Analytic Langlands correspondence over $\mathbb{R}$

with E. Frenkel and D. Kazhdan

arXiv: 2311.03743

G – connected reductive group

$G^\vee$ – Langlands dual group (corresponds to dual root datum), e.g. $G = PGL_n$, $G^\vee = SL_n$.

X – smooth projective curve over a field F. $\left(\begin{array}{l} F=\mathbb{C}: \text{ compact Riemann surface;} \\ F=\mathbb{R}: \text{ compact Riemann surface with } \tau: X \to X \text{ antiholomorphic involution} \end{array} \right)$

$Bun_G(X)$ – moduli space (stack) of principal G-bundles on X.

(Global) Langlands correspondence (for function fields) arises when we do harmonic analysis on $Bun_G(X)$, namely

diagonalize commuting Hecke operators (or functors) acting on a space of functions on $Bun_G(X)$,
(or category of sheaves)
when the eigenfunctions (eigensheaves) and eigenvalues are parametrized by data described in terms of $G^\vee$!

There are three flavors of Langlands correspondence.

1. Arithmetic (Langlands, late 1960s):

$F = \mathbb{F}_q$ is a finite field, Hecke operators act on $L^2(Bun_G(X)(F))$, eigenfunctions are parametrized by $G^\vee(\mathbb{C})$-local systems on X (= homomorphisms $\pi_1^{alg}(X) \longrightarrow G^\vee(\mathbb{C})$). (roughly speaking).

2. Geometric (Beilinson–Drinfeld, 1990s)

1-st variant: $F = \mathbb{C}$,
Hecke functors act on the derived category $\mathcal{D}\text{-mod}(\mathrm{Bun}_G(X))$ of $\mathcal{D}$-modules (or constructible sheaves),
eigenfunctions are parametrized by $G^\vee$-local systems on X (= homomorphisms $\pi_1(X) \longrightarrow G^\vee(\mathbb{C})$).

2-nd variant: F – any field,
Hecke functors act on the derived category $Sh(\mathrm{Bun}_G(X))$ of ℓ-adic constructible sheaves on X,
eigenfunctions are parametrized by ℓ-adic $G^\vee$-local systems on X (= homomorphisms $\pi_1^{alg}(X) \longrightarrow G^\vee(\overline{\mathbb{Q}}_\ell)$).

3. Analytic (2007–present;
Braverman–Kazhdan, Kontsevich, Teschner, Langlands, Nekrasov, E–Frenkel–Kazhdan, Gaiotto–Witten)

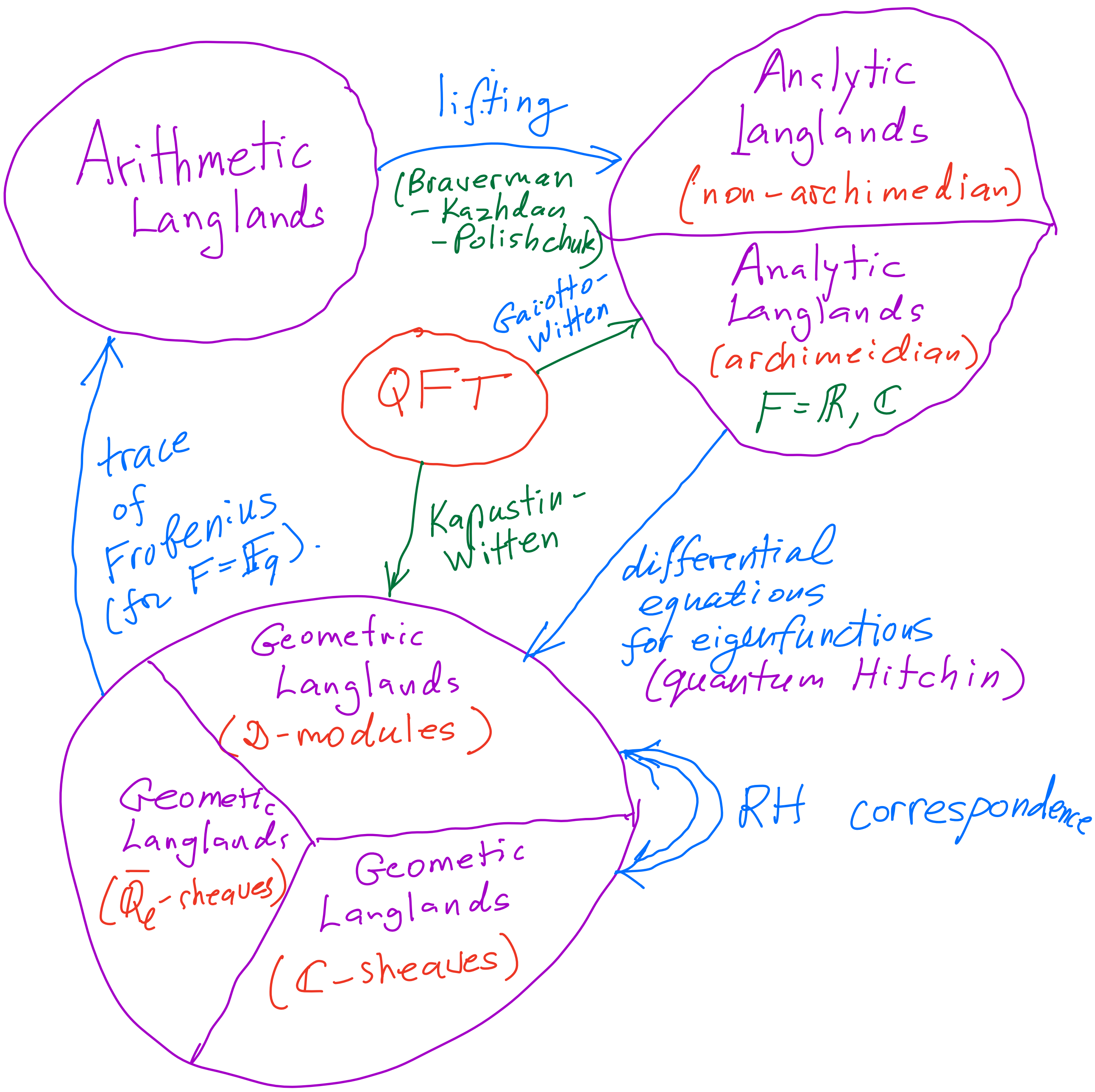
F - local field ($\mathbb{R}$, $\mathbb{C}$ or non-archimedian, $\mathbb{F}_q((t))$ or a finite extension of $\mathbb{Q}_p$).

Hecke operators act on $L^2(\mathrm{Bun}_G(X)(F))_{\frac{1}{2}\text{-densities}}$.

In this case we only have an idea how to parametrize eigenvectors by $G^\vee$-data in the archimedian case ($F = \mathbb{R}, \mathbb{C}$).

In this case eigenvectors are parametrized by $G^\vee$-opers on X (local systems with a certain specific underlying principal bundle) satisfying an appropriate topological reality conditions.

These three flavors are interrelated in various ways.



I will now focus on the analytic Langlands correspondence, starting with $F = \mathbb{C}$.

Let $\mathcal{H} = L^2(Bun_G(X)(\mathbb{C}))$

(Hilbert space of square integrable half-densities). On this space act many natural commuting operators.

1. Hecke operators $H_{x,\lambda}$, $x\in X(\mathbb{C})$, $\lambda\in\Lambda$ - dominant coweights of G

(conjecturally) compact normal operators.

2. Quantum Hitchin operators $D, \overline{D}$:

(conjecturally) unbounded normal operators.

Both types of operators are defined using the representation of $\mathrm{Bun}_G(X)$ in terms of loop group $G((t))$. Namely, pick a point $p\in X$. Let Δ be the formal neighborhood of p, $\mathcal{O}=\mathbb{C}[\Delta]$, $K=\mathrm{Frac}\,\mathcal{O}$

$\mathcal{O} \cong \mathbb{C}[[t]]$, $K \cong \mathbb{C}((t))$

then

$$\mathrm{Bun}_G(X) = G(X\setminus x)\backslash G(K)/G(\mathcal{O})$$

(we cover X by $X\setminus x$ and Δ

and glue a G-bundle out of trivial ones on these charts. So we represent a G-bundle by a transition function $g \in G((t))$ on $\Delta \cap (X \setminus x) = \Delta_*$ (punctured formal disk). This transition function is then defined up to changing trivialization on each chart).

The quantum Hitchin system is obtained by taking the Feigin-Frenkel center of $\mathcal{U}(\hat{\mathfrak{g}})$ at the critical level (for $\mathfrak{g} = \mathfrak{sl}_2$, Sugawara operators), viewing it as 2-sided invariant differential operators on $G(K)$ and descending to double quotient. The anomaly (critical level $k = -h^\vee$) results in the fact that we get diff. operators on half-densities rather than functions, as we should.

Hecke operators: We have

Affine Grassmannian $Gr_G = G(K)/G(\mathcal{O})$

it has an action of $G(\mathcal{O})$, with orbits Gr_G^λ parametrized by $\lambda \in \Lambda$ (finite dimensional).

We have convolution

$$(\psi, \gamma) \longmapsto \psi * \gamma,$$

where ψ is a half-density on $Bun_G(X)(\mathbb{C}) = G(X \setminus x) \backslash G(K)/G(\mathcal{O})$

and γ is a $G(K)$-invariant distribution on $Gr_G = G(K)/G(\mathcal{O})$.

Then $\boxed{H_{x,\lambda} \psi \overset{def}{=} \psi * \delta_\lambda}$

Hecke operator

where δ_λ is the δ-distribution of $Y_\lambda \subset G(K)/G(\mathcal{O})$.

It is not obvious that this is well defined, but it is! (the integral over Y_λ makes sense, i.e. integrand is a measure)

Example. $G = PGL_2$, $E \in Bun_G(X)$ (rank 2 bundle up to tensoring with line bundles). Let $\lambda = 1$. Then $Y_\lambda = \mathbb{P}^1$. For $s \in \mathbb{P}(E_x)$ Hecke modification of E at x using s, $HM_{x,s}(E)$, is the bundle whose local sections are sections of E with at worst first order pole at x with residue in S.

Then $H_{x,\lambda} = H_x$ is defined by

$$(H_x \psi)(E) \overset{\text{def.}}{=} \int_{\mathbb{P}^1(E_x)} \psi(HM_{x,s}(E))\, ds.$$

Variant: Ramified case.

$t_1, \dots, t_N \in X$ distinct,

$Bun_G(X, t_1, \dots, t_N)$ – moduli of G-bundles on X with trivialization at $t_1, \dots, t_N$, $\pi_1, \dots, \pi_N$ – unitary representations of $G = G(\mathbb{C})$.

Have action of G^N on $Bun_G(X, t_1, \dots, t_N)$ by changing trivializations. Can define Hilbert space bundle $\mathcal{E}$ on $Bun_G(X)$ by

$$\mathcal{E} = Bun_G(X, t_1, \dots, t_N) \times_{G^N} \pi_1 \otimes \dots \otimes \pi_N.$$

Now $\mathcal{H} = \Gamma_{L^2}(\mathrm{Bun}_G(X), \mathcal{E} \otimes |K|)$.
- hilbert space.

Still have Hitchin and Hecke operators acting on this space ($x \neq t_i$).

Example: $X = \mathbb{P}^1$, denote points by $t_0, \dots, t_{m+1}$. Then $\mathcal{H} = \bigoplus_{\chi \in \pi_1(G)} \mathcal{H}_\chi$

$$\mathcal{H}_\chi = \mathrm{Mult}(\pi^*_{m+1}, \pi_0 \otimes \cdots \otimes \pi_m).$$

For $G = PGL_2$ and $\pi_j = V_{\lambda_j}$ ($\mathrm{Re}\,\lambda_j = -1$) principal series reps, we have $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$

interpret $\mathcal{H}_0$ as the space of translation-invariant homogeneous functions of degree $\frac{1}{2}\left(\sum_{j=0}^{m} \lambda_j - \lambda_{m+1}\right)$ on $\mathbb{C}^{m+1}$ and $\mathcal{H}_1$ as such functions of degree $1 + \frac{1}{2}\sum_{j=0}^{m+1} \lambda_j$

Then the Hecke operator
$\mathcal{H}_0 \to \mathcal{H}_1$, $\mathcal{H}_1 \to \mathcal{H}_0$ is
given by

$$(H_x \psi)(y_0, \dots, y_m) = \int_{\mathbb{C}} \psi\left(\frac{t_0 - x}{s - y_0}, \dots, \frac{t_m - x}{s - y_m}\right) \prod |s - y_i|^{2\lambda_i} \, ds \, d\bar{s}.$$

Now we want to parametrize
eigenvectors of H_x and Hitchin
algebras $\mathcal{A}$, $\bar{\mathcal{A}}$. Recall that
$\operatorname{Spec} \mathcal{A} = Op_{G^\vee}(X)$, the space
of $G^\vee$-opers. So eigenvectors
will be parametrized by opers
with certain properties.

Conjecture. Eigenvectors of
Hecke and Hitchin operators
are parametrized by $G^\vee$-opers

with real monodromy, i.e. monodromy $\pi_1(X) \longrightarrow G^\vee(\mathbb{C})$ can be conjugated into split form $G^\vee(\mathbb{R})$.

In the ramified case the opers have regular singularities at t_i with prescribed residues.

Reminder: $\mathrm{Ext}^1(K_X^{-1/2}, K_X^{1/2}) = H^1(X, K_X) = H^0(X, \mathcal{O})^* = \mathbb{C}$, so $\exists$ a unique non-trivial extension $0 \to K_X^{1/2} \longrightarrow E \to K_X^{-1/2} \to 0$ which is an SL_2-bundle.

Let $E_{G^\vee}$ be the associated $G^\vee$-bundle via principal homom. $\phi: SL_2 \longrightarrow G^\vee$.

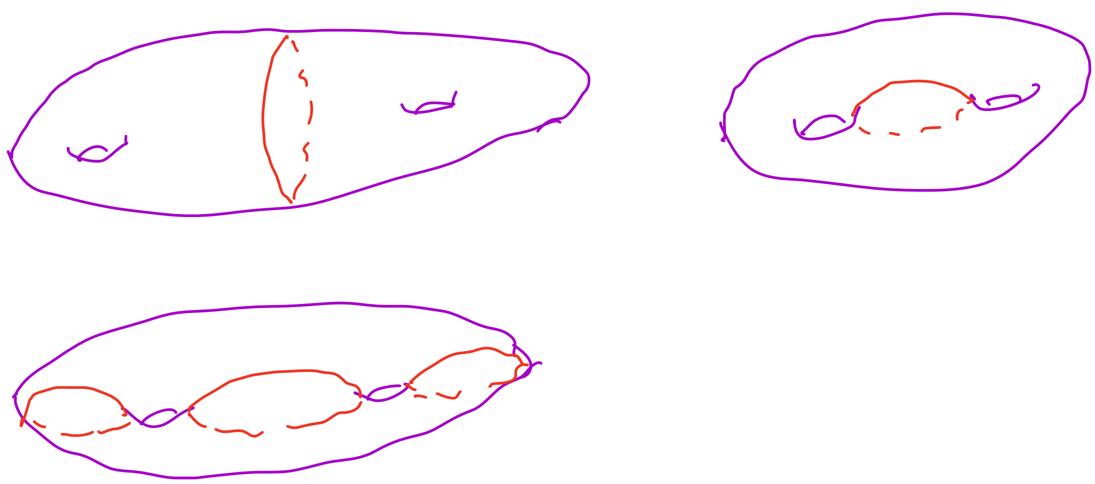
Then a $G^\vee$-oper on X is a connection of $E_{G^\vee}$.

The above conjecture is (mostly) proved in our work with Frenkel and Kazhdan for $G = PGL_2$ and $X = \mathbb{P}^1$, with punctures.

Now finally consider the real case ($F = \mathbb{R}$). In this case we have several complications coming from the fact that $\mathbb{R}$ is not algebraically closed.

Recall that in this case we have a curve X (Riemann surface) with real structure, i.e. antiholomorphic involution $\tau : X \longrightarrow X$.

This involution may have fixed points which form ovals $C_1, \dots, C_r$ (so $X(\mathbb{R}) = C_1 \cup \dots \cup C_r$



We should consider Hecke/Hitchin operators on $L^2(\mathrm{Bun}_G(X)(\mathbb{R}))$, space of real G-bundles.

To define what this means, we need to fix a real structure σ on G, which is a class $\sigma \in H^1(\mathbb{Z}/2, \mathrm{Aut}\, G)$, where

$\mathbb{Z}/2$ acts by compact conjugation (say).
Recall that $\mathrm{Aut}\, G = \mathrm{Aut}\, \Delta_G \ltimes G_{ad}$,
where Δ_G is the root datum of G.
Let s be the image of σ in
$H^1(\mathbb{Z}/2, \mathrm{Aut}\,\Delta_G) = \mathrm{Hom}(\mathbb{Z}/2, \mathrm{Aut}\,\Delta_G)$. be the inner class
of σ. It is easy to see that
the notion of real bundle
depends only on the inner class s.
So we get the moduli space

$\mathrm{Bun}_G(X)_s(\mathbb{R})$.

Now, given $P \in \mathrm{Bun}_G(X)_s(\mathbb{R})$,
for every oval C_i on X,
we get a real form σ_i
G in the inner class s.

The collections $(\sigma_1, \dots, \sigma_r)$ parametrize connected components of $Bun_G(X)_\sigma(\mathbb{R})$ (but some might be empty).

For example, for $G = PGL_2$ we have only one inner class and two real forms $PGL_2(\mathbb{R})$ and PU_2, so there are two types of contours: real and quaternionic.

Finally, if we have punctures $t_1, \dots, t_N$ then for $t_i \notin X(\mathbb{R})$ we need to fix a unitary representation of $G(\mathbb{C})$

while for $t_i \in C_j \subset X(\mathbb{R})$ we need to fix a unitary representation π_i of the real form $G^{\sigma_j}(\mathbb{R})$. This defines the Hilbert space $\mathcal{H}$ and the spectral problem we want to solve.

Now let us (conjecturally) describe the spectrum.

Case 1. τ has no fixed points.

In this case we define Langlands L-group

${}^L G = \mathbb{Z}/2 \ltimes G^\vee$ where $\mathbb{Z}/2$ acts on $G^\vee$ by $w \circ s$, w being the Cartan involution.

Conjecturally, the spectrum is parametrized by local systems $\rho: \pi_1(X/\tau) \longrightarrow {}^L G$ such that orientation-reversing paths map to the non-trivial component, and $\rho|_{\pi_1(X)}: \pi_1(X) \to G^\vee$ has an oper structure. (Gaiotto & Witten). We can show that spectrum is parametrized by a subset of this set.

2. Suppose τ has fixed points. Then the topological condition on spectral opers depends on the form G_i of G on each oval. We consider the example $G = PGL_2$

In this case we have real and quaternionic contours. Consider the case when ovals are all real and real locus cuts X into two pieces and $\lambda_i = -1$ $\forall i$

Then we have the following conditions on the monodromy $\rho_L : \pi_1(X_+) \longrightarrow SL_2(\mathbb{C})$.

Condition 1. $\rho_L(C_i)$ is unipotent for all i.

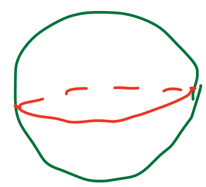
Condition 2. ρ_L lands in $SL_2(\mathbb{R})$ up to conjugation.

Opers satisfying this condition are called balanced.

Theorem. Every oper arising in the spectrum is balanced.

Finally, consider the case with punctures for $X=\mathbb{P}^1$, $G=SL_2$, when we have a single quaternionic oval $\mathbb{P}^1(\mathbb{R})$.

In this case we should put at punctures unitary (i.e. finite dimensional) representations $V_0, \dots, V_{m+1}$ of $SU(2)$, and



$$\mathcal{H} = (V_0 \otimes \cdots \otimes V_{m+1})^{SU(2)}.$$

The quantum Hitchin operators (for $t_{m+1}=\infty$) are the

Gaudin operators

$$G_i = \sum_{j \neq i} \frac{\Omega_{ij}}{t_i - t_j},$$ where

$$\Omega = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h$$

is the Casimir tensor.

So our spectral problem is the usual Gaudin model. The topological condition on the oper in this case is that it is monodromy free, and we recover the result of Feigin-Frenkel-Rybnikov that spectrum of the Gaudin model is parametrized by monodromy free opers

$$L = \partial_x^2 - \sum_{i=0}^{m} \frac{\lambda_i(\lambda_i+2)}{4(x-t_i)^2} - \sum_{i=0}^{m} \frac{\mu_i}{x-t_i},$$

$\sum_{i=0}^{m} \mu_i = 0$, residue at ∞ is $\frac{\lambda_{m+1}(\lambda_{m+1}+2)}{4}$.

The same happens for arbitrary G.

Finally, the Hecke operator in this case is the Baxter Q-operator.