{-s}(1-z)}$$

$$u = x_0 + r, \quad v = x_0 - r, \quad r = |\mathbf{x}| \equiv \sqrt{x_1^2 + \cdots + x_d^2}$$

Massless modular Hamiltonian

In terms of the wave Cauchy data, the local massless modular Hamiltonian associated with the unit space ball B is given by

$$\log \Delta_B = -2\pi\imath_0 \begin{bmatrix} 0 & \frac{1}{2}(1-r^2) \\ \frac{1}{2}(1-r^2)\nabla^2 - r\partial_r - D & 0 \end{bmatrix}$$

$D = (d-1)/2$ the scaling dimension of the free scalar field. Namely

$$\log \Delta_B = -2\pi\imath_0 \begin{bmatrix} 0 & M \\ L - D & 0 \end{bmatrix}$$

with

$$M = \text{ Multiplication operator by } \frac{1}{2}(1-r^2)$$

$$L = \text{ Legendre operator } \frac{1}{2}(1-r^2)\nabla^2 - r\partial_r$$

Entropy density of a wave packet

The classical stress-energy tensor gives the energy

$$\langle T_{00}^{(0)} \rangle_\Phi = \frac{1}{2}\big((\partial_0\Phi)^2 + |\nabla_{\mathbf{x}}\Phi|^2\big)$$

we then have

$$-(\Phi, \log \Delta_B \Phi) = 2\pi \int_{x_0=0} \frac{1-r^2}{2} \langle T_{00}^{(0)} \rangle_\Phi(x)dx + \pi D \int_{x_0=0} \Phi^2 dx$$

The entropy of a wave Φ in the unit ball B is

$$S(\Phi \| B) = 2\pi \int_B \frac{1-r^2}{2} \langle T_{00}^{(0)} \rangle_\Phi(x)dx + \pi D \int_B \Phi^2 dx$$

Massive case: numerical results by H. Bostelmann, D. Cadamuro, C. Minz

Asymptotics

With $S(\Phi\|R)$ the entropy of the wave packet Φ in the radius R ball B_R

$$\frac{S(\Phi\|R)}{R} \sim \pi E_R\,, \quad R \to \infty$$

$E_R = \int_{B_R} T^{(0)}_{00} d\mathbf{x}$, in agreement with the Bekenstein bound

$$\frac{S(\Phi\|R)}{R} \le \pi R E_R$$

On the other hand, as $R \to 0$,

$$S(\Phi\|R) = 2\pi \frac{d-1}{d} A_{d-1}(R)\Phi^2(0,0) + \cdots$$

$A_{d-1}(R)$ area of the $d-1$ dimensional sphere ∂B_R (cf. holographic thms)

Higher-dimensional Legendre operator

The Legendre operator is the one-dimensional Sturm-Liouville linear differential operator $\frac{d}{dx}(1-x^2)\frac{d}{dx}$. We consider a natural higher-dimensional generalization.

We denote by L the d-dimensional Legendre operator, on $L^2(\mathbb{R}^d)$, initialliy defined on $S(\mathbb{R}^d)$

$$L = \nabla(1-r^2)\nabla = (1-r^2)\nabla^2 - 2r\partial_r \,;$$

The quadratic form associated with L is

$$(f, Lg) = -\int_{\mathbb{R}^d} (1-r^2)\nabla\bar{f}\cdot\nabla g\,dx\,, \quad f, g \in S(\mathbb{R}^d)\,,$$

L is a Hermitian operator.

Higher-dimensional prolate operator

Let W be the operator on $L^2(\mathbb{R}^d)$ given by

$$W = \nabla(1 - r^2)\nabla - r^2 = L - r^2$$

with $D(W) = S(\mathbb{R}^d)$. W is a higher-dimensional generalisation of the prolate operator.

W is a Hermitian, being a Hermitian perturbation of L on $S(\mathbb{R}^d)$; moreover,

$$-W \geq -L \geq 0$$

on $D(W) \cap L^2(B)$

Note the equality

$$W = L + M - 1$$

with M multiplication by $(1 - r^2)$. This makes a connection with the modular Hamiltonian

• W commutes with the Fourier transformation $\mathcal{F}$:

$$\widehat{W} = W \, .$$

• Any linear combination of L and M commuting with $\mathcal{F}$ is proportional to W

• W has a natural Hermitian extension that commutes with $\mathcal{F}$ and E_B, thus with $\hat{E}_B$ and $\mathcal{F}_B$ too

Higher dimensional angle operator

The angle operator $E_B \hat{E}_B E_B$ is of trace class, indeed $E_B \hat{E}_B|_{L^2(B)}$ is the positive Hilbert-Schmidt T_B on $L^2(B)$ with kernel $k_B(x-y)$

$$k_B(z) = \frac{1}{(2\pi)^{d/2}} \int_B e^{-ix\cdot z} dx \, \chi_B(z)$$

The eigenvalues of T_B are strictly positive, with finite multiplicity

$$\lambda_1 > \lambda_2 > \cdots \lambda_k > \cdots > 0$$

The k-eigenfunctions are concentrated at level λ_k in an appropriate sense

$-E_B W$ is positive. Both W and L commute with E_B, and we consider their restrictions W_B and L_B to $L^2(B)$

Legendre and Parabolic entropies

The entropy operator $\mathcal{E}'_B$ on $L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$ corresponding to $\mathcal{E}_B$ is given by

$$\mathcal{E}'_B = \begin{bmatrix} -\pi E_B L_D & 0 \\ 0 & \pi E_B M \end{bmatrix}$$

With $f \in S(\mathbb{R}^d)$ real, we set

$$\pi(f, Mf)_B = \pi \int_B (1 - r^2) f^2 dx = \text{parabolic entropy of } f \text{ in } B\,.$$

$$-\pi(f, Lf)_B = \pi \int_B (1 - r^2) |\nabla f|^2 dx = \text{Legendre entropy of } f \text{ in } B$$

(the modular Hamiltonian gives rise to two entropies on Cauchy data, the filed entropy and the momentum entropy)

Prolate entropy

The Parabolic/Legendre entropies are the field/momentum entropies associated with a wave

Now, $-LE_B = -WE_B + ME_B - E_B$, so πWE_B is an entropy operator too; we thus define:

$$-\pi(f, Wf)_B = \pi \int_B \left((1-r^2)|\nabla f|^2 + r^2\right) dx = \textit{prolate entropy} \text{ of } f \text{ in } B\,,$$

$f \in S(\mathbb{R}^d)$ real.

Conclusion

$-\pi WE_B$ is an entropy operator on $L^2(\mathbb{R}^d)$. The sum of the prolate entropy and the parabolic entropy is equal to the sum of the Legendre entropy and the Born entropy, all with respect to B

The measure of concentration

One-dimensional case: As T_B is strictly positive and Hilbert-Schmidt, its eigenvalues can be ordered as $\lambda_1 > \lambda_2 > \cdots > 0$; they are simple.

the eigenvalues of $-W_B$ can be ordered as

$$\alpha_1 < \alpha_2 < \cdots < \infty$$

correspond to the λ_k's in inverse order. Then

$$(f_k, T_B f_k)_B = \lambda_k\,, \quad -(f_k, W_B f_k)_B = \alpha_k\,,$$

and $\pi\alpha_k$ is the prolate entropy of f_k.

lower prolate entropy $\longleftrightarrow$ higher concentration

where the concentration is both on space and in Fourier modes as above. This is intuitive since information is the opposite of entropy. *In other words, in order to maximize simultaneously both quantities* $||f||_{2,B}/||f||_2$ *and* $||\hat{f}||_{2,B}/||f||_2$ *we have to minimize the prolate entropy.*

The lucky accident is not an accident

$-\pi W E_B$ is an entropy operator on $L^2(\mathbb{R}^d)$:

$-\pi(f, Wf)_B$ is the sum of the Legendre entropy of f and $\pi||f||_B^2$ (Born entropy), minus the parabolic entropy of f, i.e.

$$-\pi(f, Wf)_B + \pi \int_B (1-r^2) f^2 dx = \pi \int_B (1-r^2)|\nabla f|^2 dx + \pi \int_B f^2 dx\,.$$

We conclude that $-\pi(f, Wf)_B$ is an entropy quantity, i.e. a measure of information, the *prolate entropy* of f w.r.t. B.

The *lucky accident*, that W commutes with the truncated Fourier transform, finds a conceptual clarification in this fact; namely, W is a natural a priori candidate to commute with $\mathcal{F}_B$