Topological order and generalized symmetry

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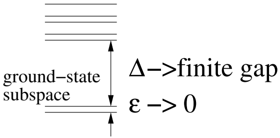
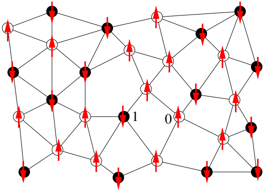
topological quantum field theory and gapped phase of matter

generalized symmetry and gapless phase of matter

Local quantum system and gapped quantum phase

- **Definition**: A **local quantum system** is described by $(\mathcal{V}_N, H_N)$
  $\mathcal{V}_N$: a Hilbert space with a tensor structure $\mathcal{V}_N = \otimes_{i=1}^N \mathcal{V}_i$
  H_N: a local Hamiltonian acting on $\mathcal{V}_N$:
  $H_N = \sum \hat{O}_{ij}$



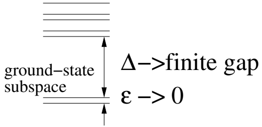


- A **gapped ground state** (a concept for $N \to \infty$ limit) has $\Delta \to$ finite non-zero and $\epsilon \to 0$. A gapped ground state is not a single vector in $\mathcal{V}_N$, but a subspace $\mathcal{V}_{\text{grnd space}} \subset \mathcal{V}_N$.
- Two gapped Hamiltonian $H(0)$ and $H(1)$ are equivalent if they are connected by a path $H(\tau)$, $\tau \in [0,1]$ of gapped Hamiltonians. Their ground states are also equivalent. The equivalence classes of gapped ground states are **gapped quantum phases of matter**
- Phases of matter is a central issue in condensed matter physics

Symmetry breaking phases and beyond $\to$ topologically ordered phases

- For a long time, we thought that non-trivial gapped phases exist only when the Hamiltonians have a symmetry: $WHW^\dagger = H$, where the unitary operators W form a **symmetry group** G_H.

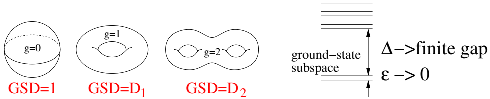
  A classification: Gapped quantum phases are classified by a pair (G_H, G_Ψ) $(G_\Psi \subset G_H)$:
  G_H is the symmetry group of Hamiltonian.
  G_Ψ is the group that acts trivially in ground state subspace



- In 1989, we realized that non-trivial gapped phases of matter exist even without symmetry, *ie* $G_H = \{\mathbf{id}\}$ $\to$ notion of **Topological order**. Examples include **chiral spin states** and **fraction quantum Hall (FQH) states**. Wen, PRB **40** 7387 (89); IJMP **4** 239 (90)

What is topological order?

- How to extract macroscopic character (topological invariants) from complicated many-body wavefunction $\Psi(x_1, \cdots, x_{10^{20}})$



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Put the gapped system on space with various topologies, and measure the ground state degeneracy → topological order

Vacuum degeneracy of chiral spin states in compactified space

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(Received 10 May 1989)

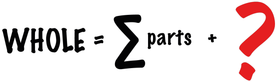
A chiral spin state is not only characterized by the T and P order parameter $E_{123} = \mathbf{S}_1 \cdot (\mathbf{S}_2 \times \mathbf{S}_3)$, it is also characterized by an integer k. In this paper we show that this integer k can be determined from the vacuum degeneracy of the chiral spin state on compactified spaces. On a Riemann surface with genus g the vacuum degeneracy of the chiral spin state is found to be $2k^g$. Among those vacuum states, some k^g states have $\langle E_{123} \rangle > 0$, while other k^g states have $\langle E_{123} \rangle < 0$. The dependence of the vacuum degeneracy on the topology of the space reflects some sort of topological ordering in the chiral spin state. In general, the topological ordering in a system is classified by topological theories.

Robust ground state degeneracy→phase of matter

Objection: Ground state degeneracy (GSD) on $S^2 \neq$ GSD on T^2 coming from the motion of center of mass. GSD is just a finite size effect, which does not reflect the thermodynamic phase of matter.

- We showed that the ground state degeneracy is **robust** against any local perturbations. GSD can change only after a phase transition → The robust ground state degeneracy characterize new phases of matter
  → **topological (=robust) order**. Wen Niu PRB 41, 9377 (90)

- The microscopic mechanism of superconductivity: electron pairing and their condensation
  The microscopic mechanism of topological order is **long range entanglement**



Wen, PRB **40** 7387 (89); IJMPB 4, 239 (90). Chen Gu Wen arXiv:1004.3835

For a many-body state $|\Psi\rangle = \sum_{m_i} \Psi(m_1, ..., m_{10^{23}})|\{m_i\}\rangle$, knowing all its overlapping parts still cannot determine the whole state $|\Psi\rangle$.

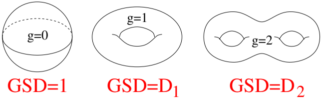
Theory of topo. order (long range entanglement)

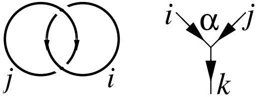
Symmetry breaking orders are described by group theory. What theory describes topological orders (long range entanglement)?

There are two approaches:

- **Ground states**: Robust degenerate ground states form vector bundles on moduli spaces of gapped Hamiltonians → **moduli bundle theory** for topological orders.

Wen, IJMPB 4, 239 (90); Wen Niu PRB 41, 9377 (90)



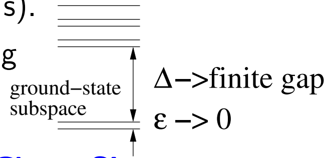


- **Excitations**: The anyons are described by their **fusion and braiding** → **modular tensor category theory** for topological orders

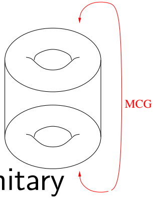
Moore Seiberg CMP **123** 177 (89). Witten, CMP **121** 352 (89)

Moduli bundle theory of topological order

The important data is the **connections** of ground-state vector bundle on moduli space (space of gapped H's).



- Non-Abelian holonomy (unitary matrix) along **contractable loops** in moduli space $\rightarrow$ a diagonal $U(1)$ factor acting on the degenerate ground states $\rightarrow$ **gravitational Chern-Simons term** $\rightarrow$ **chiral central charge** c of edge state



- Non-Abelian holonomy (unitary matrix) along **non-contractable loops** in moduli space (deform a system such that the deformed system is connected to the original system by a coordinate transformation) $\rightarrow$ S, T unitary matrices acting on the degenerate ground states $\rightarrow$ **projective representation of mapping-class-group** (which is $SL(2,\mathbb{Z})$ for torus, generated by $s:(x,y)\rightarrow(-y,x)$, $t:(x,y)\rightarrow(x+y,y)$)

Wen, PRB **40** 7387 (89); IJMPB 4, 239 (90).

Modular tensor category theory for anyons and 2+1D topological orders


- Excitation in 2+1D topological order → **Braided fusion category (or modular tensor category MTC)** → A theory for 2+1D topological orders for bosons. rational CFT → TQFT → MTC

Moore-Seiberg CMP **123** 177 (89); Witten, CMP **121** 352 (89)

- In higher dimensions, topological excitations can be **point-like**, **string-like**, *etc* , which can fuse and braid →

- Topological excitations are described by **non-degenerate braided fusion *n*-categories**.

- This leads a theory and a classification of topological orders in *n* spacial dimensions (up to invertible topological orders).

Lan Kong Wen, arXiv:1704.04221 for 3+1D

Classify 2+1D bosonic topological orders (TOs)

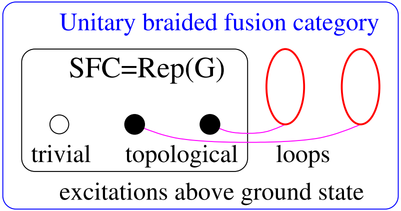
Using moduli bundle theory (*ie* $SL(2, \mathbb{Z})$ representations), plus input from modular tensor category (MTC), we can classify MTCs, *ie* 2+1D bosonic topological orders (up to invertible $E(8)$ states):

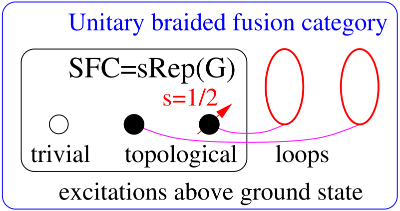
# of anyon types (rank)	1	2	3	4	5	6	7	8	9	10	11
# of 2+1D TOs	1	4	12	18	10	50	28	64	81	76	44
# of Abelian TOs	1	2	2	9	2	4	2	20	4	4	2
# of non-Abelian TOs	0	2	10	9	8	46	26	44	77	72	42
# of prime TOs	1	4	12	8	10	10	28	20	20	40	44

Rowell Stong Wang, arXiv:0712.1377: up to rank 4
Bruillard Ng Rowell Wang, arXiv:1507.05139: up to rank 5
Ng Rowell Wang Wen, arXiv:2203.14829: up to rank 6
Ng Rowell Wen, arXiv:2308.09670: up to rank 11

- **This classifies all 2+1D gapped phases for bosonic systems without symmetry, with 11 topological excitations or less.**

Classify 3+1D bosonic topological orders (*ie* non-degenerate braided fusion 2-category)

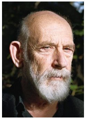
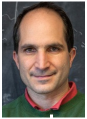




- A **3+1D topological order** in bosonic system with no symmetry has point-like and string-like excitations = **an unitary non-degenerate braided fusion 2-category**.
- The point-like excitations have trivial mutual statistics and are bosons or fermions described by **symmetric fusion category (SFC)**. Due to **Tanaka duality**, the point-like excitations are described by representations of a group G or $\mathbb{Z}_2^f \leftthreetimes G_b$.

Topological holographic principle

String holographic principle: Susskind hep-th/9409089
boundary CFT = bulk AdS gravity Maldacena hep-th/9711200

- **Holographic principle of topological order**:
  Boundary determines bulk,
  but bulk does not determine boundary

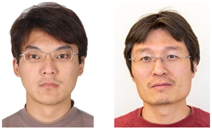

Kong Wen arXiv:1405.5858; Kong Wen Zheng arXiv:1502.01690

The excitations in a $n+1$D topological order are described by a braided fusion n-category $\mathcal{M}$. The excitations on its gapped boundary of are described by a fusion n-category $\mathcal{F}$
$\mathcal{F}$ **determines** $\mathcal{M}$: $\mathcal{Z}(\mathcal{F}) = \mathcal{M}$ ($\mathcal{Z}$ is generalized Drinfeld-center)

- **A generalization of anomaly in-flow**: Callan Harvey, NPB **250** 427 (1985)
  The theory described by fusion category $\mathcal{F}$ has a (non-invertible) gravitational anomaly (*ie* no UV completion) Kong Wen arXiv:1405.5858
  (non-invertible) grav anomaly = bulk topological order $\mathcal{M}$

Classify 3+1D AB topological orders for bosons

- Consider a 3+1D topological order $\mathcal{M}$ where all point-like excitations are bosons (**AB topological orders**): $\text{SFC} = \mathcal{R}\text{ep}(G)$.
- We condense all the point-like excitations to obtain the canonical string-only boundary of $\mathcal{M}$.
- The string-only boundary is described by a pointed fusion 2-category $\mathcal{F}$ with only trivial morphisms. Boundary determines bulk: $\mathcal{M} = \mathcal{Z}(\mathcal{F})$.
- **3+1D AB topological orders are one-to-one classifed by pointed fusion 2-categories with only trivial morphisms, or by pairs (G, ω_4), with G a finite group and $\omega_4 \in H^4(G; \mathbb{R}/\mathbb{Z})$**



Lan Kong Wen, arXiv:1704.04221

Physical implications

- 3+1D Dijkraaf-Witten theories are also classified by pairs (G, ω_4), $\omega_4 \in H^4(G; \mathbb{R}/\mathbb{Z})$.

All 3+1D AB topological orders are classified and realized by Dijkraaf-Witten theories.

Dijkgraaf Witten, Comm. Math. Phys., **129** 393, (1990)

- **All 3+1D twisted higher gauge theories are equivalent to twisted 1-gauge theories, when all point-like excitations are bosonic**

Classify 3+1D EF topological orders for bosons

- When some point-like excitations are fermions, they are described by SFC $= s\mathcal{R}ep(G_f)$, with $G_f = Z_2^f \leftthreetimes_{\epsilon_2} G_b$, $\epsilon_2 \in \mathcal{H}^2(G_b, Z_2)$.
  $\to$ **emergent-fermion (EF) topological order**
- **3+1D EF topological orders $\mathcal{C}^4_{EF}$ are classified by a subset of unitary pointed fusion 2-categories, called EF 2-categories** (describing the canonical boundary $\mathcal{F}$ from condensing all bosons).

Lan-Wen arXiv:1801.08530

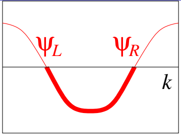
- The objects (boundary strings) and their fusion are described by a group $G_b \leftthreetimes_{\mu_2} Z_2^m$. G_b describes the **elementary** objects and $Z_2^m = \{\mathbf{1}, s_{SC}\}$ is generated by a **descendant** object s_{SC}, which is the p-wave superconducting (SC) string formed by the fermions f.

Kitaev cond-mat/0010440

- There is one simple fermionic 1-morphism f of quantum dimension 1 connecting every object g to itself (boundary fermion).
- There is one simple 1-morphism m of quantum dimension $\sqrt{2}$ connecting a pair of objects differ by s_{SC} (Majorana zero-mode).

Next adventure: a general theory for gapless state

A gapless state has emergent (and exact) symmetry: $U(1) \to U_R(1) \times U_L(1)$



- Group-like symmetries Heisenberg, Wigner, 1926
- Anomalous symmetries 't Hooft, 1980
- Higher-form symmetries Nussinov Ortiz 09; Gaiotto Kapustin Seiberg Willett 14
- Higher-group symmetries Kapustin Thorngren 2013
- Algebraic higher symmetry Kong Lan Wen Zhang Zheng 20

  algebraic (higher) symmetry = non-invertible (higher) symmetry = fusion (higher) category symmetry = ...

  Petkova Zuber 2000; Coquereaux Schieber 2001; Thorngren Wang 19; ... for 1+1D CFT
- (Non-invertible) gravitational anomalies Kong Wen 2014; Ji Wen 2019

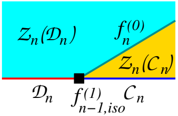
• Conjecture: **The maximal emergent (generalized) symmetry largely determine the gapless states (*ie* CFT with $\omega \sim k$).**

A classification of maximal emergent (generalized) symmetries → A classification of gapless states (CFTs). Chatterjee Ji Wen arXiv:2212.14432

What is the general theory for all those generalized symmetries, which can be beyond group and higher group?

Symmetry/Topological-Order correspondence

A **(generalized) symmetry** corresponds to:



- an **isomorphic decomposition** $\mathcal{D}_n \cong \mathcal{C}_n \boxtimes_{\mathcal{Z}_n(\mathcal{C}_n)} f_n^{(0)}$

  Kong Wen Zheng arXiv:1502.01690; Freed Moore Teleman arXiv: 2209.07471
- a **non-invertible gravitational anomaly** Ji Wen arXiv:1905.13279
- a **symmetry + dual symmetry + braiding** Ji Wen arXiv:1912.13492

  Conservation/fusion-ring of **symmetry charges** = symmetry

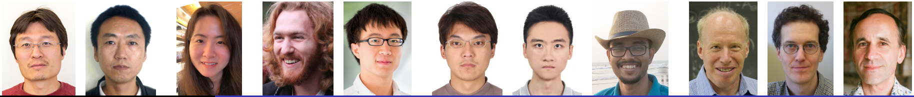
  Conservation/fusion-ring of **symmetry defects** = dual-symmetry
- a **gappable-boundary topological order** in one higher dimension

  Ji Wen arXiv:1912.13492; Kong Lan Wen Zhang Zheng arXiv:2005.14178
- a **Braided fusion higher category in trivial Witt class**

  Thorngren Wang arXiv:1912.02817 (1+1D); Kong Lan Wen Zhang Zheng arXiv:2005.14178.

  → a unified frame work to classify SSB, TO, SPT, SET phases.
- a **topological skeleton** in QFT Kong Zheng arXiv:2011.02859
- an **algebra of patch commutant operators.**

  Kong Zheng arXiv:2201.05726; Chatterjee Wen arXiv:2203.03596



Symmetry ∼ non-invertible gravitational anomaly

- A symmetry is generated by an unitary operators U that commute with the Hamiltonian: $UH = HU$.
- We consider a symmetric system (with lattice UV completion) restricted in the **symmetric sub-Hilbert space**

$$U\mathcal{V}_{\text{symmetric}} = \mathcal{V}_{\text{symmetric}}.$$

Look at a symmetry as an **algebra of local symmetric operators**

- The symmetry transformation U acts trivially within $\mathcal{V}_{\text{symmetric}}$.

How to know there is a symmetry? How to identify the symmetry?

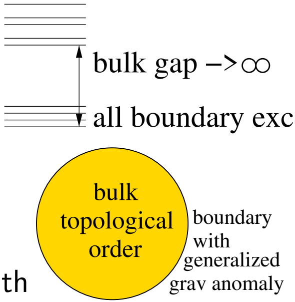
- The total Hilbert space $\mathcal{V}_{\text{tot}}$ has a tensor product decomposition $\mathcal{V}_{\text{tot}} = \otimes_i \mathcal{V}_i$, where i labels sites, due to the lattice UV completion.
- The symmetric sub-Hilbert space $\mathcal{V}_{\text{symmetric}}$ does not have a tensor product decomposition $\mathcal{V}_{\text{symmetric}} \neq \otimes_i \mathcal{V}_i$, indicating the presence of a symmetry.
- Lack of tensor product decomposition → gravitational anomaly.
→ **symmetry ≅ non-invertible gravitational anomaly**

Symmetry $\cong$ topological order in one higher dim

- **Gravitational anomaly = topo. order in one higher dim**

- The total boundary Hilbert space of a topologically ordered state has no tensor product decomposition. Yang etal arXiv:1309.4596
Lack of tensor product decomposition is described by boundary of topological order

Kong Wen arXiv:1405.5858



Systems with a (generalized) symmetry (restricted within $\mathcal{V}_{\text{symmetric}}$) can be fully and exactly simulated by boundaries of a topological order, called **symmetry-TO** (with lattice UV completion) or **symmetry TFT**.

Ji Wen arXiv:1912.13492; Kong Lan Wen Zhang Zheng arXiv:2005.14178
Apruzzi Bonetti Etxebarria Hosseini Schafer-Nameki arXiv:2112.02092

- Symmetry-TO or symmetry TFT was originally called **categorical symmetry** in Ji Wen arXiv:1912.13492; Kong etal arXiv:2005.14178
$\to$ **Symm/TO correspondence**

Summary

Gapped phases of matter in $n+1$-dim spacetime $\leftrightarrow$
Topological orders in $n+1$-dim spacetime $\leftrightarrow$
Non-degenerate braided fusion n-category $\leftrightarrow$
Non-invertible gravitational anomaly in n-dim spacetime $\leftrightarrow$
Generalized symmetry in n-dim spacetime $\leftrightarrow$
Gapless phase (CFT) in n-dim spacetime

Symmetry is a shadow of topological order in one higher dimension