$\mathbb{Z}/2$ Laplacian on S^2 talk

- The setting is the round 2-sphere in $\mathbb{R}^3$.
  a) Standard Laplacian $\Delta = \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{\sin^2\theta}\partial_\varphi{}^2$.
  b) Eigenvalues of $-\Delta$ are $\{n(n+1): n \in 0, 1, 2, \ldots\}$; multiplicities $2n+1$.
  c) Eigenvectors are spherical harmonics (restriction to S^2 of $A_{ij\cdots n}\, x_i x_j \cdots x_n$ where A is completely symmetric and trace zero on any two indices).
  d) The eigenvectors form a complete, orthonormal basis (with respect to the inner product $(f, g) \to \int_{S^2} fg$) for the space of square integrable functions on S^2.
  e) An exemplar of standard lore about $-\Delta + V$ with V being a potential function: The ground state (the constant function) is unique up to scalar multiplication.
  f) Eigenvectors are critical points of the energy function

$$f \to \int_{S^2} \langle df, df\rangle$$

  subject to the constraint that $\int_{S^2} f^2 = 1$.
  g) The eigenvalues are the critical points. This is the Raleigh-Ritz characterization of eigenvectors and eigenvalues.

- Now consider the following set up (joint work with Therese Wu):
  a) Fix a configuration $\mathfrak{p} = \{p_1, \ldots, p_{2n}\}$ of 2n points on the sphere (the space of 2n distinct, unordered points is denoted by $\mathcal{C}_{2n}$).
  b) There is a real line bundle $\mathcal{I} \to S^2 - \mathfrak{p}$ with monodromy -1 around each point. (This is why I need an even number of points.)
  c) Locally, on any given disk $D \subset S^2 - \mathfrak{p}$, a section, f, is just a function. So we know what df and Δf are.
  d) Can define a Hilbert space of sections by completing the space of sections of $\mathcal{I}$ with compact support in $S^2 - \mathfrak{p}$ using the inner product from the norm whose square is the function

$$f \to \mathcal{E}_{\mathfrak{p}}(f) \equiv \int_{S^2} \langle df, df\rangle .$$

  I will call this Hilbert space $\mathbb{H}_{\mathfrak{p}}$.
  e) I then define a eigensection to be a critical point of this function subject to the constraint that the integral of f^2 is equal to 1.
  f) These eigensections obey $-\Delta f = E f$ where E is the value of $\mathcal{E}$ at its critical point.
  g) Eigensections are smooth in the complement of $\mathfrak{p}$ and have the form

$$f \sim \mathrm{Re}(a_p z^{n_p+1/2}) + \mathcal{O}(|z|^{n_p+3/2})$$

near each $p \in \mathfrak{p}$ when written using Gaussian coordinates near the point. Here, n_p is a non-negative integer and $a_p \in \mathbb{C}-0$. The integer n_p is the degree of f at that point.

j) The eigenvalues of $\mathcal{E}_\mathfrak{p}$ are discrete, with no accumulation points. Moreover, there is a complete, basis for $\mathbb{H}_\mathfrak{p}$ consisting of eigensections.

- We can look for a minimum for $\mathcal{E}_\mathfrak{p}$ on the $\int_{S^2} f^2 = 1$ sphere in $\mathbb{H}_\mathfrak{p}$. This is a function on the configuration space C_{2n} of n points in S^2. What can one say about this function (call it $E_\mathfrak{p}$) and the corresponding eigensection $f_\mathfrak{p}$ (as a function of $\mathfrak{p}$). More generally, what about the second eigenvalue? Or k'th?
  a) First, $E_\mathfrak{p}$ is always positive because $\mathcal{I}$ doesn't have constant sections.
  b) $\inf_C E_\mathfrak{p} = 0$ and this is realized by taking the $\mathfrak{p}$ points in an ever smaller radius disk so that in the limit they coincide (eigenvalues and eigensections are continuous with respect to colliding points if you count multiplicity mod(2): Throw away even number collisions and count odd number ones as a single point).
  c) $c\,n \geq \sup_C E_\mathfrak{p} \geq c^{-1} n$.
  d) Critical points of $E_\mathfrak{p}$: The function $E_\mathfrak{p}$ is smooth where it has multiplicity 1 as an eigenvalue of $-\Delta$, but it isn't smooth where it has multiplicity > 1. Moreover, $E_\mathfrak{p}$ at its maximum (which is achieved) has multiplicity > 1 so it isn't smooth there; and infact, $dE_\mathfrak{p}$ is (technically) never zero.

- What does this mean? A better way to think about this: One would like to consider the set of pairs $\{(\mathfrak{p}, \mathbb{H}_\mathfrak{p}): \mathfrak{p} \in C_{2n}\}$ as defining a Hilbert space bundle $\mathbb{H} \to C_{2n}$ and then the assignment $(\mathfrak{p}, f) \to \mathcal{E}_\mathfrak{p}(f)$ defines a differentiable function on the unit sphere bundle. We would be looking for its critical points. Except:
  a) There is no Hilbert space bundle. This is a very simple example of an anomaly.
  b) The anomaly here is due to the following fact: One can define a universal punctured sphere fiber bundle $S \to C_{2n}$ whose fiber over $\mathfrak{p}$ is $S^2-\mathfrak{p}$. Sitting over each fiber $S^2-\mathfrak{p}$ is the corresponding line bundle $\mathcal{I}_\mathfrak{p}$. There is no universal line bundle $\mathbb{I} \to S$ whose restriction to any given fiber is this $\mathcal{I}_\mathfrak{p}$.

- You can define an $\mathbb{RP}^\infty$ bundle $\mathbb{RP} \to C_{2n}$ whose fiber at $\mathfrak{p}$ is the quotient of the unit sphere in $\mathbb{H}_\mathfrak{p}$ by $\mathbb{R}^*$. The function $(\mathfrak{p}, [\pm f]) \to \mathcal{E}_\mathfrak{p}$ comes from there.
  a) A pair $(\mathfrak{p}, f)$ is a critical point if and only if f is an eigensection that vanishes as $\mathrm{Re}(a_p z^{3/2})$ or faster at each $p \in \mathfrak{p}$. (Generic vanishing is $\mathrm{Re}(a_p z^{1/2})$).

b) At a critical value, the corresponding eigenvalue can't be the lowest $\mathcal{I}_p$ eigenvalue, there must be at least n smaller eigenvalues. (Which is a technically correct way to say that the lowest eigenvalue function $\mathfrak{p} \to E_p$ has no critical points.)
c) Actually, there might not be any critical values because C_{2n} is not compact.
d) But there is a compactification to which $\mathbb{RP}$ and $\mathcal{E}$ extend $\underline{C}_{2n} = C_{2n} \cup C_{2n-2} \cdots \cup C_0$ where lower strata just count a multiplicity k divisor as k (mod 2) (either as 0 or 1).

- You can see this almost explicitly in the case when n = 1 (so 2 points on S^2).
  a) Configuration space compactifies as the Thom space of the tangent space to $\mathbb{RP}^2$.
  b) Eigenvalues when the points come together are m(m+1) with multiplicity 2m+1.
  c) Eigenvalues when point are antipodal: $m^2 - \frac{1}{4}$ with m = 1, 2, … and multiplicity 2m.
  d) The antipodal cases with m = 2, … comprise the set of critical points of $\mathcal{E}$ on the bundle $\mathbb{RP} \to C_2$.
  d) The respective eigenvalues for the points colliding and antipodal are interleaved. As you bring the points together, half of the eigenvalues go down and half go up.
  e) Which half go up and which half go down? This depends on the arc you take the points together on!
  f) Interesting representation of Sl(2; $\mathbb{C}$) here.

- A simpler example: $\mathbb{Z}/2$ harmonic functions for C_2 on $[-1, 1] \times S^1$.