Whittaker coefficients and the Hitchin section

Sam Raskin

Yale University

Hamburg Prize in Theoretical Physics for Edward Witten

November 10th, 2023

Contents

Overview

Geometric Langlands and QFT

Geometry of $T^* \operatorname{Bun}_G$

The Nadler-Taylor theorem

Application

Overview

Overall, the goal of this talk is to say something about what J. Færgeman and I did in our joint work from last year.

Overview

Overall, the goal of this talk is to say something about what J. Færgeman and I did in our joint work from last year. Actually, the ingredient I want to discuss was proved in a stronger form by Nadler-Taylor shortly after our paper appeared, and I will emphasize their result.

Overview

Overall, the goal of this talk is to say something about what J. Færgeman and I did in our joint work from last year. Actually, the ingredient I want to discuss was proved in a stronger form by Nadler-Taylor shortly after our paper appeared, and I will emphasize their result. Then I will try to say why Færgeman and I wanted this result.

Overview

Overall, the goal of this talk is to say something about what J. Færgeman and I did in our joint work from last year. Actually, the ingredient I want to discuss was proved in a stronger form by Nadler-Taylor shortly after our paper appeared, and I will emphasize their result. Then I will try to say why Færgeman and I wanted this result. (Briefly: it was an ingredient in the main technical breakthrough before recent progress leading to the proof of the geometric Langlands conjecture.)

Overview

Overall, the goal of this talk is to say something about what J. Færgeman and I did in our joint work from last year. Actually, the ingredient I want to discuss was proved in a stronger form by Nadler-Taylor shortly after our paper appeared, and I will emphasize their result. Then I will try to say why Færgeman and I wanted this result. (Briefly: it was an ingredient in the main technical breakthrough before recent progress leading to the proof of the geometric Langlands conjecture.)

I will try to say things in QFT language today.

Overview

Overall, the goal of this talk is to say something about what J. Færgeman and I did in our joint work from last year. Actually, the ingredient I want to discuss was proved in a stronger form by Nadler-Taylor shortly after our paper appeared, and I will emphasize their result. Then I will try to say why Færgeman and I wanted this result. (Briefly: it was an ingredient in the main technical breakthrough before recent progress leading to the proof of the geometric Langlands conjecture.)

I will try to say things in QFT language today. But it is not my native tongue, and I am sorry for mistakes, misrepresentations, omissions, ignorance, etc.

Overview

Overall, the goal of this talk is to say something about what J. Færgeman and I did in our joint work from last year. Actually, the ingredient I want to discuss was proved in a stronger form by Nadler-Taylor shortly after our paper appeared, and I will emphasize their result. Then I will try to say why Færgeman and I wanted this result. (Briefly: it was an ingredient in the main technical breakthrough before recent progress leading to the proof of the geometric Langlands conjecture.)

I will try to say things in QFT language today. But it is not my native tongue, and I am sorry for mistakes, misrepresentations, omissions, ignorance, etc.

Last warning: the result looks kind of trivial in what I'll explain.

Geometric Langlands and QFT

I want to begin with a general background on objects from the geometric Langlands program that I wish to discuss.

Geometric Langlands and QFT

I want to begin with a general background on objects from the geometric Langlands program that I wish to discuss.

First, the subject is about *categories*,

Geometric Langlands and QFT

I want to begin with a general background on objects from the geometric Langlands program that I wish to discuss.

First, the subject is about *categories*, which maybe should be explained.

Geometric Langlands and QFT

I want to begin with a general background on objects from the geometric Langlands program that I wish to discuss.

First, the subject is about *categories*, which maybe should be explained. For our purposes, you can think of categories as *2d TQFTs*.

Geometric Langlands and QFT

I want to begin with a general background on objects from the geometric Langlands program that I wish to discuss.

First, the subject is about *categories*, which maybe should be explained. For our purposes, you can think of categories as *2d TQFTs*. More precisely, we have:

Geometric Langlands and QFT

I want to begin with a general background on objects from the geometric Langlands program that I wish to discuss.

First, the subject is about *categories*, which maybe should be explained. For our purposes, you can think of categories as *2d TQFTs*. More precisely, we have:

$$\{\text{2d TQFTs}\} \subseteq \{\text{categories}\}$$

Geometric Langlands and QFT

I want to begin with a general background on objects from the geometric Langlands program that I wish to discuss.

First, the subject is about *categories*, which maybe should be explained. For our purposes, you can think of categories as *2d TQFTs*. More precisely, we have:

$$\{\text{2d TQFTs}\} \subseteq \{\text{categories}\}$$

where the map sends a TQFT Z to its *category of boundary conditions*.

Geometric Langlands and QFT

I want to begin with a general background on objects from the geometric Langlands program that I wish to discuss.

First, the subject is about *categories*, which maybe should be explained. For our purposes, you can think of categories as *2d TQFTs*. More precisely, we have:

$$\{\text{2d TQFTs}\} \subseteq \{\text{categories}\}$$

where the map sends a TQFT Z to its *category of boundary conditions*. If you slightly weaken the left hand side, you can make the above into an equality.

Geometric Langlands and QFT

The main object of study today is the *category* D–mod(Bun_G) for G a complex reductive Lie group.

Geometric Langlands and QFT

The main object of study today is the *category* $\text{D–mod}(\text{Bun}_G)$ for G a complex reductive Lie group. From the physics point of view, Kapustin-Witten explained that it arises as follows.

Geometric Langlands and QFT

The main object of study today is the *category* $\text{D–mod}(\text{Bun}_G)$ for G a complex reductive Lie group. From the physics point of view, Kapustin-Witten explained that it arises as follows.

First, there is $4d$ $\mathcal{N} = 4$ Yang-Mills theory YM_G for the compact form of G.

Geometric Langlands and QFT

The main object of study today is the *category* $\text{D–mod}(\text{Bun}_G)$ for G a complex reductive Lie group. From the physics point of view, Kapustin-Witten explained that it arises as follows.

First, there is $4d$ $\mathcal{N} = 4$ Yang-Mills theory YM_G for the compact form of G. After Kapustin-Witten, this theory has a 1-dimensional family of topological twists indexed by a parameter Ψ.

Geometric Langlands and QFT

The main object of study today is the *category* $\text{D–mod}(\text{Bun}_G)$ for G a complex reductive Lie group. From the physics point of view, Kapustin-Witten explained that it arises as follows.

First, there is $4d$ $\mathcal{N} = 4$ Yang-Mills theory YM_G for the compact form of G. After Kapustin-Witten, this theory has a 1-dimensional family of topological twists indexed by a parameter Ψ. For $\Psi = 0$, we speak of the A-twist and denote the twisted theory by $\text{YM}_{G,A}$.

Geometric Langlands and QFT

The main object of study today is the *category* D–mod(Bun_G) for G a complex reductive Lie group. From the physics point of view, Kapustin-Witten explained that it arises as follows.

First, there is $4d$ $\mathcal{N} = 4$ Yang-Mills theory YM_G for the compact form of G. After Kapustin-Witten, this theory has a 1-dimensional family of topological twists indexed by a parameter Ψ. For $\Psi = 0$, we speak of the A-twist and denote the twisted theory by $\mathrm{YM}_{G,A}$. (resp. $\Psi = \infty$, B-twist, $\mathrm{YM}_{G,B}$).

Geometric Langlands and QFT

Now if we fix a compact Riemann surface X, we can compactify the theory $\mathrm{YM}_{G,A}$ on X,

Geometric Langlands and QFT

Now if we fix a compact Riemann surface X, we can compactify the theory $\mathrm{YM}_{G,A}$ on X, i.e., form the 2d TQFT $\mathrm{YM}_{G,A}(X \times -)$, which has category of boundary conditions D–mod(Bun_G).

Geometric Langlands and QFT

Now if we fix a compact Riemann surface X, we can compactify the theory $\mathsf{YM}_{G,A}$ on X, i.e., form the 2d TQFT $\mathsf{YM}_{G,A}(X \times -)$, which has category of boundary conditions D–mod(Bun_G). Here Bun_G is the space of holomorphic G-bundles on X.

Geometric Langlands and QFT

Now if we fix a compact Riemann surface X, we can compactify the theory $\mathrm{YM}_{G,A}$ on X, i.e., form the 2d TQFT $\mathrm{YM}_{G,A}(X \times -)$, which has category of boundary conditions D–mod(Bun_G). Here Bun_G is the space of holomorphic G-bundles on X.

Officially, objects $\mathcal{F}$ of D–mod($\mathcal{Y}$) are (coordinate-free) expressions of systems of linear, algebraic differential equations on $\mathcal{Y}$.

Geometric Langlands and QFT

Now if we fix a compact Riemann surface X, we can compactify the theory $\mathrm{YM}_{G,A}$ on X, i.e., form the 2d TQFT $\mathrm{YM}_{G,A}(X \times -)$, which has category of boundary conditions D–mod(Bun_G). Here Bun_G is the space of holomorphic G-bundles on X.

Officially, objects $\mathcal{F}$ of D–mod($\mathcal{Y}$) are (coordinate-free) expressions of systems of linear, algebraic differential equations on $\mathcal{Y}$. E.g., the equation $y' - y = 0$ defines a D-module exp on $\mathbb{A}^1 (= \mathbb{C})$.

Geometric Langlands and QFT

Now if we fix a compact Riemann surface X, we can compactify the theory $\mathrm{YM}_{G,A}$ on X, i.e., form the 2d TQFT $\mathrm{YM}_{G,A}(X \times -)$, which has category of boundary conditions D–mod(Bun_G). Here Bun_G is the space of holomorphic G-bundles on X.

Officially, objects $\mathcal{F}$ of D–mod$(\mathcal{Y})$ are (coordinate-free) expressions of systems of linear, algebraic differential equations on $\mathcal{Y}$. E.g., the equation $y' - y = 0$ defines a D-module exp on $\mathbb{A}^1 (= \mathbb{C})$.

For $\mathcal{F}$ a D-module, the symbols of the differential operators occurring in its construction define a closed conical subset $\mathrm{SingSupp}(\mathcal{F}) \subseteq T^*\mathcal{Y}$.

Geometric Langlands and QFT

Now if we fix a compact Riemann surface X, we can compactify the theory $\mathrm{YM}_{G,A}$ on X, i.e., form the 2d TQFT $\mathrm{YM}_{G,A}(X \times -)$, which has category of boundary conditions D–mod(Bun_G). Here Bun_G is the space of holomorphic G-bundles on X.

Officially, objects $\mathcal{F}$ of D–mod$(\mathcal{Y})$ are (coordinate-free) expressions of systems of linear, algebraic differential equations on $\mathcal{Y}$. E.g., the equation $y' - y = 0$ defines a D-module exp on $\mathbb{A}^1 (= \mathbb{C})$.

For $\mathcal{F}$ a D-module, the symbols of the differential operators occurring in its construction define a closed conical subset $\mathrm{SingSupp}(\mathcal{F}) \subseteq T^*\mathcal{Y}$. It is always coisotropic, and in the best situations, it is Lagrangian. So:

$$D\text{-modules} \rightsquigarrow \text{symplectic geometry on } T^*\mathcal{Y}$$

Geometric Langlands and QFT

Less officially, $\text{D–mod}(\mathcal{Y}) \approx \text{Fuk}(T^*\mathcal{Y})$.

Geometric Langlands and QFT

Less officially, D–mod($\mathcal{Y}$) $\approx$ Fuk($T^*\mathcal{Y}$). So I encourage you to picture D-modules as *branes*: Lagrangians Λ in $T^*\mathcal{Y}$

Geometric Langlands and QFT

Less officially, D–mod($\mathcal{Y}$) $\approx$ Fuk($T^*\mathcal{Y}$). So I encourage you to picture *D*-modules as *branes*: Lagrangians Λ in $T^*\mathcal{Y}$ (equipped with extra structure, most notably, a local system on $\Lambda^{\mathrm{smooth}}$)

Geometric Langlands and QFT

Less officially, D–mod$(\mathcal{Y}) \approx$ Fuk$(T^*\mathcal{Y})$. So I encourage you to picture D-modules as *branes*: Lagrangians Λ in $T^*\mathcal{Y}$ (equipped with extra structure, most notably, a local system on Λ^{smooth}) with morphisms having something to do with intersections of Lagrangians.

Geometric Langlands and QFT

Less officially, D–mod($\mathcal{Y}$) $\approx$ Fuk($T^*\mathcal{Y}$). So I encourage you to picture *D*-modules as *branes*: Lagrangians Λ in $T^*\mathcal{Y}$ (equipped with extra structure, most notably, a local system on Λ^{smooth}) with morphisms having something to do with intersections of Lagrangians. In the best cases, the Lagrangian is conical and can be pictured as the singular support.

Geometric Langlands and QFT

We constructed D–mod(Bun_G) by compactification from $4d$, so forgot information.

Geometric Langlands and QFT

We constructed D–mod(Bun_G) by compactification from $4d$, so forgot information. One sign of its $4d$ origins is the existence of *Hecke operators*, which are indexed by pairs $x \in X$ and a line operator in $\mathrm{YM}_G(\mathbb{S}^2_x)$ at x.

Geometric Langlands and QFT

We constructed D–mod(Bun_G) by compactification from 4d, so forgot information. One sign of its 4d origins is the existence of *Hecke operators*, which are indexed by pairs $x \in X$ and a line operator in $\mathrm{YM}_G(\mathbb{S}^2_x)$ at x. These line operators are indexed by representations of the *Langlands dual group* $\check{G}$, so a point $x \in X$ defines the *Hecke action* of $\mathrm{Rep}(\check{G})$ on D–mod(Bun_G).

Geometric Langlands and QFT

We constructed D–mod(Bun_G) by compactification from $4d$, so forgot information. One sign of its $4d$ origins is the existence of *Hecke operators*, which are indexed by pairs $x \in X$ and a line operator in $\mathrm{YM}_G(\mathbb{S}^2_x)$ at x. These line operators are indexed by representations of the *Langlands dual group* $\check{G}$, so a point $x \in X$ defines the *Hecke action* of $\mathrm{Rep}(\check{G})$ on D–mod(Bun_G). In GL, we try to understand D–mod(Bun_G) *with its Hecke symmetries*.

Geometric Langlands and QFT

There is a key boundary condition of YM_G, the *(principal) Nahm pole*/*Whittaker* boundary condition.

Geometric Langlands and QFT

There is a key boundary condition of YM_G, the *(principal) Nahm pole/Whittaker* boundary condition.

For our purposes, a boundary condition defines *dual* maps:

$$\mathrm{Vect} \to \text{D–mod}(\mathrm{Bun}_G) \text{ and } \text{D–mod}(\mathrm{Bun}_G) \to \mathrm{Vect}.$$

Geometric Langlands and QFT

There is a key boundary condition of YM_G, the *(principal) Nahm pole/Whittaker* boundary condition.

For our purposes, a boundary condition defines *dual* maps:

$$\mathrm{Vect} \to \text{D–mod}(\mathrm{Bun}_G) \text{ and } \text{D–mod}(\mathrm{Bun}_G) \to \mathrm{Vect}.$$

The former corresponds to an object $\mathcal{W}_! \in \text{D–mod}(\mathrm{Bun}_G)$ (the *Whittaker/Poincaré* sheaf)

Geometric Langlands and QFT

There is a key boundary condition of YM_G, the *(principal) Nahm pole/Whittaker* boundary condition.

For our purposes, a boundary condition defines *dual* maps:

$$\mathrm{Vect} \to \text{D–mod}(\mathrm{Bun}_G) \text{ and } \text{D–mod}(\mathrm{Bun}_G) \to \mathrm{Vect}.$$

The former corresponds to an object $\mathcal{W}_! \in \text{D–mod}(\mathrm{Bun}_G)$ (the *Whittaker/Poincaré* sheaf) and the latter is denoted $\mathrm{coeff} : \text{D–mod}(\mathrm{Bun}_G) \to \mathrm{Vect}$.

Geometric Langlands and QFT

These dual objects play a key normalizing role in GLC.

Geometric Langlands and QFT

These dual objects play a key normalizing role in GLC. In terms of S-duality $\mathrm{YM}_G \simeq \mathrm{YM}_{\check{G}}$ (or $\mathrm{YM}_{G,A} \simeq \mathrm{YM}_{\check{G},B}$),

Geometric Langlands and QFT

These dual objects play a key normalizing role in GLC. In terms of S-duality $\mathrm{YM}_G \simeq \mathrm{YM}_{\check{G}}$ (or $\mathrm{YM}_{G,A} \simeq \mathrm{YM}_{\check{G},B}$), the Nahm pole boundary condition is *dual* to the Neumann boundary condition

Geometric Langlands and QFT

These dual objects play a key normalizing role in GLC. In terms of S-duality $\mathrm{YM}_G \simeq \mathrm{YM}_{\check{G}}$ (or $\mathrm{YM}_{G,A} \simeq \mathrm{YM}_{\check{G},B}$), the Nahm pole boundary condition is *dual* to the Neumann boundary condition (see [Gaiotto-Witten]).

Geometric Langlands and QFT

These dual objects play a key normalizing role in GLC. In terms of S-duality $\mathrm{YM}_G \simeq \mathrm{YM}_{\check{G}}$ (or $\mathrm{YM}_{G,A} \simeq \mathrm{YM}_{\check{G},B}$), the Nahm pole boundary condition is *dual* to the Neumann boundary condition (see [Gaiotto-Witten]).

Mathematically, this means the following.

Geometric Langlands and QFT

These dual objects play a key normalizing role in GLC. In terms of S-duality $\mathsf{YM}_G \simeq \mathsf{YM}_{\check{G}}$ (or $\mathsf{YM}_{G,A} \simeq \mathsf{YM}_{\check{G},B}$), the Nahm pole boundary condition is *dual* to the Neumann boundary condition (see [Gaiotto-Witten]).

Mathematically, this means the following. For example, if σ an irreducible $\check{G}$-local system, GLC predicts there exists a *unique* Hecke eigensheaf $\mathcal{F}_\sigma \in$ D–mod(Bun_G) with eigenvalue σ and *subject to the normalization* $\mathrm{coeff}(\mathcal{F}_\sigma) \simeq \mathbb{C}$.

Geometric Langlands and QFT

These dual objects play a key normalizing role in GLC. In terms of S-duality $\mathrm{YM}_G \simeq \mathrm{YM}_{\check{G}}$ (or $\mathrm{YM}_{G,A} \simeq \mathrm{YM}_{\check{G},B}$), the Nahm pole boundary condition is *dual* to the Neumann boundary condition (see [Gaiotto-Witten]).

Mathematically, this means the following. For example, if σ an irreducible $\check{G}$-local system, GLC predicts there exists a *unique* Hecke eigensheaf $\mathcal{F}_\sigma \in \mathrm{D\text{–}mod}(\mathrm{Bun}_G)$ with eigenvalue σ and *subject to the normalization* $\mathrm{coeff}(\mathcal{F}_\sigma) \simeq \mathbb{C}$.

More generally, for any σ, there should be a unique *tempered* eigensheaf $\mathcal{F}_\sigma$ subject to the same normalization.

Geometric Langlands and QFT

These dual objects play a key normalizing role in GLC. In terms of S-duality $\mathrm{YM}_G \simeq \mathrm{YM}_{\check{G}}$ (or $\mathrm{YM}_{G,A} \simeq \mathrm{YM}_{\check{G},B}$), the Nahm pole boundary condition is *dual* to the Neumann boundary condition (see [Gaiotto-Witten]).

Mathematically, this means the following. For example, if σ an irreducible $\check{G}$-local system, GLC predicts there exists a *unique* Hecke eigensheaf $\mathcal{F}_\sigma \in \text{D–mod}(\mathrm{Bun}_G)$ with eigenvalue σ and *subject to the normalization* $\mathrm{coeff}(\mathcal{F}_\sigma) \simeq \mathbb{C}$.

More generally, for any σ, there should be a unique *tempered* eigensheaf $\mathcal{F}_\sigma$ subject to the same normalization.

The technical problem we solved: why are there no (tempered) eigensheaves with $\mathrm{coeff}(\mathcal{F}_\sigma) = 0$? In other words, why is $\text{D–mod}(\mathrm{Bun}_G)$ not "too big" for the GLC to hold?

Geometry of $T^* \operatorname{Bun}_G$

Let us describe the basic aspects of geometry of $T^* \operatorname{Bun}_G$.

Geometry of $T^* \operatorname{Bun}_G$

Let us describe the basic aspects of geometry of $T^* \operatorname{Bun}_G$.

First, a standard analysis shows that:

$$T^* \operatorname{Bun}_G = \operatorname{Higgs}_G$$

Geometry of $T^* \operatorname{Bun}_G$

Let us describe the basic aspects of geometry of $T^* \operatorname{Bun}_G$.

First, a standard analysis shows that:

$$T^* \operatorname{Bun}_G = \operatorname{Higgs}_G \ = \{\mathcal{P}_G \in \operatorname{Bun}_G, \phi \in \operatorname{Ad}(\mathcal{P}_G).\}$$

Geometry of $T^* \operatorname{Bun}_G$

Let us describe the basic aspects of geometry of $T^* \operatorname{Bun}_G$.

First, a standard analysis shows that:

$$T^* \operatorname{Bun}_G = \operatorname{Higgs}_G \ = \{\mathcal{P}_G \in \operatorname{Bun}_G, \phi \in \operatorname{Ad}(\mathcal{P}_G).\}$$

There is a *characteristic polynomial* map from $T^* \operatorname{Bun}_G = \operatorname{Higgs}_G$ to the *Hitchin base* Hitch_G

Geometry of $T^* \operatorname{Bun}_G$

Let us describe the basic aspects of geometry of $T^* \operatorname{Bun}_G$.

First, a standard analysis shows that:

$$T^* \operatorname{Bun}_G = \operatorname{Higgs}_G = \{\mathcal{P}_G \in \operatorname{Bun}_G, \phi \in \operatorname{Ad}(\mathcal{P}_G).\}$$

There is a *characteristic polynomial* map from $T^* \operatorname{Bun}_G = \operatorname{Higgs}_G$ to the *Hitchin base* $\operatorname{Hitch}_G = \prod_i \Gamma(X, \Omega_X^{\otimes d_i})$ where i runs over the exponents of G.

Geometry of $T^* \operatorname{Bun}_G$

Let us describe the basic aspects of geometry of $T^* \operatorname{Bun}_G$.

First, a standard analysis shows that:

$$T^* \operatorname{Bun}_G = \operatorname{Higgs}_G = \{\mathcal{P}_G \in \operatorname{Bun}_G, \phi \in \operatorname{Ad}(\mathcal{P}_G).\}$$

There is a *characteristic polynomial* map from $T^* \operatorname{Bun}_G = \operatorname{Higgs}_G$ to the *Hitchin base* $\operatorname{Hitch}_G = \prod_i \Gamma(X, \Omega_X^{\otimes d_i})$ where i runs over the exponents of G.

For example, if $G = SL_2$, we have $\mathcal{E}$ of rank 2, $\phi : \mathcal{E} \to \mathcal{E} \otimes \Omega^1$,

Geometry of $T^* \operatorname{Bun}_G$

Let us describe the basic aspects of geometry of $T^* \operatorname{Bun}_G$.

First, a standard analysis shows that:

$$T^* \operatorname{Bun}_G = \operatorname{Higgs}_G = \{\mathcal{P}_G \in \operatorname{Bun}_G, \phi \in \operatorname{Ad}(\mathcal{P}_G).\}$$

There is a *characteristic polynomial* map from $T^* \operatorname{Bun}_G = \operatorname{Higgs}_G$ to the *Hitchin base* $\operatorname{Hitch}_G = \prod_i \Gamma(X, \Omega_X^{\otimes d_i})$ where i runs over the exponents of G.

For example, if $G = SL_2$, we have $\mathcal{E}$ of rank 2, $\phi : \mathcal{E} \to \mathcal{E} \otimes \Omega^1$, and the Hitchin map assigns the 2-form $\det(\phi) : \Lambda^2 \mathcal{E} \to \Lambda^2 \mathcal{E} \otimes \Omega_X^1 = \Lambda^2 \mathcal{E} \otimes \Omega_X^{\otimes 2}$.

Geometry of $T^* \operatorname{Bun}_G$: the global nilpotent cone

By definition, the *global nilpotent cone* $\operatorname{Nilp} \subseteq T^* \operatorname{Bun}_G$ is the preimage of $0 \in \operatorname{Hitch}_G$ under the Hitchin map.

Geometry of $T^* \operatorname{Bun}_G$: the global nilpotent cone

By definition, the *global nilpotent cone* $\operatorname{Nilp} \subseteq T^* \operatorname{Bun}_G$ is the preimage of $0 \in \operatorname{Hitch}_G$ under the Hitchin map. This is a favorite conical Lagrangian in $T^* \operatorname{Bun}_G$.

Geometry of $T^* \operatorname{Bun}_G$: the global nilpotent cone

By definition, the *global nilpotent cone* $\operatorname{Nilp} \subseteq T^* \operatorname{Bun}_G$ is the preimage of $0 \in \operatorname{Hitch}_G$ under the Hitchin map. This is a favorite conical Lagrangian in $T^* \operatorname{Bun}_G$.

In joint work with Arinkin, Gaitsgory, Kazhdan, Rozenblyum, and Varshavsky, we showed important relations between the Hecke action and Nilp.

Geometry of $T^* \operatorname{Bun}_G$: the global nilpotent cone

By definition, the *global nilpotent cone* $\operatorname{Nilp} \subseteq T^* \operatorname{Bun}_G$ is the preimage of $0 \in \operatorname{Hitch}_G$ under the Hitchin map. This is a favorite conical Lagrangian in $T^* \operatorname{Bun}_G$.

In joint work with Arinkin, Gaitsgory, Kazhdan, Rozenblyum, and Varshavsky, we showed important relations between the Hecke action and Nilp. Namely:

Geometry of $T^* \operatorname{Bun}_G$: the global nilpotent cone

By definition, the *global nilpotent cone* $\operatorname{Nilp} \subseteq T^* \operatorname{Bun}_G$ is the preimage of $0 \in \operatorname{Hitch}_G$ under the Hitchin map. This is a favorite conical Lagrangian in $T^* \operatorname{Bun}_G$.

In joint work with Arinkin, Gaitsgory, Kazhdan, Rozenblyum, and Varshavsky, we showed important relations between the Hecke action and Nilp. Namely: (i) suitably understood, every Hecke eigensheaf has singular support in Nilp,

Geometry of $T^* \mathrm{Bun}_G$: the global nilpotent cone

By definition, the *global nilpotent cone* $\mathrm{Nilp} \subseteq T^* \mathrm{Bun}_G$ is the preimage of $0 \in \mathrm{Hitch}_G$ under the Hitchin map. This is a favorite conical Lagrangian in $T^* \mathrm{Bun}_G$.

In joint work with Arinkin, Gaitsgory, Kazhdan, Rozenblyum, and Varshavsky, we showed important relations between the Hecke action and Nilp. Namely: (i) suitably understood, every Hecke eigensheaf has singular support in Nilp, i.e., lies in $\mathrm{D\text{–}mod}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, (ii) objects of $\mathrm{D\text{–}mod}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ have topological counterparts

Geometry of $T^* \mathrm{Bun}_G$: the global nilpotent cone

By definition, the *global nilpotent cone* $\mathrm{Nilp} \subseteq T^* \mathrm{Bun}_G$ is the preimage of $0 \in \mathrm{Hitch}_G$ under the Hitchin map. This is a favorite conical Lagrangian in $T^* \mathrm{Bun}_G$.

In joint work with Arinkin, Gaitsgory, Kazhdan, Rozenblyum, and Varshavsky, we showed important relations between the Hecke action and Nilp. Namely: (i) suitably understood, every Hecke eigensheaf has singular support in Nilp, i.e., lies in $\mathrm{D\text{–}mod}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, (ii) objects of $\mathrm{D\text{–}mod}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ have topological counterparts (they are *regular* holonomic),

Geometry of $T^* \operatorname{Bun}_G$: the global nilpotent cone

By definition, the *global nilpotent cone* $\operatorname{Nilp} \subseteq T^* \operatorname{Bun}_G$ is the preimage of $0 \in \operatorname{Hitch}_G$ under the Hitchin map. This is a favorite conical Lagrangian in $T^* \operatorname{Bun}_G$.

In joint work with Arinkin, Gaitsgory, Kazhdan, Rozenblyum, and Varshavsky, we showed important relations between the Hecke action and Nilp. Namely: (i) suitably understood, every Hecke eigensheaf has singular support in Nilp, i.e., lies in $\text{D–mod}_{\operatorname{Nilp}}(\operatorname{Bun}_G)$, (ii) objects of $\text{D–mod}_{\operatorname{Nilp}}(\operatorname{Bun}_G)$ have topological counterparts (they are *regular* holonomic), so $\text{D–mod}_{\operatorname{Nilp}}(\operatorname{Bun}_G) \simeq \operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G)$,

Geometry of $T^* \operatorname{Bun}_G$: the global nilpotent cone

By definition, the *global nilpotent cone* $\operatorname{Nilp} \subseteq T^* \operatorname{Bun}_G$ is the preimage of $0 \in \operatorname{Hitch}_G$ under the Hitchin map. This is a favorite conical Lagrangian in $T^* \operatorname{Bun}_G$.

In joint work with Arinkin, Gaitsgory, Kazhdan, Rozenblyum, and Varshavsky, we showed important relations between the Hecke action and Nilp. Namely: (i) suitably understood, every Hecke eigensheaf has singular support in Nilp, i.e., lies in $\text{D–mod}_{\operatorname{Nilp}}(\operatorname{Bun}_G)$, (ii) objects of $\text{D–mod}_{\operatorname{Nilp}}(\operatorname{Bun}_G)$ have topological counterparts (they are *regular* holonomic), so $\text{D–mod}_{\operatorname{Nilp}}(\operatorname{Bun}_G) \simeq \operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G)$, and (iii) in some sense, we can recover all of $\text{D–mod}(\operatorname{Bun}_G)$ from the Hecke action by knowing $\operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G)$.

Geometry of $T^* \operatorname{Bun}_G$: the global nilpotent cone

By definition, the *global nilpotent cone* $\operatorname{Nilp} \subseteq T^* \operatorname{Bun}_G$ is the preimage of $0 \in \operatorname{Hitch}_G$ under the Hitchin map. This is a favorite conical Lagrangian in $T^* \operatorname{Bun}_G$.

In joint work with Arinkin, Gaitsgory, Kazhdan, Rozenblyum, and Varshavsky, we showed important relations between the Hecke action and Nilp. Namely: (i) suitably understood, every Hecke eigensheaf has singular support in Nilp, i.e., lies in $\operatorname{D-mod}_{\operatorname{Nilp}}(\operatorname{Bun}_G)$, (ii) objects of $\operatorname{D-mod}_{\operatorname{Nilp}}(\operatorname{Bun}_G)$ have topological counterparts (they are *regular* holonomic), so $\operatorname{D-mod}_{\operatorname{Nilp}}(\operatorname{Bun}_G) \simeq \operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G)$, and (iii) in some sense, we can recover all of $\operatorname{D-mod}(\operatorname{Bun}_G)$ from the Hecke action by knowing $\operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G)$.

Summary:

Geometry of $T^* \operatorname{Bun}_G$: the global nilpotent cone

By definition, the *global nilpotent cone* $\operatorname{Nilp} \subseteq T^* \operatorname{Bun}_G$ is the preimage of $0 \in \operatorname{Hitch}_G$ under the Hitchin map. This is a favorite conical Lagrangian in $T^* \operatorname{Bun}_G$.

In joint work with Arinkin, Gaitsgory, Kazhdan, Rozenblyum, and Varshavsky, we showed important relations between the Hecke action and Nilp. Namely: (i) suitably understood, every Hecke eigensheaf has singular support in Nilp, i.e., lies in $\text{D–mod}_{\operatorname{Nilp}}(\operatorname{Bun}_G)$, (ii) objects of $\text{D–mod}_{\operatorname{Nilp}}(\operatorname{Bun}_G)$ have topological counterparts (they are *regular* holonomic), so $\text{D–mod}_{\operatorname{Nilp}}(\operatorname{Bun}_G) \simeq \operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G)$, and (iii) in some sense, we can recover all of $\text{D–mod}(\operatorname{Bun}_G)$ from the Hecke action by knowing $\operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G)$.

Summary: for practical purposes, we can essentially think of typical objects of $\text{D–mod}(\operatorname{Bun}_G)$ as branes supported on Nilp.

Geometry of $T^* \mathrm{Bun}_G$: the Kostant-Hitchin section

The Hitchin map has a canonical section, called the *global Kostant slice* or the *Hitchin section*.

Geometry of $T^* \operatorname{Bun}_G$: the Kostant-Hitchin section

The Hitchin map has a canonical section, called the *global Kostant slice* or the *Hitchin section*. It is defined using Kostant's normal form for an element of $\mathfrak{g}$ with prescribed characteristic polynomial.

Geometry of $T^* \mathrm{Bun}_G$: the Kostant-Hitchin section

The Hitchin map has a canonical section, called the *global Kostant slice* or the *Hitchin section*. It is defined using Kostant's normal form for an element of $\mathfrak{g}$ with prescribed characteristic polynomial.

For example, for $G = SL_2$, our section sends $\omega \in \Gamma(\Omega_X^{\otimes 2})$ to

$\mathcal{P}_G = \Omega_X^{\otimes \frac{1}{2}} \oplus \Omega_X^{\otimes - \frac{1}{2}}$

Geometry of $T^* \mathrm{Bun}_G$: the Kostant-Hitchin section

The Hitchin map has a canonical section, called the *global Kostant slice* or the *Hitchin section*. It is defined using Kostant's normal form for an element of $\mathfrak{g}$ with prescribed characteristic polynomial.

For example, for $G = SL_2$, our section sends $\omega \in \Gamma(\Omega_X^{\otimes 2})$ to $\mathcal{P}_G = \Omega_X^{\otimes \frac{1}{2}} \oplus \Omega_X^{\otimes -\frac{1}{2}}$ and $\phi = \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}$.

Geometry of $T^* \mathrm{Bun}_G$: the Kostant-Hitchin section

The Hitchin map has a canonical section, called the *global Kostant slice* or the *Hitchin section*. It is defined using Kostant's normal form for an element of $\mathfrak{g}$ with prescribed characteristic polynomial.

For example, for $G = SL_2$, our section sends $\omega \in \Gamma(\Omega_X^{\otimes 2})$ to $\mathcal{P}_G = \Omega_X^{\otimes \frac{1}{2}} \oplus \Omega_X^{\otimes -\frac{1}{2}}$ and $\phi = \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}$.

In general, there is a space Bun_N^Ω, a $\check{\rho}$-twisted form of Bun_N, with a map $\psi : \mathrm{Bun}_N^\Omega \to \mathbb{A}^1$,

Geometry of $T^* \mathrm{Bun}_G$: the Kostant-Hitchin section

The Hitchin map has a canonical section, called the *global Kostant slice* or the *Hitchin section*. It is defined using Kostant's normal form for an element of $\mathfrak{g}$ with prescribed characteristic polynomial.

For example, for $G = SL_2$, our section sends $\omega \in \Gamma(\Omega_X^{\otimes 2})$ to $\mathcal{P}_G = \Omega_X^{\otimes \frac{1}{2}} \oplus \Omega_X^{\otimes - \frac{1}{2}}$ and $\phi = \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}$.

In general, there is a space Bun_N^Ω, a $\check{\rho}$-twisted form of Bun_N, with a map $\psi : \mathrm{Bun}_N^\Omega \to \mathbb{A}^1$, and the Hitchin section consists of:

Geometry of $T^* \mathrm{Bun}_G$: the Kostant-Hitchin section

The Hitchin map has a canonical section, called the *global Kostant slice* or the *Hitchin section*. It is defined using Kostant's normal form for an element of $\mathfrak{g}$ with prescribed characteristic polynomial.

For example, for $G = SL_2$, our section sends $\omega \in \Gamma(\Omega_X^{\otimes 2})$ to $\mathcal{P}_G = \Omega_X^{\otimes \frac{1}{2}} \oplus \Omega_X^{\otimes -\frac{1}{2}}$ and $\phi = \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}$.

In general, there is a space Bun_N^{Ω}, a $\check{\rho}$-twisted form of Bun_N, with a map $\psi : \mathrm{Bun}_N^{\Omega} \to \mathbb{A}^1$, and the Hitchin section consists of:

$$\mathrm{Kost}_G :=$$

Geometry of $T^* \mathrm{Bun}_G$: the Kostant-Hitchin section

The Hitchin map has a canonical section, called the *global Kostant slice* or the *Hitchin section*. It is defined using Kostant's normal form for an element of $\mathfrak{g}$ with prescribed characteristic polynomial.

For example, for $G = SL_2$, our section sends $\omega \in \Gamma(\Omega_X^{\otimes 2})$ to $\mathcal{P}_G = \Omega_X^{\otimes \frac{1}{2}} \oplus \Omega_X^{\otimes -\frac{1}{2}}$ and $\phi = \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}$.

In general, there is a space Bun_N^{Ω}, a $\check{\rho}$-twisted form of Bun_N, with a map $\psi : \mathrm{Bun}_N^{\Omega} \to \mathbb{A}^1$, and the Hitchin section consists of:

$$\mathrm{Kost}_G := \{\mathcal{P}_N \in \mathrm{Bun}_N^{\Omega}, \phi \in T^*_{\mathcal{P}_G} \mathrm{Bun}_G \mid$$

Geometry of $T^* \mathrm{Bun}_G$: the Kostant-Hitchin section

The Hitchin map has a canonical section, called the *global Kostant slice* or the *Hitchin section*. It is defined using Kostant's normal form for an element of $\mathfrak{g}$ with prescribed characteristic polynomial.

For example, for $G = SL_2$, our section sends $\omega \in \Gamma(\Omega_X^{\otimes 2})$ to $\mathcal{P}_G = \Omega_X^{\otimes \frac{1}{2}} \oplus \Omega_X^{\otimes -\frac{1}{2}}$ and $\phi = \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}$.

In general, there is a space Bun_N^Ω, a $\check{\rho}$-twisted form of Bun_N, with a map $\psi : \mathrm{Bun}_N^\Omega \to \mathbb{A}^1$, and the Hitchin section consists of:

$$\mathrm{Kost}_G := \{\mathcal{P}_N \in \mathrm{Bun}_N^\Omega, \phi \in T^*_{\mathcal{P}_G} \mathrm{Bun}_G \mid \phi|_{T|^*_{\mathcal{P}_N} \mathrm{Bun}_N^\Omega} = d\psi(\mathcal{P}_N)\}.$$

Geometry of $T^* \operatorname{Bun}_G$: the Kostant-Hitchin section

The Hitchin map has a canonical section, called the *global Kostant slice* or the *Hitchin section*. It is defined using Kostant's normal form for an element of $\mathfrak{g}$ with prescribed characteristic polynomial.

For example, for $G = SL_2$, our section sends $\omega \in \Gamma(\Omega_X^{\otimes 2})$ to $\mathcal{P}_G = \Omega_X^{\otimes \frac{1}{2}} \oplus \Omega_X^{\otimes -\frac{1}{2}}$ and $\phi = \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}$.

In general, there is a space $\operatorname{Bun}_N^\Omega$, a $\check{\rho}$-twisted form of Bun_N, with a map $\psi : \operatorname{Bun}_N^\Omega \to \mathbb{A}^1$, and the Hitchin section consists of:

$$\operatorname{Kost}_G := \{\mathcal{P}_N \in \operatorname{Bun}_N^\Omega, \phi \in T^*_{\mathcal{P}_G} \operatorname{Bun}_G \mid \phi|_{T|^*_{\mathcal{P}_N} \operatorname{Bun}_N^\Omega} = d\psi(\mathcal{P}_N)\}.$$

Points of Kost_G are also called *classical* opers or *oper Higgs bundles*.

Geometry of $T^* \operatorname{Bun}_G$: the Kostant-Hitchin section

The Hitchin map has a canonical section, called the *global Kostant slice* or the *Hitchin section*. It is defined using Kostant's normal form for an element of $\mathfrak{g}$ with prescribed characteristic polynomial.

For example, for $G = SL_2$, our section sends $\omega \in \Gamma(\Omega_X^{\otimes 2})$ to $\mathcal{P}_G = \Omega_X^{\otimes \frac{1}{2}} \oplus \Omega_X^{\otimes - \frac{1}{2}}$ and $\phi = \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}$.

In general, there is a space $\operatorname{Bun}_N^\Omega$, a $\check{\rho}$-twisted form of Bun_N, with a map $\psi : \operatorname{Bun}_N^\Omega \to \mathbb{A}^1$, and the Hitchin section consists of:

$$\operatorname{Kost}_G := \{\mathcal{P}_N \in \operatorname{Bun}_N^\Omega, \phi \in T^*_{\mathcal{P}_G} \operatorname{Bun}_G \mid \phi|_{T|^*_{\mathcal{P}_N} \operatorname{Bun}_N^\Omega} = d\psi(\mathcal{P}_N)\}.$$

Points of Kost_G are also called *classical* opers or *oper Higgs bundles*.

The Hitchin section is Lagrangian in $T^* \operatorname{Bun}_G$.

Geometry of $T^* \mathrm{Bun}_G$: the Kostant-Hitchin section

Officially, the Whittaker sheaf $\mathcal{W}_!$ and its dual coeff are defined using push-pull along $\mathbb{A}^1 = \mathbb{C} \leftarrow \mathrm{Bun}_N^\Omega \to \mathrm{Bun}_G$, starting with $\exp \in \text{D–mod}(\mathbb{A}^1)$, i.e., the same data as defined the Kostant slice.

Geometry of $T^* \mathrm{Bun}_G$: the Kostant-Hitchin section

Officially, the Whittaker sheaf $\mathcal{W}_!$ and its dual coeff are defined using push-pull along $\mathbb{A}^1 = \mathbb{C} \leftarrow \mathrm{Bun}_N^\Omega \rightarrow \mathrm{Bun}_G$, starting with $\exp \in \text{D–mod}(\mathbb{A}^1)$, i.e., the same data as defined the Kostant slice. You can convince yourself that the Hitchin section is the Lagrangian brane supporting $\mathcal{W}_!$.

Geometry of $T^* \mathrm{Bun}_G$: the Kostant-Hitchin section

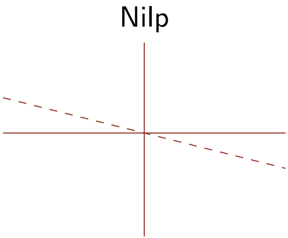
Officially, the Whittaker sheaf $\mathcal{W}_!$ and its dual coeff are defined using push-pull along $\mathbb{A}^1 = \mathbb{C} \leftarrow \mathrm{Bun}_N^\Omega \rightarrow \mathrm{Bun}_G$, starting with $\exp \in \text{D–mod}(\mathbb{A}^1)$, i.e., the same data as defined the Kostant slice. You can convince yourself that the Hitchin section is the Lagrangian brane supporting $\mathcal{W}_!$. Note that this is a *non-conical* Lagrangian, which formally complicates things.

A picture

We have the following basic, completely tautological picture:

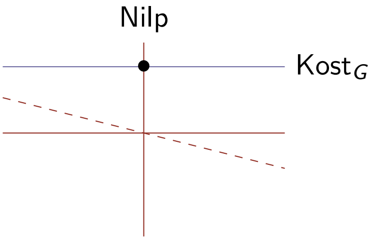
A picture

We have the following basic, completely tautological picture:



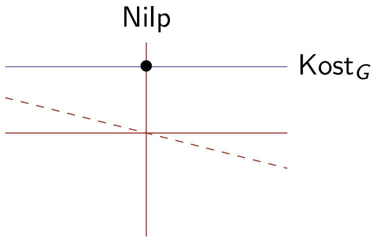
A picture

We have the following basic, completely tautological picture:



A picture

We have the following basic, completely tautological picture:



This picture suggests that we should expect Whittaker coefficients of D-modules/sheaves with nilpotent singular support to behave especially nicely.

The Nadler-Taylor theorem

We have:

The Nadler-Taylor theorem

We have:

Theorem (Nadler-Taylor)

For $\mathcal{F} \in \mathsf{Shv}_{\mathsf{Nilp}}(\mathsf{Bun}_G)$, $\mathsf{coeff}(\mathcal{F})$ *is the microstalk of* $\mathcal{F}$ *at the intersection point* $\mathsf{Nilp} \cap \mathsf{Kost}_G$.

The Nadler-Taylor theorem

We have:

Theorem (Nadler-Taylor)

For $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, $\mathrm{coeff}(\mathcal{F})$ *is the microstalk of* $\mathcal{F}$ *at the intersection point* $\mathrm{Nilp} \cap \mathrm{Kost}_G$*. In other words, if we convert* $\mathcal{F}$ *to a local system* $\mu(\mathcal{F})$ *on* $\mathrm{Nilp}^{\mathrm{smooth}}$, $\mathrm{coeff}(\mathcal{F})$ *is its fiber at the point* $\mathrm{Nilp} \cap \mathrm{Kost}_G \subseteq \mathrm{Nilp}^{\mathrm{smooth}}$.

Corollary

For $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, $\mathrm{coeff}(\mathcal{F}) \neq 0$ *if and only if the point* $\mathrm{Nilp} \cap \mathrm{Kost}_G$ *is in* $\mathrm{SingSupp}(\mathcal{F})$.

The Nadler-Taylor theorem

We have:

Theorem (Nadler-Taylor)

For $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, $\mathrm{coeff}(\mathcal{F})$ *is the microstalk of* $\mathcal{F}$ *at the intersection point* $\mathrm{Nilp} \cap \mathrm{Kost}_G$*. In other words, if we convert* $\mathcal{F}$ *to a local system* $\mu(\mathcal{F})$ *on* $\mathrm{Nilp}^{\mathrm{smooth}}$, $\mathrm{coeff}(\mathcal{F})$ *is its fiber at the point* $\mathrm{Nilp} \cap \mathrm{Kost}_G \subseteq \mathrm{Nilp}^{\mathrm{smooth}}$.

Corollary

For $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, $\mathrm{coeff}(\mathcal{F}) \neq 0$ *if and only if the point* $\mathrm{Nilp} \cap \mathrm{Kost}_G$ *is in* $\mathrm{SingSupp}(\mathcal{F})$*. Also...*

The Nadler-Taylor theorem

We have:

Theorem (Nadler-Taylor)

For $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, $\mathrm{coeff}(\mathcal{F})$ *is the microstalk of* $\mathcal{F}$ *at the intersection point* $\mathrm{Nilp} \cap \mathrm{Kost}_G$*. In other words, if we convert* $\mathcal{F}$ *to a local system* $\mu(\mathcal{F})$ *on* $\mathrm{Nilp}^{\mathrm{smooth}}$, $\mathrm{coeff}(\mathcal{F})$ *is its fiber at the point* $\mathrm{Nilp} \cap \mathrm{Kost}_G \subseteq \mathrm{Nilp}^{\mathrm{smooth}}$.

Corollary

For $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, $\mathrm{coeff}(\mathcal{F}) \neq 0$ *if and only if the point* $\mathrm{Nilp} \cap \mathrm{Kost}_G$ *is in* $\mathrm{SingSupp}(\mathcal{F})$*. Also... this functor is exact, commutes with Verdier duality, yields the characteristic cycle at this point.*

The Nadler-Taylor theorem

We have:

Theorem (Nadler-Taylor)

For $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, $\mathrm{coeff}(\mathcal{F})$ *is the microstalk of* $\mathcal{F}$ *at the intersection point* $\mathrm{Nilp} \cap \mathrm{Kost}_G$*. In other words, if we convert* $\mathcal{F}$ *to a local system* $\mu(\mathcal{F})$ *on* $\mathrm{Nilp}^{\mathrm{smooth}}$, $\mathrm{coeff}(\mathcal{F})$ *is its fiber at the point* $\mathrm{Nilp} \cap \mathrm{Kost}_G \subseteq \mathrm{Nilp}^{\mathrm{smooth}}$.

Corollary

For $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, $\mathrm{coeff}(\mathcal{F}) \neq 0$ *if and only if the point* $\mathrm{Nilp} \cap \mathrm{Kost}_G$ *is in* $\mathrm{SingSupp}(\mathcal{F})$*. Also... this functor is exact, commutes with Verdier duality, yields the characteristic cycle at this point.*

Færgeman and I obtained this corollary using some hard results in geometric Langlands.

The Nadler-Taylor theorem

We have:

Theorem (Nadler-Taylor)

For $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, $\mathrm{coeff}(\mathcal{F})$ *is the microstalk of* $\mathcal{F}$ *at the intersection point* $\mathrm{Nilp} \cap \mathrm{Kost}_G$*. In other words, if we convert* $\mathcal{F}$ *to a local system* $\mu(\mathcal{F})$ *on* $\mathrm{Nilp}^{\mathrm{smooth}}$, $\mathrm{coeff}(\mathcal{F})$ *is its fiber at the point* $\mathrm{Nilp} \cap \mathrm{Kost}_G \subseteq \mathrm{Nilp}^{\mathrm{smooth}}$*.*

Corollary

For $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, $\mathrm{coeff}(\mathcal{F}) \neq 0$ *if and only if the point* $\mathrm{Nilp} \cap \mathrm{Kost}_G$ *is in* $\mathrm{SingSupp}(\mathcal{F})$*. Also... this functor is exact, commutes with Verdier duality, yields the characteristic cycle at this point.*

Færgeman and I obtained this corollary using some hard results in geometric Langlands. Nadler-Taylor gave a geometric mechanism and obtained the full theorem above, which we only conjectured.

Our application

Recall the goal: given a non-zero eigensheaf $\mathcal{F}_\sigma$ with σ an irreducible $\check{G}$-local system, show that $\mathrm{coeff}(\mathcal{F}_\sigma) \neq 0$. We do this as follows:

1. By AGKRRV, $\mathrm{SingSupp}(\mathcal{F}_\sigma) \subseteq \mathrm{Nilp}$.

Our application

Recall the goal: given a non-zero eigensheaf $\mathcal{F}_\sigma$ with σ an irreducible $\check{G}$-local system, show that $\mathrm{coeff}(\mathcal{F}_\sigma) \neq 0$. We do this as follows:

1. By AGKRRV, $\mathrm{SingSupp}(\mathcal{F}_\sigma) \subseteq \mathrm{Nilp}$.
2. $\mathrm{SingSupp}(\mathcal{F}_\sigma)$ cannot lie only in $\mathrm{Nilp}^{\mathrm{irreg}} \subseteq \mathrm{Nilp}$.

Our application

Recall the goal: given a non-zero eigensheaf $\mathcal{F}_\sigma$ with σ an irreducible $\check{G}$-local system, show that $\mathrm{coeff}(\mathcal{F}_\sigma) \neq 0$. We do this as follows:

1. By AGKRRV, $\mathrm{SingSupp}(\mathcal{F}_\sigma) \subseteq \mathrm{Nilp}$.
2. $\mathrm{SingSupp}(\mathcal{F}_\sigma)$ cannot lie only in $\mathrm{Nilp}^{\mathrm{irreg}} \subseteq \mathrm{Nilp}$. This is a general theorem about *tempered* objects of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ (again: temperedness is automatic for σ irreducible).

Our application

Recall the goal: given a non-zero eigensheaf $\mathcal{F}_\sigma$ with σ an irreducible $\check{G}$-local system, show that $\mathrm{coeff}(\mathcal{F}_\sigma) \neq 0$. We do this as follows:

1. By AGKRRV, $\mathrm{SingSupp}(\mathcal{F}_\sigma) \subseteq \mathrm{Nilp}$.
2. $\mathrm{SingSupp}(\mathcal{F}_\sigma)$ cannot lie only in $\mathrm{Nilp}^{\mathrm{irreg}} \subseteq \mathrm{Nilp}$. This is a general theorem about *tempered* objects of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ (again: temperedness is automatic for σ irreducible). (We prove this using some geometric representation theory techniques, some of which are a little subtle.)

Our application

Recall the goal: given a non-zero eigensheaf $\mathcal{F}_\sigma$ with σ an irreducible $\check{G}$-local system, show that $\mathrm{coeff}(\mathcal{F}_\sigma) \neq 0$. We do this as follows:

1. By AGKRRV, $\mathrm{SingSupp}(\mathcal{F}_\sigma) \subseteq \mathrm{Nilp}$.
2. $\mathrm{SingSupp}(\mathcal{F}_\sigma)$ cannot lie only in $\mathrm{Nilp}^{\mathrm{irreg}} \subseteq \mathrm{Nilp}$. This is a general theorem about *tempered* objects of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ (again: temperedness is automatic for σ irreducible). (We prove this using some geometric representation theory techniques, some of which are a little subtle.)
3. Therefore, there exists a point $\phi \in \mathrm{SingSupp}(\mathcal{F}_\sigma)$ which is a nilpotent Higgs field that is regular nilpotent at the generic point of X.

Our application

Recall the goal: given a non-zero eigensheaf $\mathcal{F}_\sigma$ with σ an irreducible $\check{G}$-local system, show that $\mathrm{coeff}(\mathcal{F}_\sigma) \neq 0$. We do this as follows:

1. By AGKRRV, $\mathrm{SingSupp}(\mathcal{F}_\sigma) \subseteq \mathrm{Nilp}$.
2. $\mathrm{SingSupp}(\mathcal{F}_\sigma)$ cannot lie only in $\mathrm{Nilp}^{\mathrm{irreg}} \subseteq \mathrm{Nilp}$. This is a general theorem about *tempered* objects of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ (again: temperedness is automatic for σ irreducible). (We prove this using some geometric representation theory techniques, some of which are a little subtle.)
3. Therefore, there exists a point $\phi \in \mathrm{SingSupp}(\mathcal{F}_\sigma)$ which is a nilpotent Higgs field that is regular nilpotent at the generic point of X. There is a divisor $D = \sum \check{\lambda}_i x_i$ with each $\check{\lambda}_i$ a dominant coweight

Our application

Recall the goal: given a non-zero eigensheaf $\mathcal{F}_\sigma$ with σ an irreducible $\check{G}$-local system, show that $\mathrm{coeff}(\mathcal{F}_\sigma) \neq 0$. We do this as follows:

1. By AGKRRV, $\mathrm{SingSupp}(\mathcal{F}_\sigma) \subseteq \mathrm{Nilp}$.
2. $\mathrm{SingSupp}(\mathcal{F}_\sigma)$ cannot lie only in $\mathrm{Nilp}^{\mathrm{irreg}} \subseteq \mathrm{Nilp}$. This is a general theorem about *tempered* objects of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ (again: temperedness is automatic for σ irreducible). (We prove this using some geometric representation theory techniques, some of which are a little subtle.)
3. Therefore, there exists a point $\phi \in \mathrm{SingSupp}(\mathcal{F}_\sigma)$ which is a nilpotent Higgs field that is regular nilpotent at the generic point of X. There is a divisor $D = \sum \check{\lambda}_i x_i$ with each $\check{\lambda}_i$ a dominant coweight – i.e., corresponding to an irreducible representation of $\check{G}$! –

Our application

Recall the goal: given a non-zero eigensheaf $\mathcal{F}_\sigma$ with σ an irreducible $\check{G}$-local system, show that $\mathrm{coeff}(\mathcal{F}_\sigma) \neq 0$. We do this as follows:

1. By AGKRRV, $\mathrm{SingSupp}(\mathcal{F}_\sigma) \subseteq \mathrm{Nilp}$.
2. $\mathrm{SingSupp}(\mathcal{F}_\sigma)$ cannot lie only in $\mathrm{Nilp}^{\mathrm{irreg}} \subseteq \mathrm{Nilp}$. This is a general theorem about *tempered* objects of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ (again: temperedness is automatic for σ irreducible). (We prove this using some geometric representation theory techniques, some of which are a little subtle.)
3. Therefore, there exists a point $\phi \in \mathrm{SingSupp}(\mathcal{F}_\sigma)$ which is a nilpotent Higgs field that is regular nilpotent at the generic point of X. There is a divisor $D = \sum \check{\lambda}_i x_i$ with each $\check{\lambda}_i$ a dominant coweight – i.e., corresponding to an irreducible representation of $\check{G}$! – which measures the failure of ϕ to be regular nilpotent at *every* point of X.

Our application

4. Applying the Hecke functor corresponding to D (and using the eigensheaf property), we see that there exists a point $\phi' \in \mathrm{SingSupp}(\mathcal{F})$ with ϕ' being regular nilpotent.

Our application

4. Applying the Hecke functor corresponding to D (and using the eigensheaf property), we see that there exists a point $\phi' \in \mathrm{SingSupp}(\mathcal{F})$ with ϕ' being regular nilpotent. (This is proved with some properties of singular support and some quite simple geometry of Lie groups.)

Our application

4. Applying the Hecke functor corresponding to D (and using the eigensheaf property), we see that there exists a point $\phi' \in \mathrm{SingSupp}(\mathcal{F})$ with ϕ' being regular nilpotent. (This is proved with some properties of singular support and some quite simple geometry of Lie groups.) The set of such Higgs fields is smooth, connected, and Lagrangian, so every such point lies in $\mathrm{SingSupp}(\mathcal{F})$.

Our application

4. Applying the Hecke functor corresponding to D (and using the eigensheaf property), we see that there exists a point $\phi' \in \mathrm{SingSupp}(\mathcal{F})$ with ϕ' being regular nilpotent. (This is proved with some properties of singular support and some quite simple geometry of Lie groups.) The set of such Higgs fields is smooth, connected, and Lagrangian, so every such point lies in $\mathrm{SingSupp}(\mathcal{F})$.
5. Therefore, $\mathrm{Nilp} \cap \mathrm{Kost}_G$ lies in $\mathrm{SingSupp}(\mathcal{F})$.

Our application

4. Applying the Hecke functor corresponding to D (and using the eigensheaf property), we see that there exists a point $\phi' \in \mathrm{SingSupp}(\mathcal{F})$ with ϕ' being regular nilpotent. (This is proved with some properties of singular support and some quite simple geometry of Lie groups.) The set of such Higgs fields is smooth, connected, and Lagrangian, so every such point lies in $\mathrm{SingSupp}(\mathcal{F})$.
5. Therefore, $\mathrm{Nilp} \cap \mathrm{Kost}_G$ lies in $\mathrm{SingSupp}(\mathcal{F})$. Now the corollary to Nadler-Taylor implies that $\mathrm{coeff}(\mathcal{F}) \neq 0$, so we win.

Another application

Suppose G is simply-connected for convenience, so Bun_G is connected.

Another application

Suppose G is simply-connected for convenience, so Bun_G is connected. For σ irreducible and *Schurian*

Another application

Suppose G is simply-connected for convenience, so Bun_G is connected. For σ irreducible and *Schurian* – its centralizer in $\check{G}$ should be trivial –

Another application

Suppose G is simply-connected for convenience, so Bun_G is connected. For σ irreducible and *Schurian* – its centralizer in $\check{G}$ should be trivial – we show that any normalized eigensheaf is *an irreducible perverse sheaf*.

Another application

Suppose G is simply-connected for convenience, so Bun_G is connected. For σ irreducible and *Schurian* – its centralizer in $\check{G}$ should be trivial – we show that any normalized eigensheaf is *an irreducible perverse sheaf*. Essentially, we can detect its subquotients using coeff and Hecke functors, and there is no room for anything non-trivial to appear.

Another application

Suppose G is simply-connected for convenience, so Bun_G is connected. For σ irreducible and *Schurian* – its centralizer in $\check{G}$ should be trivial – we show that any normalized eigensheaf is *an irreducible perverse sheaf*. Essentially, we can detect its subquotients using coeff and Hecke functors, and there is no room for anything non-trivial to appear.

In particular, this applies to the eigensheaves constructed by Beilinson-Drinfeld via quantization of Hitchin's fibration.

Thanks!