The Massless String Spectrum on $AdS_3 \times S^3$ from the Supergroup

Sebastian Gerigk
ETH Zurich

based on work with M. Gaberdiel (arXiV:1107.2660)

Why String Theory on AdS_3?

- Superstring Theory on AdS-spaces is conjectured to be dual to conformal field theories.
  [Maldacena `98]

- In pure NS-backgrounds, there exists a well-established RNS-formulation of string theory on $AdS_3 \times S^3$ in terms of a (world sheet)-supersymmetric $SL(2) \times SU(2)$ WZW model.
  [Giveon, Kutasov, Seiberg `98]
  [Kutasov, Seiberg `99]

- Formulation including RR-flux is desirable.
  $\Rightarrow$ Target space supersymmetric formulation is needed.
  $\Rightarrow$ *Need to find a „superspace-version“ of $SL(2) \times SU(2) \simeq AdS_3 \times S^3$!*

Lie Superalgebras

[Kac `77]

- A Lie superalgebra is given by a Lie algebra $\mathfrak{g}^{(0)}$ and a set of fermionic generators $\mathfrak{g}^{(1)}$ such that

$$\left[\mathfrak{g}^{(0)}, \mathfrak{g}^{(1)}\right] \subset \mathfrak{g}^{(1)}\,, \qquad \left\{\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}\right\} \subset \mathfrak{g}^{(0)}$$

- In other words, the fermionic generators form a representation of $\mathfrak{g}^{(0)}$ with respect to the adjoint action.

- The anticommutator needs to be compatible with the $\mathfrak{g}^{(0)}$-action. (generalised Jacobi identity)

$$[a, \{b, c\}] = \{[a, b]\,, c\} + \{b, [a, c]\}$$

Adding SUSY to $AdS_3 \times S^3$

- As manifolds we can locally identify $AdS_3 \times S^3$ with (a real form of) the semisimple Lie algebra $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$.

- This Lie algebra allows a consistent supersymmetric structure, *i.e.* it appears as the bosonic subalgebra of a Lie superalgebra.

- The resulting Lie superalgebra is known as $\mathfrak{psl}(2|2)$.

Superstrings on $AdS_3 \times S^3$

[Berkovits, Vafa, Witten `99]

- The *hybrid formulation* gives a description of the superstring that is target space supersymmetric in 6 dimensions.
- Superstrings on $AdS_3 \times S^3$ (including RR-flux) are described by a
  - a non-linear sigma-model on $PSL(2|2)$
  - + an (twisted) N=2 superconformal structure on the world sheet.
- The WZW-point corresponds to the case of pure NSNS-flux.

topic of this talk! ↗

The Path to the Physical States

- Goal: Give a description of the physical string spectrum in the hybrid formulation.
- 1st Step: Understand the representations of $\mathfrak{psl}(2|2)$.
- 2nd Step: Determine the space of states of the PSL(2|2) WZW-model.
- 3rd Step: Find the cohomology of physical string states.
- *Today, we will restrict ourselves to the massless sector of string states!*

Representations: Kac-Modules

[Kac `77]

creation operators

- $\mathfrak{psl}(2|2)$ has a decomposition of the form $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$

annihilation operators

- First take a representation $\mathcal{V}(j_1, j_2)$ of the bosonic subalgebra.
  Note: We are interested in representations with $j_1 < 0\,,\;\; j_2 \geq 0\,.$
- Let all elements in $\mathfrak{g}_1$ act trivially on $\mathcal{V}(j_1, j_2)\,.$
- The Kac-module $\mathcal{K}(j_1, j_2)$ is obtained by treating $\mathfrak{g}_{-1}$ as fermionic creation operators.
- Warning: Kac-modules may not be fully reducible!

Atypical Kac-Modules

- For $\mathfrak{psl}(2|2)$, Kac-Modules are reducible if the second Casimir vanishes,

$$j_1 + j_2 + 1 = 0\,.$$

[Kac `77]
[Götz, Quella, Schomerus `07]

- These are referred to as *atypical*, and denoted $\mathcal{K}(j) \equiv \mathcal{K}(-j-1, j)\,.$

$$\mathcal{K}(j): \quad \begin{tikzcd}[column sep=small] & \mathcal{L}(j) \arrow[dl, no head] \arrow[dr, no head] & \\ \mathcal{L}(j-\tfrac{1}{2}) \arrow[dr, no head] & & \mathcal{L}(j+\tfrac{1}{2}) \arrow[dl, no head] \\ & \mathcal{L}(j) & \end{tikzcd} \qquad \text{for } j \geq 1$$

Projective Covers

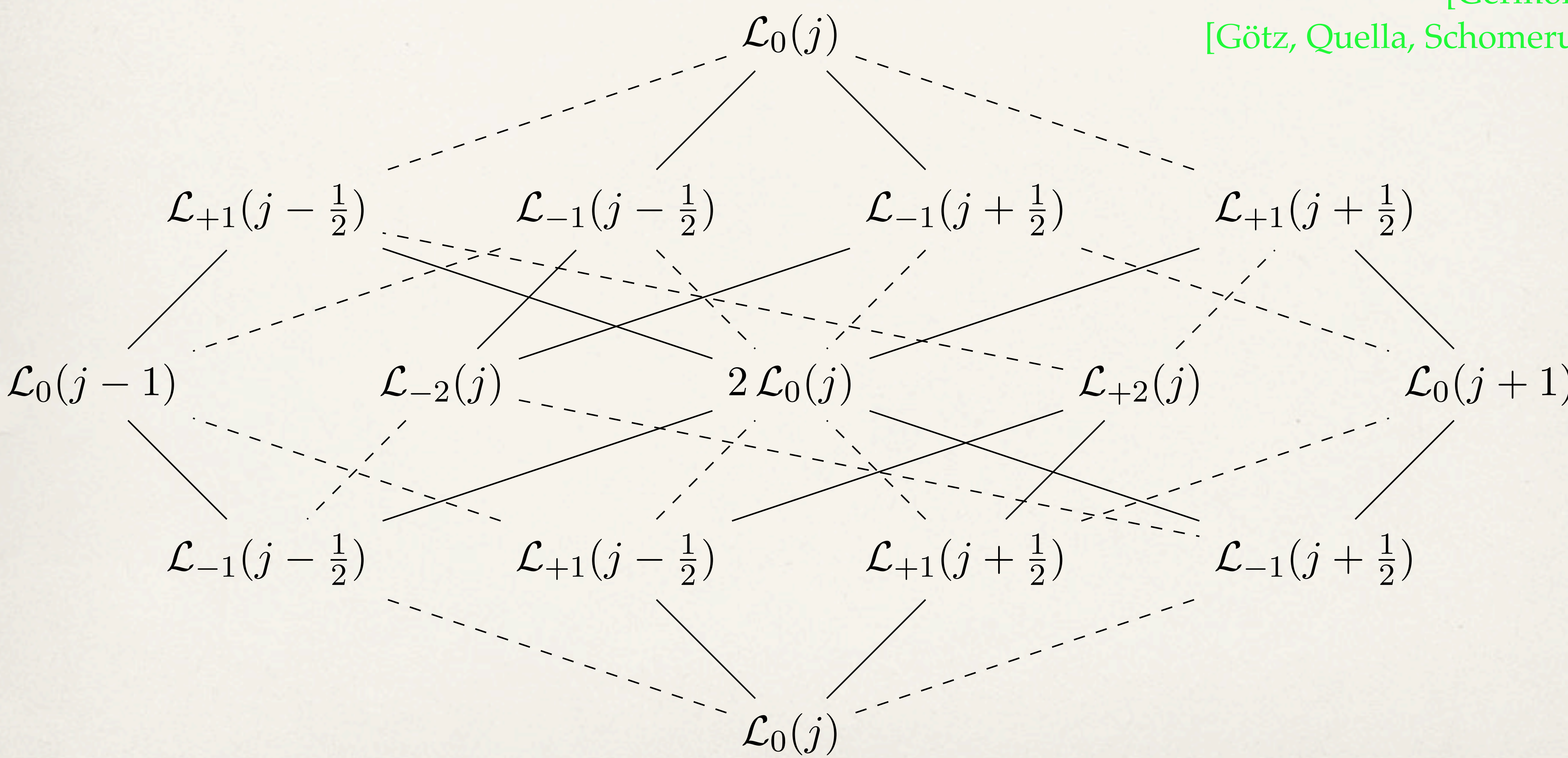
- The *projective cover* $\mathcal{P}(j_1, j_2)$ of an irreducible representation $\mathcal{L}(j_1, j_2)$ is in some sense the largest indecomposable representation covering $\mathcal{L}(j_1, j_2)$.

- $\mathcal{P}(j_1, j_2)$ covers every representation that itself is covering $\mathcal{L}(j_1, j_2)$.

- In particular, $\mathcal{P}(j_1, j_2)$ covers the Kac-module $\mathcal{K}(j_1, j_2)$.

The Projective Cover $\mathcal{P}(j)$

[Zou `96]
[Germoni `98]
[Götz, Quella, Schomerus `07]



The Ansatz for the Spectrum

- The PSL(2|2) WZW model is a logarithmic CFT. [Götz, Quella, Schomerus `07] [Quella, Schomerus `07]
- Motivated by the spectra of known logarithmic CFTs, we propose the space of states to be of the form [Gaberdiel, Runkel `07] [Gaberdiel, Runkel, Wood `09]

$$\mathcal{H}^{(0)} = \hat{\mathcal{H}}/\mathcal{N} \ , \qquad \text{where} \qquad \hat{\mathcal{H}} = \bigoplus_{(j_1,j_2)} \mathcal{P}(j_1,j_2) \otimes \overline{\mathcal{P}(j_1,j_2)} \ ,$$

- The subrepresentation $\mathcal{N}$ has to be chosen such that w.r.t. the left action of $\mathfrak{psl}(2|2)$ we have the identification

$$\mathcal{H}^{(0)} = \bigoplus_{(j_1,j_2)} \mathcal{P}(j_1,j_2) \otimes \overline{\mathcal{L}(j_1,j_2)} \ ,$$

How to Define $\mathcal{N}$?

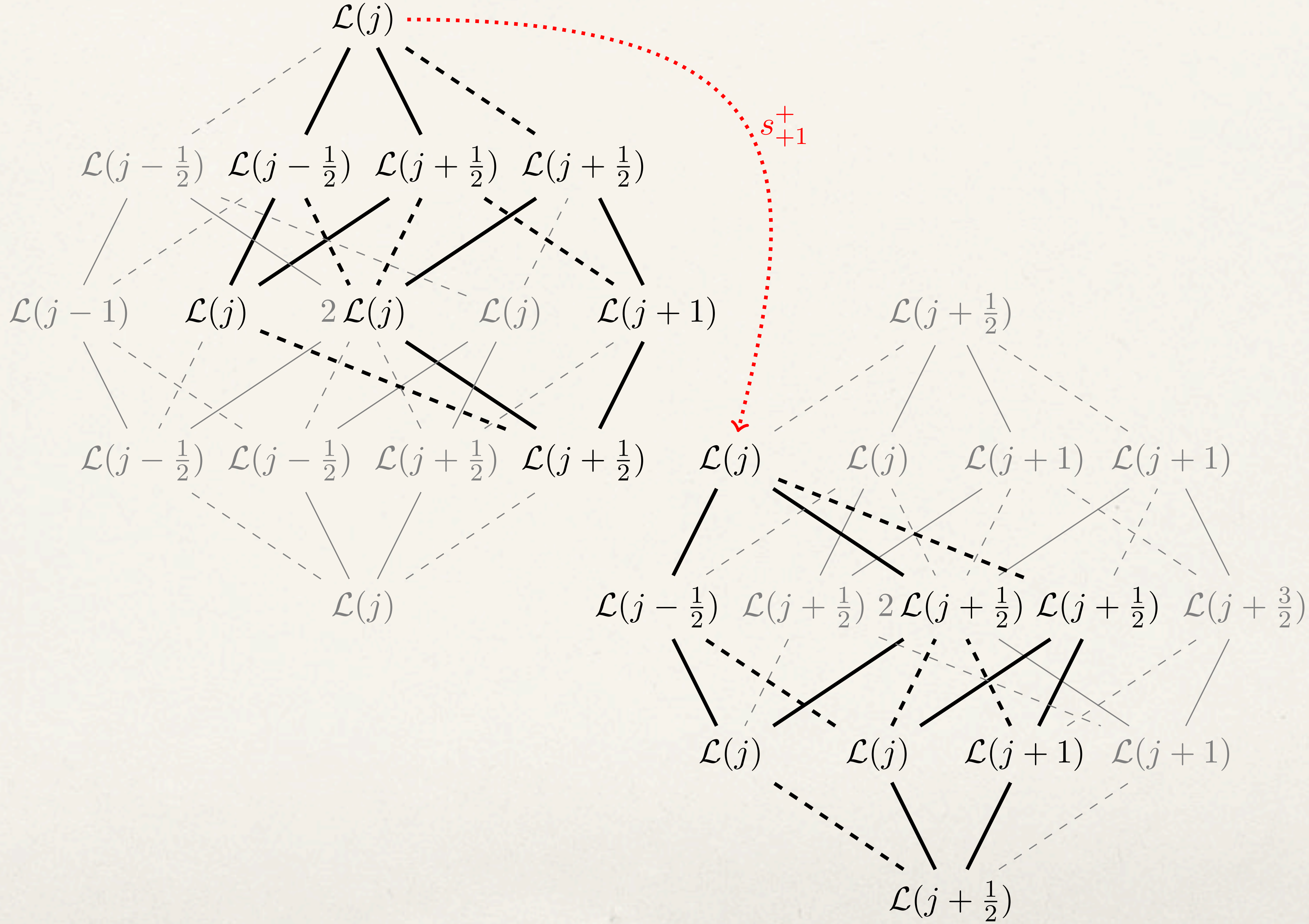
- There is a natural way to obtain submodules: <u>kernels</u> and <u>images</u> of homomorphisms.

- The basic homomorphisms between projective covers are

$$s_\sigma^\pm : \mathcal{P}(j) \to \mathcal{P}(j + \tfrac{\sigma}{2}) \ ,$$

mapping the head of $\mathcal{P}(j)$ to one of the heads of the maximal subrepresentation of $\mathcal{P}(j + \frac{\sigma}{2})$.

Basic Homomorphisms



How to Define $\mathcal{N}$?

- Submodules consistent with the requirement before are

$$\mathcal{N}_\sigma^\pm(j) = \left(s_\sigma^\pm \otimes \overline{\mathrm{id}} - \mathrm{id} \otimes \overline{(s_\sigma^\pm)^\vee}\right) \left(\mathcal{P}(j - \tfrac{\sigma}{2}) \otimes \overline{\mathcal{P}(j)}\right)$$

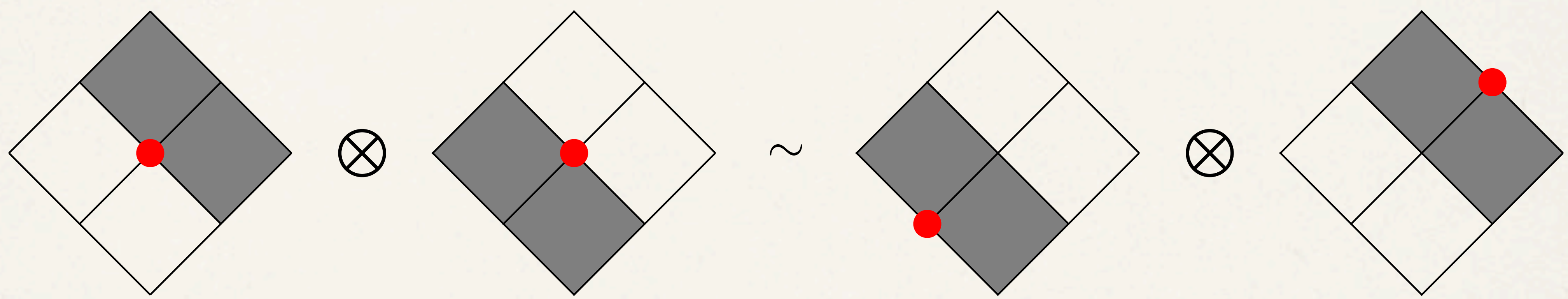
- Note that the induced equivalence relation identifies states in different direct summands of $\hat{\mathcal{H}}$!

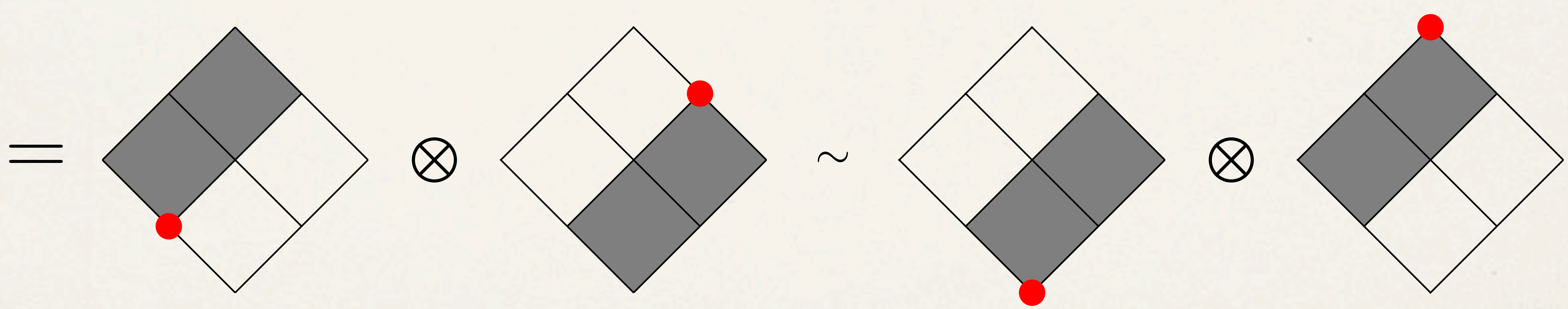
$$\left(\mathcal{P}(j) \otimes \overline{\mathcal{P}(j)}\right) \supset \mathcal{M}_\sigma^\pm(j - \tfrac{\sigma}{2}) \otimes \overline{\mathcal{P}(j)} \sim \mathcal{P}(j - \tfrac{\sigma}{2}) \otimes \overline{\mathcal{M}_{-\sigma}^\mp(j)} \subset \left(\mathcal{P}(j - \tfrac{\sigma}{2}) \otimes \overline{\mathcal{P}(j - \tfrac{\sigma}{2})}\right) .$$

This looks pretty formal, but here's a really nice way to think about it....

An Illustration of the Equivalence

$\mathcal{P}(j) \otimes \overline{\mathcal{P}(j)}$ $\qquad$ $\mathcal{P}(j + \frac{\sigma}{2}) \otimes \overline{\mathcal{P}(j + \frac{\sigma}{2})}$





$\mathcal{P}(j + \frac{\sigma}{2}) \otimes \overline{\mathcal{P}(j + \frac{\sigma}{2})}$ $\qquad$ $\mathcal{P}(j + \frac{\sigma+\sigma'}{2}) \otimes \overline{\mathcal{P}(j + \frac{\sigma+\sigma'}{2})}$

The Ansatz for the Spectrum

- The PSL(2|2) WZW model is a logarithmic CFT.
- Motivated by the spectra of known logarithmic CFTs, we propose the space of states to be of the form

$$\mathcal{H}^{(0)} = \hat{\mathcal{H}}/\mathcal{N} \ , \qquad \text{where} \qquad \hat{\mathcal{H}} = \bigoplus_{(j_1,j_2)} \mathcal{P}(j_1,j_2) \otimes \overline{\mathcal{P}(j_1,j_2)} \ ,$$

- The subrepresentation $\mathcal{N}$ has to be chosen such that w.r.t. the left action of $\mathfrak{psl}(2|2)$ we have the identification

$$\mathcal{H}^{(0)} = \bigoplus_{(j_1,j_2)} \mathcal{P}(j_1,j_2) \otimes \overline{\mathcal{L}(j_1,j_2)} \ , \quad \checkmark$$

Lift of the Hybrid-BRST-Operator

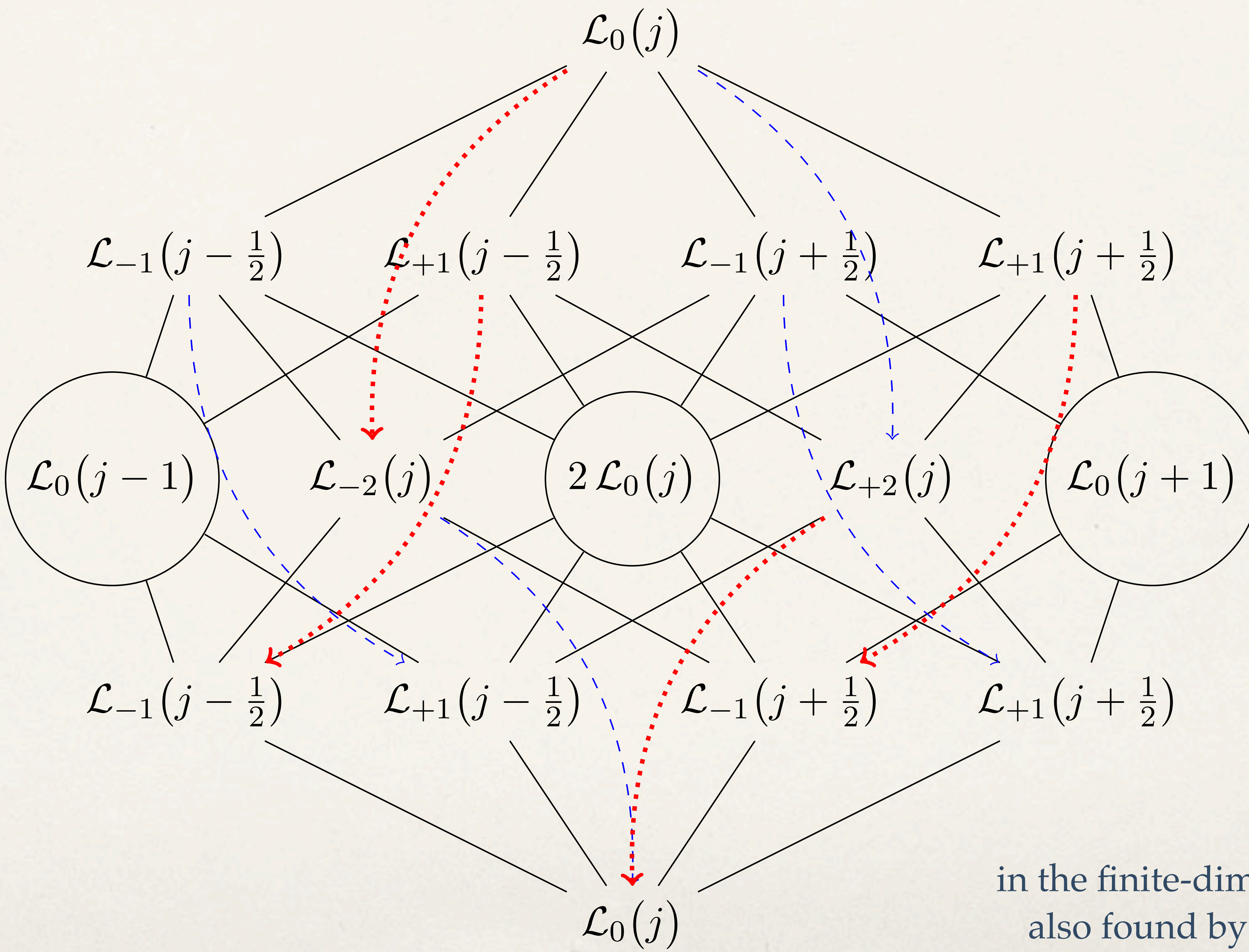
- The hybrid BRST-operator does not induce a homomorphism on $\mathcal{P}(j)$:

$$\left[Q, \mathfrak{g}^{(1)}\right]\Big|_{\mathcal{P}(j)} \propto C_2 \neq 0\,.$$

- But there exists a unique homomorphism $\mathcal{P}(j) \to \mathcal{P}(j)$ that reduces to the action of the hybrid BRST-operator Q on $\mathcal{K}(j)$.

- Thus in the presented description of the space of states, one should take the cohomology with respect to *the lifted BRST-operator.*

The BRST Cohomology



in the finite-dimensional case
also found by [Troost `11]

Low-Lying String States

$(j_2, \bar{j}_2)$	$\mathfrak{psl}(2\|2)$-rep	# in $\mathfrak{psl}(2\|2)$-rep	# in $\mathcal{H}$	$\sum$
$(0,0)_{S^3}$	$\mathcal{L}(0) \otimes \overline{\mathcal{L}(0)}$	4	4	6
	$\mathcal{L}(\frac{1}{2}) \otimes \overline{\mathcal{L}(\frac{1}{2})}$	1	2	
$(0,\frac{1}{2})_{S^3}$	$\mathcal{L}(0) \otimes \overline{\mathcal{L}(0)}$	2	2	8
	$\mathcal{L}(\frac{1}{2}) \otimes \overline{\mathcal{L}(\frac{1}{2})}$	2	4	
	$\mathcal{L}(0) \otimes \overline{\mathcal{L}(1)}$	2	2	
$(\frac{1}{2},\frac{1}{2})_{S^3}$	$\mathcal{L}(0) \otimes \overline{\mathcal{L}(0)}$	1	1	13
	$\mathcal{L}(\frac{1}{2}) \otimes \overline{\mathcal{L}(\frac{1}{2})}$	4	8	
	$\mathcal{L}(0) \otimes \overline{\mathcal{L}(1)}$	1	1	
	$\mathcal{L}(1) \otimes \overline{\mathcal{L}(0)}$	1	1	
	$\mathcal{L}(1) \otimes \overline{\mathcal{L}(1)}$	1	2	
$(0,1)_{S^3}$	$\mathcal{L}(0) \otimes \overline{\mathcal{L}(1)}$	4	4	7
	$\mathcal{L}(\frac{1}{2}) \otimes \overline{\mathcal{L}(\frac{1}{2})}$	1	2	
	$\mathcal{L}(\frac{1}{2}) \otimes \overline{\mathcal{L}(\frac{3}{2})}$	1	1	

Spectrum in agreement with the supergravity answer!

[Deger, Kaya, Sezgin, Sundell `98]
[de Boer `98]

Table 1. *Decompostion of $\mathcal{H}_{\text{phys}}$ under $\mathfrak{so}(4)$. The first column denotes the $\mathfrak{so}(4)$ representations, the second enumerates the irreducible $\mathfrak{psl}(2|2)$ representations which contain the relevant $\mathfrak{so}(4)$ representation. The third column lists its multiplicity within the $\mathfrak{psl}(2|2)$ representation, and the fourth its overall multiplicity in $\mathcal{H}_{\text{phys}}$. Finally, the last column sums the multiplicities from the different $\mathfrak{psl}(2|2)$ representations.*

Conclusions and Outlook

- A description of the space of massless string states as the quotient of a full space of states was proposed.
- The physical subsector was determined by taking the cohomology of a lifted hybrid BRST operator and shown to agree with the supergravity answer.
- *Massive States?*
- *RR-Deformations away from the WZW-point in moduli space?*