Langlands duality in quantum topology

David Jordan
University of Edinburgh

Theoretical Physics Symposium 2023

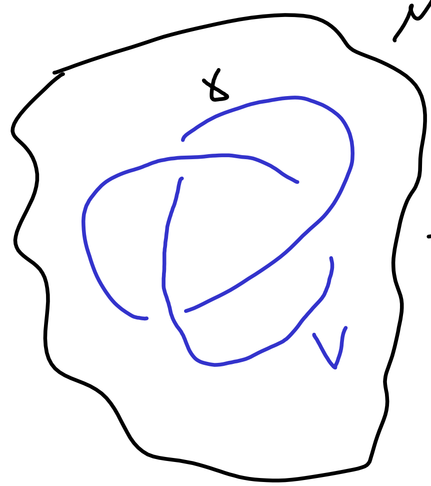
Expository references:

arXiv: 2302.14734 Langlands duality for skein modules
arXiv: 2309.06543 Encyclopedia article on quantum topology

From Witten's 1989 "Quantum Field Theory and the Jones Polynomial":

Wilson loop observables in SL(2) Chern-Simons theory:

M^3 $\quad q \in \mathbb{C}^\times \quad \nabla \in Ch_{SL_2}(M^3) =$ moduli space of flat connections.

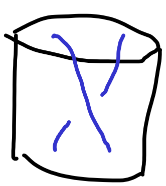
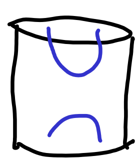
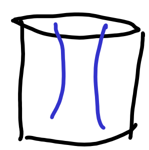


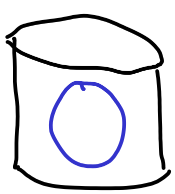
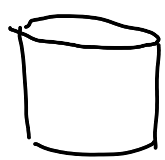
$\overset{q=1}{\rightsquigarrow} \; Tr_V(\nabla_\gamma) \in \mathrm{Functions}(Ch_{SL_2}(M^3))$

$\overset{q\neq 1}{\rightsquigarrow} \; Tr^q_V(\nabla_\gamma)$ Wilson loop observables

---

"A two dimensional vector space has the marvelous property that any three vectors obey a relation of linear dependence." - Edward Witten

(*)  $= q^{1/2}$  $+ \; q^{-1/2}$ 

(**)  $= (q+q^{-1})$ 

$\boxed{V\otimes V = S^2(V)\oplus\Lambda^2(V) \Rightarrow \dim(\mathrm{End}(V\otimes V)) = 2}$

$$Sk^q_{SL_2}(M^3) = \frac{\mathbb{C}\cdot\langle \text{links}\rangle}{(*),\ (**)}$$

e.g.

$Sk^q_{SL_2}(S^3) = \mathbb{C} \Leftrightarrow \exists$ Jones polynomial.

Quantum topology was born from Witten's construction of the Jones polynomial and the Witten-Reshetikhin-Turaev invariants of 3-manifolds after Atiyah-Segal.

Quantum topology is now fully self-perpetuating, and encompasses:
- topological field theory
- quantum groups
- categorification
- skein theory

Many quantum topologists work entirely without interaction with the physical origins. However, QFT (and in particular Edward Witten) still has a lot to teach us!

25 years after the introduction of skein theory, Witten made a fundamental yet surprising (at the time) conjecture about skein theory, which we later proved with Gunningham and Safronov:

For any closed 3-manifold, and generic q,
the G-skein module of M has finite dimension.

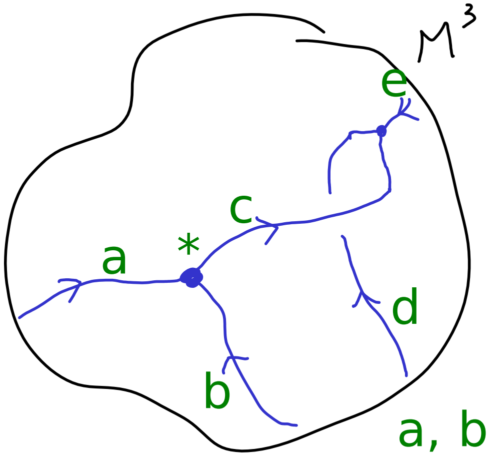
I stress that this defines (apparently) new, integer valued invariants of 3-manifolds, and finite-dimensional representations of their mapping class groups.

5 years earlier - 20 years after the WRT invariants - Kapustin and Witten gave a QFT-inspired explanation for the geometric Langlands duality conjectures of Beilinson-Drinfeld. Inspired by this, we conjecture (w/ Ben-Zvi, G & S):

$$Sk_G^q(M^3) \cong Sk_{G^L}^{q^L}(M^3)$$

In English: A skein is a quantity of yarn; a web, a weave, a tangle.
In Physics: a model for Wilson loop/network observables.
In Topology: a model for (first) non-abelian simplicial homology.
In Geometry: a model for functions on the moduli of flat G-bundles.

Simplicial homology:

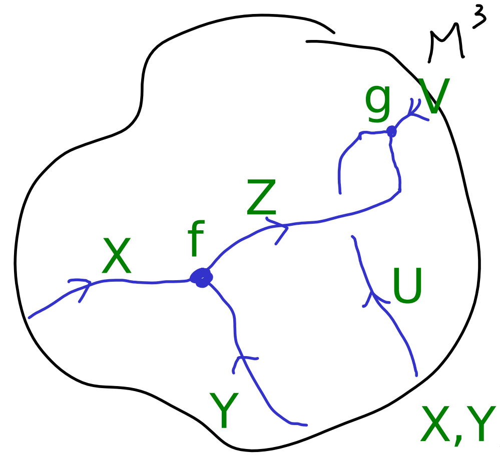


a, b, c, d, e ∈ A

(*): a+b=c

$$H_1(M,A) = \frac{Z_1(M,A)}{dZ_2(M,A)}$$

Skein module:



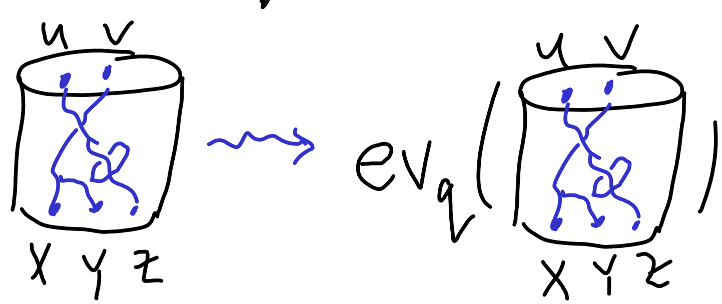
X,Y,Z,U,V ∈ Rep(G)

f: X⊗Y→Z

$$Sk^q_G(M) = \frac{\text{G-skeins}}{\text{skein relns}_q}$$

Reshetikhin-Turaev evaluation:

$$\text{skein relns} = \ker(ev_q) \hookrightarrow \text{G-skeins} \xrightarrow{ev_q} \mathrm{Hom}_G(\text{Inputs, Outputs})$$



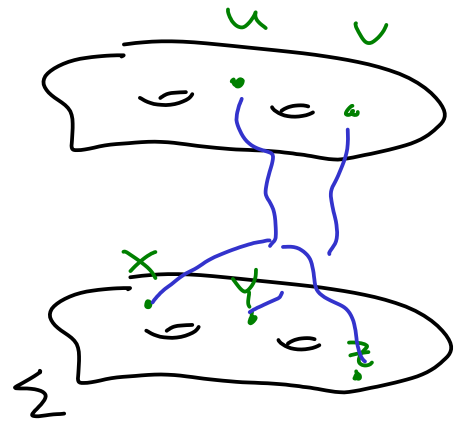
$\in \mathrm{Hom}_G(X\otimes Y\otimes Z, U\otimes V)$

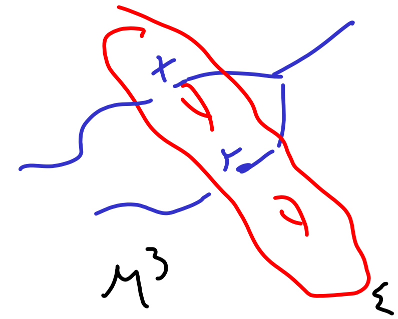
Important mathematical structures on skein modules:

(3,2) TQFT (Walker):

$$M^3 \longmapsto Sk^q_G(M^3) \in Vect$$

$$\Sigma \longmapsto Sk^q_G(\Sigma) \in Cat$$





In fact, it's fully extended (via cobordism hypothesis) <--> Rep G_q

Can be described via factorization algebras, factorization homology:
<--> Wants to be an algebra of topological observables in some QFT.

The 4D TQFT defined by them is fully extended down to a point, but is a priori divergent in dimension 4.

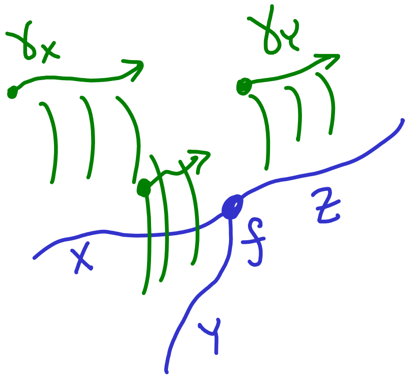
However, Witten's conjecture tells us that $Z(M^3 \times S^1) = \dim Sk^q_G(M^3)$ is convergent.

We'll see later however that it cannot be an oriented fully extended 4D TQFT.

Skeins as functions on flat G-bundles:

$\boxed{q=1}$

$\nabla \in Ch_G(M^3)$



$\rightsquigarrow \quad tr_{labels}(\nabla_\Gamma) \in \text{Functions}\left(Ch_G(M^3)\right)$

$Sk_G^{q=1}(M^3) \cong$ Polynomial functions on the moduli space $Ch_G(M^3)$ of flat G-bundles.

$Sk_G^q(\Sigma \times I)$ is a flat deformation in q. In fact it is free as a $C[q,q^{-1}]$-module.

Conjecture (Witten 2014) ---> Theorem (Gunningham, J, Safronov 2019)

By contrast, if q is generic, and if M is closed, then $Sk_G^q(M^3)$ is finite-dimensional.

Ideas of proof:

- Inspired by complexified instanton Floer homology (Abouzaid-Manolescu)

- Given a Heegaard splitting of M^3, we obtain a tensor decomposition:

$$Sk^q_G(M^3) \cong Sk^q_G(H_g) \otimes_{Sk^q_G(\Sigma_g)} Sk^q_G(H_g^\gamma) \qquad \gamma \in Map(\Sigma_g)$$

- At q=1, $Ch_G(\Sigma_g)$ is symplectic, and each $Ch_G(H_g)$ is Lagrangian.

- The expected dimension is zero, but the intersection may be non-transverse.

- We use Kashiwara-Schapira's theory of deformation quantization to rewrite (*) as a vector space of solutions to a holonomic set of PDE's on G^g

- This passes through Alekseev-Grosse-Schomerus algebras as "gauged" skeins.

- Holonomic systems of PDE's have finite-dimensional solution spaces.

Remark: Derived skein modules exist (Ayala) and have finite dimension in each homological degree (by the same proof above).

Related conjecture (Gunningham-J-Vazirani):

The skein category attached to a surface has a compact generator, i.e. it is equvalent to modules for an algebra.

Langlands duality conjecture:

$$Sk_G^q(M^3) \cong Sk_{G^L}^{q^L}(M^3)$$

Note: $PGL_N = SL_N^L$

Some dimensions are known:

	S^3	$S^2 \times S^1$	$L(r,s)$	T^3	$\Sigma_g \times S^1$	$T^2 \times_\gamma S^1$
PGL_N	1	1	#pts?	$\boxed{\sum_{N=d\cdot e\cdot f} P(d)\cdot e^3 \cdot \mu(f)}$		
SL_N	1	1				
PGL_2	1	1	$\boxed{\#pts}$	$\boxed{9}$	$\boxed{2^{2g+1}+2g-1}$	✓
SL_2	1	1				

SL2 Lens: Hoste-Przytycki
SL2 $\Sigma_g \times S^1$: Gilmer-Masbaum, Detcherry-Wolff
SL2 T^3: Carrega, Gilmer,

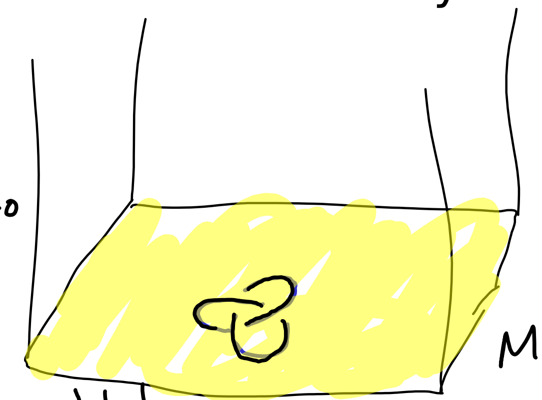
SLN: Gunningham-J-Vazirani-Yang,
PGL2,PGLN: Gunningham-J-Safronov
SL2 mapping tori: Kinnear

Consider the Markus/Kapustin-Witten twist of N=4 SYM in 4D.

$$\rightsquigarrow \text{4D TQFT} \quad Z_{G,\Psi} \xleftrightarrow{\text{S-duality}} Z_{G^L,\Psi^L}, \quad \Psi \in \mathbb{CP}^1, \quad \Psi \cdot \Psi^L = -1$$

Chern-Simons at the boundary of N=4 SYM

$M^3 \times \mathbb{R}_{\geq 0}$



M^3

Dirichlet

$$\rightsquigarrow Sk^q_G(M^3) \longleftrightarrow Z_{G,q}(M^3)$$

$$q = e^{\pi i \Psi}$$

$Z_{G,\Psi}$ | Z_{G^L,Ψ^L}

B-side Dirichlet | Dirichlet B-side

generic Ψ | S-duality | generic Ψ

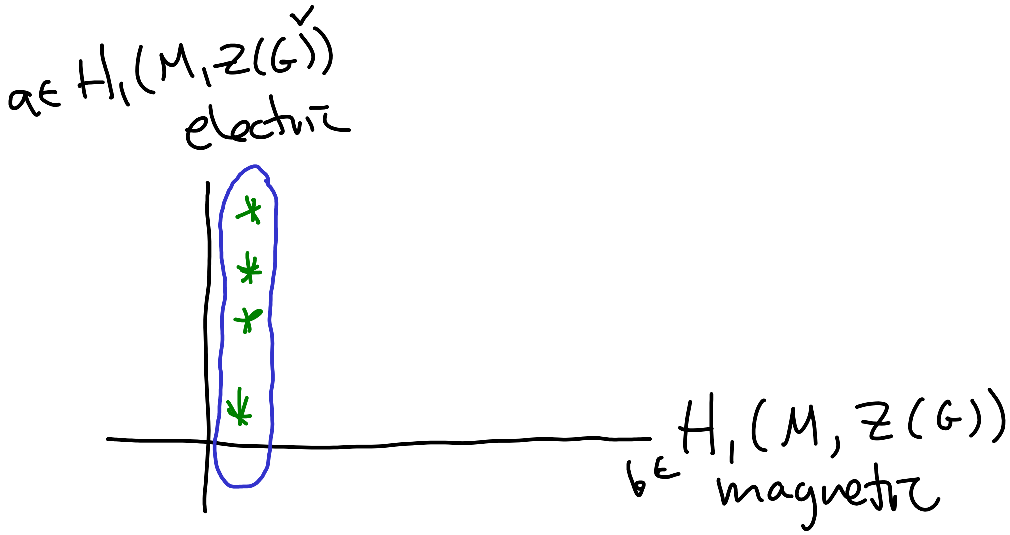
A-side Dirichlet^L | Dirichlet^L A-side

Further evidence: Electric-magnetic 1-form symmetries

Recall that simply connected G has an "electric" 1-form Z(G) symmetry.

This allows us to grade and twist the skein module

$$Sk^q_{G^{s.c}}(M^3)^{a,b}$$



Similarly the adjoint form has a "magnetic" 1-form $\pi_1(G)$ symmetry.

$$Sk^q_{G^{ad}}(M^3)^{b,a} \cong Sk^q_{G^{sc}}(M^3)^{a,b}$$

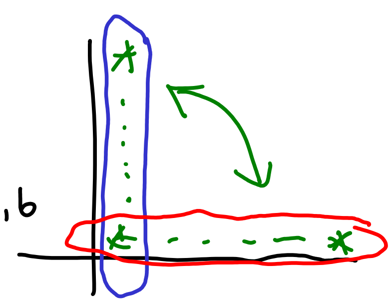
canonically via gauging the electric 1-form symmetry.

Langlands duality predicts instead:

$$Sk^q_{G^{ad}}(M^3)^{a,b} \cong Sk^q_{G^{s.c}}(M^3)^{a,b} \rightsquigarrow$$

Reformulation

$$Sk^q_{G^{s.c}}(M^3)^{a,b}$$



Future questions and directions:

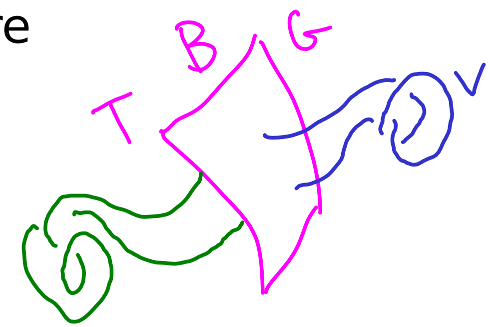
- Deeper, possibly enumerative meaning of skein modules and their dimensions.

- Extension of the duality to all $\Psi, \Psi^L \in \mathbb{CP}^1$.

    - $\Psi, \Psi^L \in \mathbb{Q} \Leftrightarrow q, q^L$ roots of unity
    - $\Psi, \Psi^L = 0, \infty$

- De Rham geometric Langlands for 3-manifolds? A-side construction?

- Skein theory in the presence of defects, quantum A-polynomial, AJ conjecture



- non-semisimple and geometric structures when q is a root of unity.

Congratulations Edward, and thank you for the many years of job security!!!