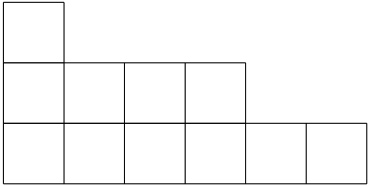
Hypergeometric integrals, hook formulas, and Whittaker vectors

Giovanni Felder, ETH Zurich

Hamburg, 10 November 2023

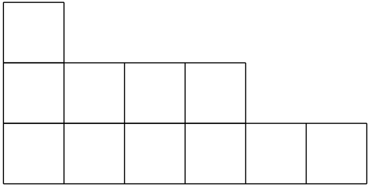
based on joint work with Andrey Smirnov, Vitaly Tarasov and Alexander Varchenko

- Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r \geq 0)$ be a partition of $N \in \mathbb{N}$. We also denote by λ the Young diagram of size $|\lambda| = N$ with rows of length $\lambda_1 \ldots, \lambda_r$.



The Young diagram $\lambda = (6, 4, 1)$

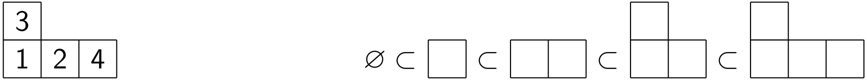
- Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r \geq 0)$ be a partition of $N \in \mathbb{N}$. We also denote by λ the Young diagram of size $|\lambda| = N$ with rows of length $\lambda_1 \ldots, \lambda_r$.



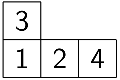
The Young diagram $\lambda = (6, 4, 1)$

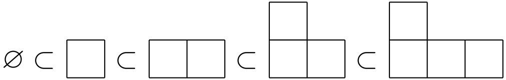
- The classical hook length formula relates the number of standard Young tableaux of shape λ to the product of lengths of hooks.

- A standard Young tableau of shape λ is a bijection $\lambda \to \{1, \ldots, N\}$ on the set of boxes which is increasing in both directions.
- It can be thought as a path $\varnothing \subset \lambda_1 \subset \cdots \subset \lambda_N = \lambda$ of Young diagtrams obtained from the empty diagram by adding one box at a time.

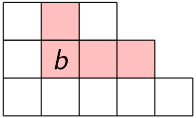


- A standard Young tableau of shape λ is a bijection $\lambda \to \{1, \ldots, N\}$ on the set of boxes which is increasing in both directions.
- It can be thought as a path $\varnothing \subset \lambda_1 \subset \cdots \subset \lambda_N = \lambda$ of Young diagtrams obtained from the empty diagram by adding one box at a time.





- The hook $H(b)$ of a box $b \in \lambda$ is the subset consisting of b and all boxes of λ above b and to its right. Its cardinality is the hook length ℓ_b.



Theorem

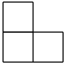
(Frame, Robinson, Thrall 1953) For any partition λ of N the number f^λ of Young tableaux of shape λ is

$$f^\lambda = \frac{N!}{\prod_{b\in\lambda} \ell_b}.$$

Theorem

(Frame, Robinson, Thrall 1953) For any partition λ of N the number f^λ of Young tableaux of shape λ is

$$f^\lambda = \frac{N!}{\prod_{b\in\lambda} \ell_b}.$$

- For example,  has a hook of length 3 and two hooks of length 1. It has thus $3!/3\cdot 1\cdot 1 = 2$ tableaux, namely

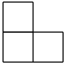
2	
1	3

3	
1	2

Theorem

(Frame, Robinson, Thrall 1953) For any partition λ of N the number f^λ of Young tableaux of shape λ is

$$f^\lambda = \frac{N!}{\prod_{b\in\lambda} \ell_b}.$$

- For example,  has a hook of length 3 and two hooks of length 1. It has thus $3!/3 \cdot 1 \cdot 1 = 2$ tableaux, namely

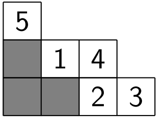
2	
1	3

3	
1	2

- f^λ is the dimension of the irreducible representation of S_N labeled by λ.

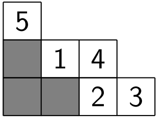
- A skew Young diagram λ/μ is a pair $\mu \subset \lambda$ consisting of a diagram and a subdiagram. The size of λ/μ is $N = |\lambda \setminus \mu|$.

- A skew Young diagram λ/μ is a pair $\mu \subset \lambda$ consisting of a diagram and a subdiagram. The size of λ/μ is $N = |\lambda \setminus \mu|$.
- A standard Young tableau of shape λ/μ is a bijection $\lambda \setminus \mu \to \{1, \ldots, N\}$ increasing in both direction.



Alternatively, it is a path $\mu = \mu_0 \subset \mu_1 \subset \cdots \subset \mu_N = \lambda$ of embedded Young diagrams from μ to λ such that $|\mu_i - \mu_{i-1}| = 1$.

- A skew Young diagram λ/μ is a pair $\mu \subset \lambda$ consisting of a diagram and a subdiagram. The size of λ/μ is $N = |\lambda \setminus \mu|$.
- A standard Young tableau of shape λ/μ is a bijection $\lambda \setminus \mu \to \{1, \ldots, N\}$ increasing in both direction.



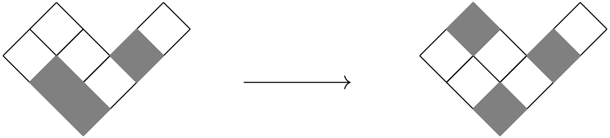
Alternatively, it is a path $\mu = \mu_0 \subset \mu_1 \subset \cdots \subset \mu_N = \lambda$ of embedded Young diagrams from μ to λ such that $|\mu_i - \mu_{i-1}| = 1$.

- H. Naruse gave a subtraction free combinatorial formula for the number of standard Young tableaux of skew shape in terms of hook lengths of excited diagrams.

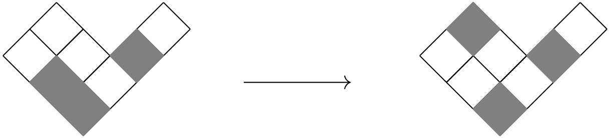
- To define this notion it is better to rotate Young diagrams by 45°.

- To define this notion it is better to rotate Young diagrams by 45°.
- Let ν be a subset of λ. A box of ν is called active if its upper left and upper right neighbours belong to λ but are not in ν.

- To define this notion it is better to rotate Young diagrams by 45°.
- Let ν be a subset of λ. A box of ν is called active if its upper left and upper right neighbours belong to λ but are not in ν.
- An elementary excitation (or ladder move) of a subset ν of a Young diagram λ is a subset obtained by moving one active box of ν up one step.

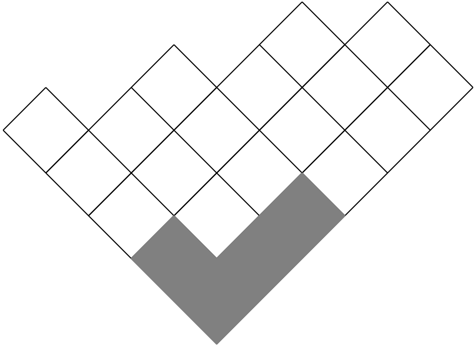


- To define this notion it is better to rotate Young diagrams by 45°.
- Let ν be a subset of λ. A box of ν is called active if its upper left and upper right neighbours belong to λ but are not in ν.
- An elementary excitation (or ladder move) of a subset ν of a Young diagram λ is a subset obtained by moving one active box of ν up one step.

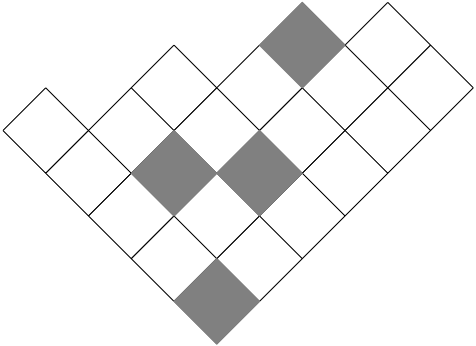


Definition

(Kreiman 2005, Ikeda, Naruse 2009) An excited diagram of the skew diagram λ/μ is a subset of λ obtained from μ by a sequence of elementary excitations.



The skew diagram $\lambda/\mu = (6,6,5,3,1)/(3,1)$



An excited diagram of λ/μ.

Theorem (Naruse 2014)

Let λ/μ be a skew Young diagram of size $N = |\lambda - \mu|$.
The number of standard Young tableaux of shape λ/μ is

$$f^{\lambda/\mu} = \sum_{\nu \in E(\lambda/\mu)} \frac{N!}{\prod_{b \in \lambda \setminus \nu} \ell_b}$$

The summation is over the set of excited diagrams of λ/μ

Theorem (Naruse 2014)

Let λ/μ be a skew Young diagram of size $N = |\lambda - \mu|$.
The number of standard Young tableaux of shape λ/μ is

$$f^{\lambda/\mu} = \sum_{\nu \in E(\lambda/\mu)} \frac{N!}{\prod_{b \in \lambda \setminus \nu} \ell_b}$$

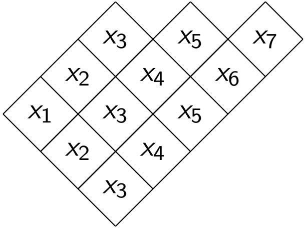
The summation is over the set of excited diagrams of λ/μ

- It will be convenient to define the rational numbers $g^{\lambda/\mu} = f^{\lambda/\mu}/N!$. Then $g^{\lambda/\lambda} = 1$ and since $E(\lambda/\varnothing) = \{\varnothing\}$ we recover the classical hook length formula

$$g^{\lambda/\varnothing} = \frac{f^{\lambda/\varnothing}}{N!} = \frac{1}{\prod_{b \in \lambda} \ell_b}.$$

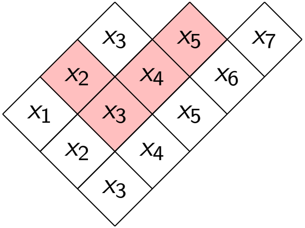
- These formulae are specialization at $x_1 = \cdots = x_{n-1} = 1$ of identities between rational functions in several "equivariant" variables.

- These formulae are specialization at $x_1 = \cdots = x_{n-1} = 1$ of identities between rational functions in several "equivariant" variables.
- Let $I_{r,n-r}$ be the set of Young diagrams fitting in an $r \times (n-r)$ rectangle. Assign variables $x_1, \ldots, x_{n-1}$ to boxes of $\lambda \in I_{r,n-r}$ from left to right, the same variable is assigned to boxes above each other. Let $x(b) \in \{x_1, \ldots, x_{n-1}\}$ be the variable assigned to $b \in \lambda$.



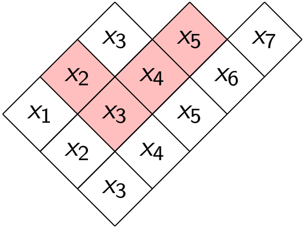
- The hook weight of $b \in \lambda$ is

$$\ell_b(x) = \sum_{b' \in H(b)} x(b') = x_i + x_{i+1} + \cdots + x_j.$$



- The hook weight of $b \in \lambda$ is

$$\ell_b(x) = \sum_{b' \in H(b)} x(b') = x_i + x_{i+1} + \cdots + x_j.$$



- The weight of a skew diagram λ/μ is

$$w_{\lambda/\mu}(x) = \sum_{b \in \lambda \setminus \mu} x(b) = \sum k_i x_i$$

where k_i is the number of boxes in $\lambda \setminus \mu$ labeled by x_i.

Naruse's multivariate hook formula

Theorem (Naruse 2014)

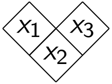
Let λ/μ be a skew diagram of size N.

$$\sum_{\mu=\mu_0 \subset \mu_1 \subset \cdots \subset \mu_N = \lambda} \frac{1}{\prod_{i=1}^{N} w_{\lambda/\mu_i}(x)} = \sum_{\nu \in E(\lambda/\mu)} \frac{1}{\prod_{b \in \lambda \setminus \nu} \ell_b(x)}$$

The summation on the left is over standard Young tableaux of shape λ/μ, i.e., paths from μ to λ such that $|\mu_i - \mu_{i-1}| = 1$

Example

$\lambda = (2,1)$, $\mu = \varnothing$.



$$\frac{1}{(x_1+x_2+x_3)(x_1+x_3)x_3} + \frac{1}{(x_1+x_2+x_3)(x_1+x_3)x_1} = \frac{1}{x_1(x_1+x_2+x_3)x_3}.$$

We can reformulate this theorem by saying that the right-hand side of

$$g^{\lambda/\mu}(x) = \sum_{\nu \in E(\lambda/\mu)} \frac{1}{\prod_{b \in \lambda \setminus \nu} \ell_b(x)}$$

is the solution of the Pieri-type recurrence relation

$$g^{\lambda/\mu}(x) = \frac{1}{w_{\lambda/\mu}(x)} \sum_{\mu' \to \mu} g^{\lambda/\mu'}(x),$$

where the sum is over Young subdiagrams $\mu' \subset \lambda$ obtained for μ by adding one box, with initial condition

$$g^{\lambda/\lambda}(x) = 1.$$

Next

- Equivariant Schubert calculus. This is the context where the formulae were discovered.
- Whittaker vectors in tensor products of dual Verma modules with fundamental modules.
- Multidimensional hypergeometric functions and 3D mirror symmetry.

- The torus $T = U(1)^n \subset U(n)$ acts on the Grassmannian $X = \mathrm{Gr}_r(\mathbb{C}^n)$ with isolated fixed points p_λ labeled by Young diagrams $\lambda \in I_{r,n-r}$.

- The torus $T = U(1)^n \subset U(n)$ acts on the Grassmannian $X = \mathrm{Gr}_r(\mathbb{C}^n)$ with isolated fixed points p_λ labeled by Young diagrams $\lambda \in I_{r,n-r}$.
- The equivariant cohomology $H_T(X)$ of X is the free module over $H_T(pt) = \mathbb{Z}[t_1, \ldots, t_n]$ with basis the Schubert classes $[X_\lambda] = [\overline{B^- p_\lambda}]$.

- The torus $T = U(1)^n \subset U(n)$ acts on the Grassmannian $X = \mathrm{Gr}_r(\mathbb{C}^n)$ with isolated fixed points p_λ labeled by Young diagrams $\lambda \in I_{r,n-r}$.
- The equivariant cohomology $H_T(X)$ of X is the free module over $H_T(pt) = \mathbb{Z}[t_1, \ldots, t_n]$ with basis the Schubert classes $[X_\lambda] = [\overline{B^- p_\lambda}]$.
- The inclusion maps $i_\lambda \colon \{p_\lambda\} \to X$ of fixed points define a monomorphism

$$i^* \colon H_T(X) \to H_T(X^T) = \oplus_{p \in X^T} \mathbb{Z}[t_1, \ldots, t_n]$$

of graded algebras over $\mathbb{Z}[t_1, \ldots, t_n]$ with $\deg t_i = 2$.

- The torus $T = U(1)^n \subset U(n)$ acts on the Grassmannian $X = \mathrm{Gr}_r(\mathbb{C}^n)$ with isolated fixed points p_λ labeled by Young diagrams $\lambda \in I_{r,n-r}$.
- The equivariant cohomology $H_T(X)$ of X is the free module over $H_T(pt) = \mathbb{Z}[t_1, \dots, t_n]$ with basis the Schubert classes $[X_\lambda] = [\overline{B^- p_\lambda}]$.
- The inclusion maps $i_\lambda \colon \{p_\lambda\} \to X$ of fixed points define a monomorphism

$$i^* \colon H_T(X) \to H_T(X^T) = \oplus_{p \in X^T} \mathbb{Z}[t_1, \dots, t_n]$$

  of graded algebras over $\mathbb{Z}[t_1, \dots, t_n]$ with $\deg t_i = 2$.

Proposition (Okounkov 1996, Molev–Sagan 1999, Knutson–Tao 2003, Mihalcea 2005, Naruse 2014)

For all $\mu \subset \lambda \in I_{r,n-r}$, $g^{\lambda/\mu}(x) = i_\mu^*[X_\lambda]/i_\lambda^*[X_\lambda]$, $x_i = t_{i+1} - t_i$ solves the Pieri-type recurrence relation.

Proposition

For all $\mu \subset \lambda \in I_{r,n-r}$, $g^{\lambda/\mu}(x) = i_\mu^*[X_\lambda]/i_\lambda^*[X_\lambda]$, $x_i = t_{i+1} - t_i$ solves the Pieri-type recurrence relation.

Proposition

For all $\mu \subset \lambda \in I_{r,n-r}$, $g^{\lambda/\mu}(x) = i_\mu^*[X_\lambda]/i_\lambda^*[X_\lambda]$, $x_i = t_{i+1} - t_i$ solves the Pieri-type recurrence relation.

- The recurrence relation for $i_\mu^*[X_\lambda] = C_{\lambda\mu}^\mu(t)$ follows from the Chevalley formula for equivariant Littlewood–Richardson coefficients.

Proposition

For all $\mu \subset \lambda \in I_{r,n-r}$, $g^{\lambda/\mu}(x) = i_\mu^*[X_\lambda]/i_\lambda^*[X_\lambda]$, $x_i = t_{i+1} - t_i$ solves the Pieri-type recurrence relation.

- The recurrence relation for $i_\mu^*[X_\lambda] = C_{\lambda\mu}^{\mu}(t)$ follows from the Chevalley formula for equivariant Littlewood–Richardson coefficients.
- On the other hand one has the AJS/Billey formula for $i_\mu^*[X_\lambda]$ (Andersen–Jantzen–Soergel 1994, Billey 1999, Kumar 2002), which can be recast into a sum over excited diagrams.

Proposition

For all $\mu \subset \lambda \in I_{r,n-r}$, $g^{\lambda/\mu}(x) = i_\mu^*[X_\lambda]/i_\lambda^*[X_\lambda]$, $x_i = t_{i+1} - t_i$ solves the Pieri-type recurrence relation.

- The recurrence relation for $i_\mu^*[X_\lambda] = C_{\lambda\mu}^{\mu}(t)$ follows from the Chevalley formula for equivariant Littlewood–Richardson coefficients.
- On the other hand one has the AJS/Billey formula for $i_\mu^*[X_\lambda]$ (Andersen–Jantzen–Soergel 1994, Billey 1999, Kumar 2002), which can be recast into a sum over excited diagrams.
- This generalizes to generalized flag manifolds G/P and to equivariant K-theory and their quantum version.

- Let $\mathfrak{n}^- \subset \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ be the maximal nilpotent of lower triangular matrices. It is generated by $f_i = E_{i+1,i}$, $(i = 1, \ldots, n-1)$. Let $\eta\colon \mathfrak{n}^- \to \mathbb{C}$ be the character such that $\eta(f_i) = -1$ for all i.

- Let $\mathfrak{n}^- \subset \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ be the maximal nilpotent of lower triangular matrices. It is generated by $f_i = E_{i+1,i}$, $(i = 1, \ldots, n-1)$. Let $\eta \colon \mathfrak{n}^- \to \mathbb{C}$ be the character such that $\eta(f_i) = -1$ for all i.
- A Whittaker vector (for the character η) in a $\mathfrak{g}$-module V is a vector $v \in V$ such that

$$xv = \eta(x)v \quad \text{for all } x \in \mathfrak{n}^-.$$

Whittaker vectors (Kostant 1978)

- Let $\mathfrak{n}^- \subset \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ be the maximal nilpotent of lower triangular matrices. It is generated by $f_i = E_{i+1,i}$, $(i = 1, \ldots, n-1)$. Let $\eta\colon \mathfrak{n}^- \to \mathbb{C}$ be the character such that $\eta(f_i) = -1$ for all i.
- A Whittaker vector (for the character η) in a $\mathfrak{g}$-module V is a vector $v \in V$ such that

$$xv = \eta(x)v \quad \text{for all } x \in \mathfrak{n}^-.$$

- The space $\mathrm{Wh}(V)$ of Whittaker vectors in V is a module over the centre $Z = Z(U\mathfrak{g})$ of the universal enveloping algebra of $\mathfrak{g}$.

- Let $\mathfrak{n}^- \subset \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ be the maximal nilpotent of lower triangular matrices. It is generated by $f_i = E_{i+1,i}$, $(i = 1, \ldots, n-1)$. Let $\eta \colon \mathfrak{n}^- \to \mathbb{C}$ be the character such that $\eta(f_i) = -1$ for all i.
- A Whittaker vector (for the character η) in a $\mathfrak{g}$-module V is a vector $v \in V$ such that

$$xv = \eta(x)v \quad \text{for all } x \in \mathfrak{n}^-.$$

- The space $\mathrm{Wh}(V)$ of Whittaker vectors in V is a module over the centre $Z = Z(U\mathfrak{g})$ of the universal enveloping algebra of $\mathfrak{g}$.
- If $v \in \mathrm{Wh}(V) \smallsetminus \{0\}$ and $zv = \chi(z)v$ for all $z \in Z$ for some character $\chi \colon Z \to \mathbb{C}$ of the commutative algebra Z, then one says that v has infinitesimal character χ.

- Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the Gauss decomposition, $\rho \in \mathfrak{h}^*$ the half-sum of positive roots.

Example: Whittaker vectors in dual Verma modules

- Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the Gauss decomposition, $\rho \in \mathfrak{h}^*$ the half-sum of positive roots.
- The Verma module M_t of highest weight $t \in \mathfrak{h}^* \cong \mathbb{C}^n$ is generated by a vector v_t of weight t killed by $\mathfrak{n}^+$ and is free over $U(\mathfrak{n}^-)$.

- Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the Gauss decomposition, $\rho \in \mathfrak{h}^*$ the half-sum of positive roots.
- The Verma module M_t of highest weight $t \in \mathfrak{h}^* \cong \mathbb{C}^n$ is generated by a vector v_t of weight t killed by $\mathfrak{n}^+$ and is free over $U(\mathfrak{n}^-)$.

Lemma

The space of Whittaker vectors in the dual module $M'_{t-\rho} = \mathrm{Hom}_{\mathbb{C}}(M_{t-\rho}, \mathbb{C})$ is 1-dimensional, spanned by ψ such that

$$\psi(f_{i_1} \cdots f_{i_k} v_{t-\rho}) = 1 \quad \text{for all } 1 \leq i_1, \ldots, i_k \leq n-1.$$

- Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the Gauss decomposition, $\rho \in \mathfrak{h}^*$ the half-sum of positive roots.
- The Verma module M_t of highest weight $t \in \mathfrak{h}^* \cong \mathbb{C}^n$ is generated by a vector v_t of weight t killed by $\mathfrak{n}^+$ and is free over $U(\mathfrak{n}^-)$.

Lemma

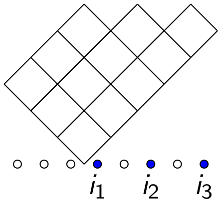
The space of Whittaker vectors in the dual module $M'_{t-\rho} = \mathrm{Hom}_{\mathbb{C}}(M_{t-\rho}, \mathbb{C})$ is 1-dimensional, spanned by ψ such that

$$\psi(f_{i_1} \cdots f_{i_k} v_{t-\rho}) = 1 \quad \text{for all } 1 \leq i_1, \ldots, i_k \leq n-1.$$

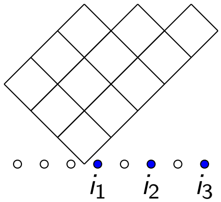
- The centre Z acts on $M'_{t-\rho}$ via a character $\chi(t)\colon Z \to \mathbb{C}$. In particular ψ has infinitesimal character $\chi(t)$.

- Let $U_r = \bigwedge^r \mathbb{C}^n$ be the r-th fundamental module of $\mathfrak{g} = \mathfrak{gl}_n$, $(r = 1, \ldots, n-1)$. It has a basis $u_\mu = e_{i_1} \wedge \cdots \wedge e_{i_r}$ in 1-1 correspondence with Young diagrams $\mu \in I_{r,n}$ fitting in a $r \times (n-r)$-rectangle. Let $\mathrm{wt}(\mu) \in \mathfrak{h}^*$ denote the weight of u_μ.

- Let $U_r = \bigwedge^r \mathbb{C}^n$ be the r-th fundamental module of $\mathfrak{g} = \mathfrak{gl}_n$, $(r = 1, \ldots, n-1)$. It has a basis $u_\mu = e_{i_1} \wedge \cdots \wedge e_{i_r}$ in 1-1 correspondence with Young diagrams $\mu \in I_{r,n}$ fitting in a $r \times (n-r)$-rectangle. Let $\mathrm{wt}(\mu) \in \mathfrak{h}^*$ denote the weight of u_μ.
- Example: $r = 3, n = 8$, $u_\mu = e_4 \wedge e_6 \wedge e_8$, $\mathrm{wt}(\mu) = (0, 0, 0, 1, 0, 1, 0, 1)$.



- Let $U_r = \bigwedge^r \mathbb{C}^n$ be the r-th fundamental module of $\mathfrak{g} = \mathfrak{gl}_n$, $(r = 1, \ldots, n-1)$. It has a basis $u_\mu = e_{i_1} \wedge \cdots \wedge e_{i_r}$ in 1-1 correspondence with Young diagrams $\mu \in I_{r,n}$ fitting in a $r \times (n-r)$-rectangle. Let $\mathrm{wt}(\mu) \in \mathfrak{h}^*$ denote the weight of u_μ.
- Example: $r = 3, n = 8$, $u_\mu = e_4 \wedge e_6 \wedge e_8$, $\mathrm{wt}(\mu) = (0,0,0,1,0,1,0,1)$.



- It follows from results of Kostant that $\mathrm{Wh}(M'_{t-\rho} \otimes U_r)$ has dimension $\dim(U_r) = \binom{n}{r}$. How does the centre Z act?

Theorem (FSTV 2023)

Let $t \in \mathfrak{h}^*$ be generic and let $x_i = t_{i+1} - t_i$ $(i = 1, \ldots, n-1)$. For $\lambda \in I_{r,n-r}$ there is a unique Whittaker vector $\beta_\lambda \in M'_{t-\rho} \otimes U_r \cong \mathrm{Hom}_{\mathbb{C}}(M_{t-\rho}, U_r)$ such that

$$\beta_\lambda(v_{t-\rho}) = \sum_{\mu \subset \lambda} g^{\lambda/\mu}(t) u_\mu, \quad g^{\lambda/\mu}(t) = \sum_{\nu \in E(\lambda/\mu)} \frac{1}{\prod_{b \in \lambda \smallsetminus \nu} \ell_b(x)},$$

It is a simultaneaous eigenvector for the action of Z with infinitesimal character $\chi(t - \mathrm{wt}(\lambda))$. The vectors β_λ form a basis of the space of Whittaker vectors.

Theorem (FSTV 2023)

Let $t \in \mathfrak{h}^*$ be generic and let $x_i = t_{i+1} - t_i$ $(i = 1, \dots, n-1)$. For $\lambda \in I_{r,n-r}$ there is a unique Whittaker vector $\beta_\lambda \in M'_{t-\rho} \otimes U_r \cong \mathrm{Hom}_{\mathbb{C}}(M_{t-\rho}, U_r)$ such that

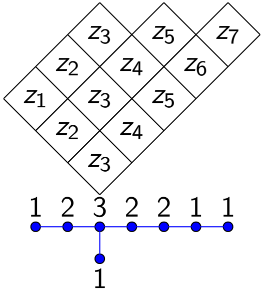
$$\beta_\lambda(v_{t-\rho}) = \sum_{\mu \subset \lambda} g^{\lambda/\mu}(t) u_\mu, \quad g^{\lambda/\mu}(t) = \sum_{\nu \in E(\lambda/\mu)} \frac{1}{\prod_{b \in \lambda \setminus \nu} \ell_b(x)},$$

It is a simultaneaous eigenvector for the action of Z with infinitesimal character $\chi(t - \mathrm{wt}(\lambda))$. The vectors β_λ form a basis of the space of Whittaker vectors.

- Sketch of proof The condition for β to be a Whittaker vector with an infinitesimal character can be translated into the Pieri-type recurrence relation for $g^{\lambda/\mu}(x)$

- To a Nakajima quiver variety X one associates hypergeometric integrals called vertex function $V(X)$ and capping operator $I(X)$ predicting by 3d mirror symmetry to encode the enumerative geometry of quasi-maps (with different boundary conditions) from $\mathbb{P}^1$ to the 3d mirror dual $X^!$. (Okounkov 2015, Aganagic–Okounkov 2017)

- To a Nakajima quiver variety X one associates hypergeometric integrals called vertex function $V(X)$ and capping operator $I(X)$ predicting by 3d mirror symmetry to encode the enumerative geometry of quasi-maps (with different boundary conditions) from $\mathbb{P}^1$ to the 3d mirror dual $X^!$. (Okounkov 2015, Aganagic–Okounkov 2017)
- To a Young diagram λ one associates a quiver and a (0-dimensional) Nakajima variety $X_\lambda = T^*Rep_{v,w}////G_v = \mu^{-1}(0)//G_v$.



- Thus we have a vertex function V_λ and a capping operator I_λ. Since the cohomology of X_λ is one-dimensional one expect them to be proportional.

- Thus we have a vertex function V_λ and a capping operator I_λ. Since the cohomology of X_λ is one-dimensional one expect them to be proportional.

Theorem (FSTV 2003)

$$V_\lambda(z,\kappa) = \frac{1}{\prod_{b\in\lambda}(\ell_b(z)+1)} I_\lambda(z,\kappa).$$

- Thus we have a vertex function V_λ and a capping operator I_λ. Since the cohomology of X_λ is one-dimensional one expect them to be proportional.

Theorem (FSTV 2003)

$$V_\lambda(z,\kappa) = \frac{1}{\prod_{b\in\lambda}(\ell_b(z)+1)} I_\lambda(z,\kappa).$$

- These hypergeometric integrals first appeared in the study of solutions of the Knizhnik–Zamolodchikov equation.

- The theory of hypergeometric solutions of the Knizhnik-Zamolodchikov (Schechtman–Varchenko 1991) provides in particular integral formulas for singular vectors in $M_t \otimes U_r$.

- The theory of hypergeometric solutions of the Knizhnik-Zamolodchikov (Schechtman–Varchenko 1991) provides in particular integral formulas for singular vectors in $M_t \otimes U_r$.
- For generic $t \in \mathfrak{h}^*$ and $\lambda \in I_{r,n-r}$ there is a singular vector, unique up to normalization, of the form

$$\chi_\lambda = \sum_{\mu \leq \lambda} \sum_{I \in A(\lambda/\mu)} c_I^{\lambda/\mu}(t)\, f_{i_1} \cdots f_{i_k} v_t \otimes u_\mu, \quad c_\varnothing^{\lambda/\lambda} \neq 0.$$

The sum is over $I = (i_1, \ldots, i_k)$ such that $\mathrm{wt}(\mu) = \mathrm{wt}(\lambda) + \alpha_{i_1} + \cdots + \alpha_{i_k}$

- The theory of hypergeometric solutions of the Knizhnik-Zamolodchikov (Schechtman–Varchenko 1991) provides in particular integral formulas for singular vectors in $M_t \otimes U_r$.
- For generic $t \in \mathfrak{h}^*$ and $\lambda \in I_{r,n-r}$ there is a singular vector, unique up to normalization, of the form

$$\chi_\lambda = \sum_{\mu \leq \lambda} \sum_{I \in A(\lambda/\mu)} c_I^{\lambda/\mu}(t)\, f_{i_1} \cdots f_{i_k} v_t \otimes u_\mu, \qquad c_\varnothing^{\lambda/\lambda} \neq 0.$$

  The sum is over $I = (i_1, \dots, i_k)$ such that $\mathrm{wt}(\mu) = \mathrm{wt}(\lambda) + \alpha_{i_1} + \cdots + \alpha_{i_k}$
- Pick any generic $\kappa \in \mathbb{C}$. The coefficients are hypergeometric integrals of the form

$$c_I^{\lambda/\mu}(t) = \int_\gamma \Phi_\lambda(s,t)^{\frac{1}{\kappa}} W_I(s) \prod ds_{i,j}$$

- The coefficients are hypergeometric integrals of the form

$$c_I^{\lambda/\mu}(t) = \int_\gamma \Phi_\lambda(s,t)^{\frac{1}{\kappa}} W_I(s) \prod ds_{i,j}$$

- The coefficients are hypergeometric integrals of the form

$$c_I^{\lambda/\mu}(t) = \int_\gamma \Phi_\lambda(s,t)^{\frac{1}{\kappa}} W_I(s) \prod ds_{i,j}$$

- Let $\mathrm{wt}(\lambda) = \varpi_r - \sum_{i=1}^{n-1} k_i \alpha_i$ Then we have k_i integration variables $s_{i,j}$ $(j = 1, \ldots, k_i)$ associated with the simple root α_i (one integration variable for each box of λ). Let $k = \sum_{k_i} = |\lambda|$. The master function is

$$\Phi_\lambda(s,t) = \prod_{(i,j)} s_{i,j}^{-(\alpha_i,t)} (s_{i,j}-1)^{-(\alpha_i,\varpi_r)} \prod_{(i,j)<(i',j')} (s_{i,j}-s_{i',j'})^{(\alpha_i,\alpha_{i'})}$$

- The coefficients are hypergeometric integrals of the form

$$c_I^{\lambda/\mu}(t) = \int_\gamma \Phi_\lambda(s,t)^{\frac{1}{\kappa}} W_I(s) \prod ds_{i,j}$$

- Let $\mathrm{wt}(\lambda) = \varpi_r - \sum_{i=1}^{n-1} k_i \alpha_i$ Then we have k_i integration variables $s_{i,j}$ ($j = 1, \ldots, k_i$) associated with the simple root α_i (one integration variable for each box of λ). Let $k = \sum_{k_i} = |\lambda|$. The master function is

$$\Phi_\lambda(s,t) = \prod_{(i,j)} s_{i,j}^{-(\alpha_i,t)} (s_{i,j} - 1)^{-(\alpha_i,\varpi_r)} \prod_{(i,j)<(i',j')} (s_{i,j} - s_{i',j'})^{(\alpha_i,\alpha_{i'})}$$

- The weight functions $W_I(s)$ are certain rational functions and $\gamma \in H_k(C, L_\kappa)^-$ is a $\prod_{i=1} S_{k_i}$-antiinvariant cycle with coefficients in the local system on a complement of hyperplanes in $\mathbb{C}^k$ defined by the many-valued function $\Phi_\lambda^{1/\kappa}$.

- The integrals $c_I^{\lambda/\mu}(t)$ are complicated objects but their sums $c^{\lambda/\mu}(t) = \sum_{I \in A(\lambda/\mu)} c_I^{\lambda/\mu}(t)$ turn out to be much simpler.

- The integrals $c_I^{\lambda/\mu}(t)$ are complicated objects but their sums $c^{\lambda/\mu}(t) = \sum_{I \in A(\lambda/\mu)} c_I^{\lambda/\mu}(t)$ turn out to be much simpler.

Theorem (FSTV 2023)

Let $t' = t - \rho - \mathrm{wt}(\lambda)$. Then $c^{\lambda/\mu}(t') = g^{\lambda/\mu}(x) c^{\lambda/\lambda}(t'), \quad x_i = t_{i+1} - t_i$. In particular,

$$c^{\lambda/\varnothing}(t') = \prod_{b \in \lambda} \frac{1}{\ell_b(x)} c^{\lambda/\lambda}(t').$$

- The integrals $c_I^{\lambda/\mu}(t)$ are complicated objects but their sums $c^{\lambda/\mu}(t) = \sum_{I \in A(\lambda/\mu)} c_I^{\lambda/\mu}(t)$ turn out to be much simpler.

Theorem (FSTV 2023)

Let $t' = t - \rho - \mathrm{wt}(\lambda)$. Then $c^{\lambda/\mu}(t') = g^{\lambda/\mu}(x) c^{\lambda/\lambda}(t'), \quad x_i = t_{i+1} - t_i$. In particular,

$$c^{\lambda/\varnothing}(t') = \prod_{b \in \lambda} \frac{1}{\ell_b(x)} c^{\lambda/\lambda}(t').$$

- The hypergeometric integrals $c^{\lambda/\varnothing}$ and $c^{\lambda/\lambda}$ are the putative enumerative invariants of $X_\lambda^!$ More precisely,

- The integral $c^{\lambda/\varnothing}(t')$ is (up to a shift of variables) the vertex function. It simplifies to

$$V_\lambda(t', \kappa) = \int_\gamma \Phi_\lambda(s, t')^{\frac{1}{\kappa}} \prod \frac{ds_{i,j}}{s_{i,j}}$$

- The integral $c^{\lambda/\varnothing}(t')$ is (up to a shift of variables) the vertex function. It simplifies to

$$V_\lambda(t',\kappa) = \int_\gamma \Phi_\lambda(s,t')^{\frac{1}{\kappa}} \prod \frac{ds_{i,j}}{s_{i,j}}$$

- The (properly normalized) integral $c^{\lambda/\lambda}(t')$ is the capping operator

$$I_\lambda(t'.\kappa) = \int_\gamma \Phi_\lambda(s,t')^{\frac{1}{\kappa}} W_\varnothing(s) \prod ds_{i,j}.$$

- The integral $c^{\lambda/\varnothing}(t')$ is (up to a shift of variables) the vertex function. It simplifies to

$$V_\lambda(t',\kappa) = \int_\gamma \Phi_\lambda(s,t')^{\frac{1}{\kappa}} \prod \frac{ds_{i,j}}{s_{i,j}}$$

- The (properly normalized) integral $c^{\lambda/\lambda}(t')$ is the capping operator

$$I_\lambda(t'.\kappa) = \int_\gamma \Phi_\lambda(s,t')^{\frac{1}{\kappa}} W_\varnothing(s) \prod ds_{i,j}.$$

Thanks for your attention!