$O(T_F^2)$ Contributions to the Heavy Flavor DIS Wilson Coefficients at $O(\alpha_s^3)$

Fabian Wissbrock, DESY

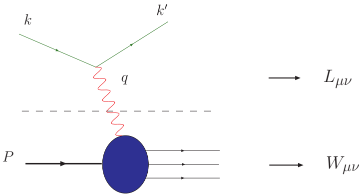
in collaboration with Jakob Ablinger, Isabella Bierenbaum, Johannes Blümlein, Sebastian Klein and Carsten Schneider

[Nuclear Physics B 844 (2011), pp. 26-54, arXiv:1008.3347] and in preparation

DESY Theory Workshop September 28, 2011

Contents

Introduction



- kinematic quantities: $Q^2 := -q^2, \quad x := \frac{Q^2}{2pq}, \quad \nu := \frac{Pq}{M}$
- differential cross-section: $\frac{d\sigma}{dQ^2\,dx} \sim W_{\mu\nu} L^{\mu\nu}$

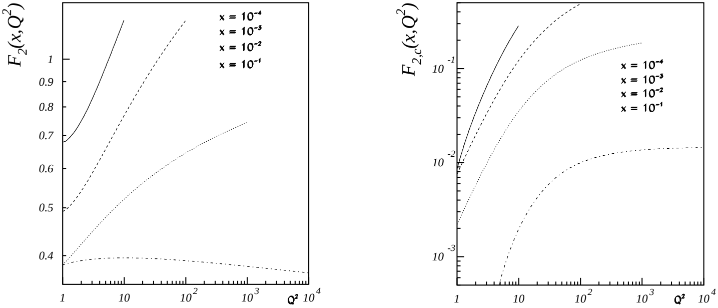
$$W_{\mu\nu}(q, P, s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P, s \rangle$$

$$\text{unpol.} \left\{ = \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x, Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x, Q^2)\right.$$

$$\text{pol.} \left\{ \quad - \frac{M}{2Pq}\varepsilon_{\mu\nu\alpha\beta} q^\alpha \left[s^\beta g_1(x, Q^2) + \left(s^\beta - \frac{sq}{Pq}p^\beta\right) g_2(x, Q^2)\right] .\right.$$

Structure Functions: $F_{2,L}$
contain light and heavy quark contributions

Heavy flavor contributions to F_2



LO contributions: massless vs. massive (PDFs from [Alekhin, Melnikov, Petriello, 2006.])
→ different scaling violations,
→ massive contributions at lower values of x are of order 20% – 35%.

Representation for F_2 at $Q^2 > 10m^2$

- in the asymptotic region F_L is known for general values of N to NNLO
  [Blümlein, De Freitas, van Neerven, Klein, 2006.]
- F_2 for N_F massless and one heavy quark flavor:
  [Bierenbaum, Blümlein, Klein, 2009.]

$$
\begin{aligned}
F_{(2,L)}^{Q\overline{Q}}(x, N_F+1, Q^2, m^2) = & \sum_{k=1}^{N_F} e_k^2 \Bigg\{ L_{q,(2,L)}^{\rm NS}\left(x, N_F+1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes \left[f_k(x,\mu^2,N_F) + f_{\bar{k}}(x,\mu^2,N_F)\right] \\
& + \frac{1}{N_F}\left[L_{q,(2,L)}^{\rm PS}\left(x, N_F+1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes \Sigma(x,\mu^2,N_F) + L_{g,(2,L)}^{\rm S}\left(x, N_F+1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes G(x,\mu^2,N_F)\right]\Bigg\} \\
& + e_Q^2\left[H_{q,(2,L)}^{\rm PS}\left(x, N_F+1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes \Sigma(x,\mu^2,N_F) + H_{g,(2,L)}^{\rm S}\left(x, N_F+1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes G(x,\mu^2,N_F)\right]
\end{aligned}
$$

- $\otimes$ denotes the Mellin convolution

$$
[A \otimes B](x) = \int_0^1 \int_0^1 dx_1\, dx_2\; \delta(x - x_1 x_2) A(x_1) B(x_2) \; .
$$

- The Mellin convolution is diagonalized by applying a Mellin transformation

$$
\mathbf{M}[f](N) := \int_0^1 dz z^{N-1} f(z) \; .
$$

- The asymptotic representation for the heavy flavor Wilson coefficients for $F_2(x, Q^2)$ becomes effective at $Q^2 \geq 10 \cdot m^2$

DESY

Heavy flavor Wilson Coefficients

- In this limit the massive Wilson coefficients up to $O(a_s^3)$ read

$$
\begin{aligned}
L_{q,(2,L)}^{\mathsf{NS}}(N_F+1) &= a_s^2\Big[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)\;\delta_2+\hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\Big]\\
&+ a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+1)\;\delta_2+A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)+\hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big]\\
L_{q,(2,L)}^{\mathsf{PS}}(N_F+1) &= a_s^3\Big[\;A_{qq,Q}^{(3),\mathsf{PS}}(N_F+1)\;\delta_2+A_{gq,Q}^{(2)}(N_F)\;\;N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F)\Big]\\
L_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s^2A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+a_s^3\Big[\;A_{qg,Q}^{(3)}(N_F+1)\;\delta_2\\
&+A_{gg,Q}^{(1)}(N_F+1)\;\;N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)+A_{gg,Q}^{(2)}(N_F+1)\;\;N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\\
&+\;A_{Qg}^{(1)}(N_F+1)\;\;N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)+N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\;,\\
H_{q,(2,L)}^{\mathsf{PS}}(N_F+1) &= a_s^2\Big[\;A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\;\delta_2+\;\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big]+a_s^3\Big[\;A_{Qq}^{(3),\mathsf{PS}}(N_F+1)\;\delta_2\\
&+\;\tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+1)+A_{gq,Q}^{(2)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\\
&+A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\;C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\Big]\;,\\
H_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s\Big[\;A_{Qg}^{(1)}(N_F+1)\;\delta_2+\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\Big]+a_s^2\Big[\;A_{Qg}^{(2)}(N_F+1)\;\delta_2\\
&+\;A_{Qg}^{(1)}(N_F+1)\;C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)+\;A_{gg,Q}^{(1)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\\
&+\;\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\Big]+\;a_s^3\Big[\;A_{Qg}^{(3)}(N_F+1)\;\delta_2+\;A_{Qg}^{(2)}(N_F+1)\;C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\\
&+\;A_{gg,Q}^{(2)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+\;A_{Qg}^{(1)}(N_F+1)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+1)\\
&+\;\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big\}+\;A_{gg,Q}^{(1)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)+\;\tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\Big]
\end{aligned}
$$

Massive OME's

- Fixed moments $N = 2 \ldots 10\ (12, 14)$ are known [Bierenbaum, Blümlein, Klein, 2009]
- The renormalization prescription has been worked out by [Bierenbaum, Blümlein, Klein, 2009]
- The general structure of the unrenormalized OME's is known
- Example:

$$
\begin{aligned}
\hat{A}_{Qg}^{(3)} = {} & \left(\frac{\hat{m}^2}{\mu^2}\right)^{3\varepsilon/2} \Bigg[\frac{\hat{\gamma}_{qg}^{(0)}}{6\varepsilon^3} \Bigg((N_F+1)\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)} + \gamma_{qq}^{(0)}\Big[\gamma_{qq}^{(0)} - 2\gamma_{gg}^{(0)} - 6\beta_0 - 8\beta_{0,Q}\Big] + 8\beta_0^2 \\
& + 28\beta_{0,Q}\beta_0 + 24\beta_{0,Q}^2 + \gamma_{gg}^{(0)}\Big[\gamma_{gg}^{(0)} + 6\beta_0 + 14\beta_{0,Q}\Big]\Bigg) + \frac{1}{6\varepsilon^2}\Bigg(\hat{\gamma}_{qg}^{(1)}\Big[2\gamma_{qq}^{(0)} - 2\gamma_{gg}^{(0)} \\
& - 8\beta_0 - 10\beta_{0,Q}\Big] + \hat{\gamma}_{qg}^{(0)}\Big[\hat{\gamma}_{qq}^{(1),\mathrm{PS}}\{1-2N_F\} + \gamma_{qq}^{(1),\mathrm{NS}} + \hat{\gamma}_{qq}^{(1),\mathrm{NS}} + 2\hat{\gamma}_{gg}^{(1)} - \gamma_{gg}^{(1)} - 2\beta_1 \\
& - 2\beta_{1,Q}\Big] + 6\delta m_1^{(-1)}\hat{\gamma}_{qg}^{(0)}\Big[\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 3\beta_0 + 5\beta_{0,Q}\Big]\Bigg) + \frac{1}{\varepsilon}\Bigg(\frac{\boxed{\hat{\gamma}_{qg}^{(2)}}}{3} - N_F\frac{\boxed{\hat{\tilde{\gamma}}_{qg}^{(2)}}}{3} \\
& + \hat{\gamma}_{qg}^{(0)}\Big[a_{gg,Q}^{(2)} - N_F a_{Qq}^{(2),\mathrm{PS}}\Big] + a_{Qg}^{(2)}\Big[\gamma_{qq}^{(0)} - \gamma_{gg}^{(0)} - 4\beta_0 - 4\beta_{0,Q}\Big] + \frac{\hat{\gamma}_{qg}^{(0)}\zeta_2}{16}\Big[\gamma_{gg}^{(0)}\Big\{2\gamma_{qq}^{(0)} \\
& - \gamma_{gg}^{(0)} - 6\beta_0 + 2\beta_{0,Q}\Big\} - (N_F+1)\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)} + \gamma_{qq}^{(0)}\Big\{-\gamma_{qq}^{(0)} + 6\beta_0\Big\} - 8\beta_0^2 \\
& + 4\beta_{0,Q}\beta_0 + 24\beta_{0,Q}^2\Big] + \frac{\delta m_1^{(-1)}}{2}\Big[-2\hat{\gamma}_{qg}^{(1)} + 3\delta m_1^{(-1)}\hat{\gamma}_{qg}^{(0)} + 2\delta m_1^{(0)}\hat{\gamma}_{qg}^{(0)}\Big] \\
& + \delta m_1^{(0)}\hat{\gamma}_{qg}^{(0)}\Big[\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 2\beta_0 + 4\beta_{0,Q}\Big] - \delta m_2^{(-1)}\hat{\gamma}_{qg}^{(0)}\Bigg) + \boxed{a_{Qg}^{(3)}}\Bigg]
\end{aligned}
$$

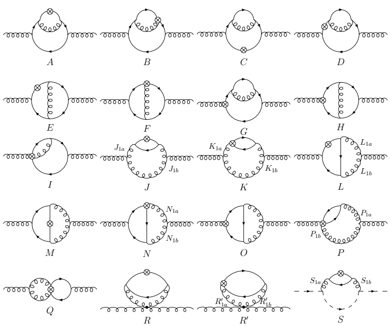
The 3-loop logarithmic contributions

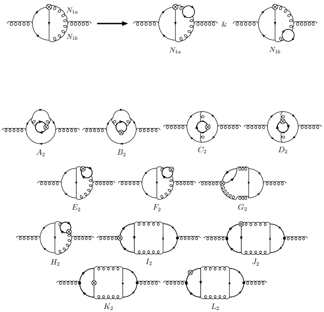
- General structure of renormalized OMEs:

$$A_{ij}^{(3)}\left(\frac{m^2}{Q^2}\right) = a_{ij}^{(3),3}\ln^3\left(\frac{m^2}{Q^2}\right) + a_{ij}^{(3),2}\ln^2\left(\frac{m^2}{Q^2}\right) + a_{ij}^{(3),1}\ln\left(\frac{m^2}{Q^2}\right) + a_{ij}^{(3),0} \quad (1)$$

- all logarithmic contributions are known [Ablinger,Bierenbaum, Blümlein, Klein, Wißbrock 2011] (explicit N- and $x-$space representation & analytic continuation $N \in \mathbb{C}$)
- but: in the relevant kinematic region there is no logarithmic dominance
- terms $a_{ij}^{(3),0}$ are needed to describe the correct behaviour of the structure functions

$O\left(\alpha_s^3 T_F^2 N_F C_{A,F}\right)$: Contributing diagrams





- 289 Diagrams $\propto N_F T_F^2$ contribute
- 167 Diagrams $\propto T_F^2$ contribute
- due to symmetry some diagrams are identical

Evaluation of Feynman integrals

- Typical Feynman parameter integral after momentum integration

$$I_1 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 dx_1 dx_2 dx_4 dx_5 x_1^{2+\varepsilon} x_2^{1-\varepsilon/2} x_5^{1-\varepsilon} (1-x_1)^{\varepsilon/2} (1-x_5)^2 (x_4 - x_5 x_4 + x_2 x_5)^N \times \left(1 - x_5\left(1 - \frac{1}{1-x_1}\right)\right)^{3/2\varepsilon}$$

- Performing the integral yields a linear combination of sums over B-functions and Hypergeometric ${}_PF_Q$s

$$I_1 = \frac{\Gamma(1-\varepsilon)\,\Gamma(3+\varepsilon)}{6(N+1)} \left\{ \sum_{j=1}^{N+1} \binom{1+N}{j} (-1)^j\, B(2-\varepsilon+j,2)\, B(1+j,2-\varepsilon/2)\, {}_3F_2\left[{-3/2\,\varepsilon, 2, 3+\varepsilon \atop 4+j-\varepsilon, 4}, 1 \right] + B(3+N-\varepsilon,2)\, B(1,3+N-\varepsilon/2)\, {}_3F_2\left[{-3/2\varepsilon, 2, 3+\varepsilon \atop 5-\varepsilon, 4}; 1 \right] \right\}$$

- where the ${}_PF_Q$ is defined by

$${}_PF_Q\left[{a_1, \ldots, a_P \atop b_1, \ldots, b_Q}; z \right] = \sum_{i=0}^{\infty} \frac{(a_1)_i \ldots (a_P)_i}{(b_1)_i \ldots (b_Q)_i} \frac{z^i}{\Gamma(i+1)} .$$

Mathematical structures

- Now: perform a series expansion in ε and evaluate the remaining sums
- Up to 4 (in)finite sums occur, which are computed using modern summation methods encoded in SIGMA [C. Schneider, 2007]
- results are given in terms of ζ_2, ζ_3 and harmonic Sums $S_{\vec{a}}(N)$

$$S_{a_1,\dots,a_m}(N) = \sum_{n_1=1}^{N} \sum_{n_2=1}^{n_1} \dots \sum_{n_m=1}^{n_{m-1}} \frac{(\mathrm{sign}(a_1))^{n_1}}{n_1^{|a_1|}} \frac{(\mathrm{sign}(a_2))^{n_2}}{n_2^{|a_2|}} \dots \frac{(\mathrm{sign}(a_m))^{n_m}}{n_m^{|a_m|}}$$

- in intermediary steps also generalized harmonic Sums occur

$$\widetilde{S}_{m_1,\dots}(x_1,\dots;N) = \sum_{i_1}^{N} \frac{x_1^{i_1}}{i_1^{m_1}} \sum_{i_2=1}^{i_1-1} \frac{x_2^{i_2}}{i_2^{m_2}} \widetilde{S}_{m_3,\dots}(x_3,\dots;i_2) + \widetilde{S}_{m_1+m_2,m_3,\dots}(x_1 \cdot x_2, x_3,\dots;N)$$

[Moch, Uwer, Weinzierl, 2002]

- algebraic and structural relations for these sums have been worked out [Ablinger, Blümlein, Schneider, 2011]

Results for the contributions $\propto N_F T_F^2 C_{F,A}$

$$
\begin{aligned}
\hat{a}_{Qg}^{(3),0} = \; & N_F T_F^2 C_A \Biggl\{ \frac{16(N^2+N+2)}{27N(N+1)(N+2)} \Bigl[108 S_{-2,1,1} - 78 S_{2,1,1} - 90 S_{-3,1} + 72 S_{2,-2} - 6 S_{3,1} \\
& -108 S_{-2,1} S_1 + 42 S_{2,1} S_1 - 6 S_{-4} + 90 S_{-3} S_1 + 118 S_3 S_1 + 120 S_4 + 18 S_{-2} S_2 + 54 S_{-2} S_1^2 \\
& +33 S_2 S_1^2 + 15 S_2^2 + 2 S_1^4 + 18 S_{-2}\zeta_2 + 9 S_2 \zeta_2 + 9 S_1^2 \zeta_2 - 42 S_1 \zeta_3 \Bigr] \\
& +32 \frac{5N^4+14N^3+53N^2+82N+20}{27N(N+1)^2(N+2)^2} \Bigl[6 S_{-2,1} - 5 S_{-3} - 6 S_{-2} S_1 \Bigr] \\
& -\frac{64(5N^4+11N^3+50N^2+85N+20)}{27N(N+1)^2(N+2)^2} S_{2,1} - \frac{16(40N^4+151N^3+544N^2+779N+214)}{27N(N+1)^2(N+2)^2} S_2 S_1 \\
& -\frac{32(65N^6+429N^5+1155N^4+725N^3+370N^2+496N+648)}{81(N-1)N^2(N+1)^2(N+2)^2} S_3 \\
& -\frac{16(20N^4+107N^3+344N^2+439N+134)}{81N(N+1)^2(N+2)^2} S_1^3 + \frac{Q_1(N)}{81(N-1)N^3(N+1)^3(N+2)^3} S_2 \\
& +\frac{32(47N^6+278N^5+1257N^4+2552N^3+1794N^2+284N+448)}{81N(N+1)^3(N+2)^3} S_{-2} \\
& +\frac{8(22N^6+271N^5+2355N^4+6430N^3+6816N^2+3172N+1256)}{81N(N+1)^3(N+2)^3} S_1^2 \\
& +\frac{Q_2(N)}{243(N-1)N^2(N+1)^4(N+2)^4} S_1 + \frac{448(N^2+N+1)(N^2+N+2)}{9(N-1)N^2(N+1)^2(N+2)^2} \zeta_3 \\
& -\frac{16(5N^4+20N^3+59N^2+76N+20)}{9N(N+1)^2(N+2)^2} S_1 \zeta_2 - \frac{Q_3(N)}{9(N-1)N^3(N+1)^3(N+2)^3} \zeta_2 \\
& -\frac{Q_4(N)}{243(N-1)N^5(N+1)^5(N+2)^5} \Biggr\}
\end{aligned}
$$

$$
\begin{aligned}
&+N_F T_F^2 C_F \Bigg\{ \frac{16(N^2+N+2)}{27N(N+1)(N+2)} \Big[144 S_{2,1,1} - 72 S_{3,1} - 72 S_{2,1}S_1 + 48 S_4 - 16 S_3 S_1 \\
&-24 S_2^2 - 12 S_2 S_1^2 - 2 S_1^4 - 9 S_1^2 \zeta_2 + 42 S_1 \zeta_3\Big] + 32 \frac{10N^3+49N^2+83N+24}{81N^2(N+1)(N+2)} \Big[3 S_2 S_1 + S_1^3\Big] \\
&-\frac{128(N^2-3N-2)}{3N^2(N+1)(N+2)} S_{2,1} - \frac{Q_5(N)}{81(N-1)N^3(N+1)^3(N+2)^2} S_3 \\
&+\frac{Q_6(N)}{27(N-1)N^4(N+1)^4(N+2)^3} S_2 - \frac{32(10N^4+185N^3+789N^2+521N+141)}{81N^2(N+1)^2(N+2)} S_1^2 \\
&-\frac{16(230N^5-924N^4-5165N^3-7454N^2-10217N-2670)}{243N^2(N+1)^3(N+2)} S_1 \\
&+\frac{16(5N^3+11N^2+28N+12)}{9N^2(N+1)(N+2)} S_1 \zeta_2 - \frac{Q_7(N)}{9(N-1)N^3(N+1)^3(N+2)^2} \zeta_3 \\
&+\frac{Q_8(N)}{9(N-1)N^4(N+1)^4(N+2)^3} \zeta_2 + \frac{Q_9(N)}{243(N-1)N^6(N+1)^6(N+2)^5} \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{qg}^{(2)} \;=\; & \frac{N_F^2 T_F^2}{(N+1)(N+2)}\Bigg\{C_A\Bigg[(N^2+N+2)\left(\frac{128}{3N}S_{2,1}+\frac{128}{3N}S_{-3}+\frac{64}{9N}S_3+\frac{32}{9N}S_1^3\right.\\
& \left.-\frac{32}{3N}S_2S_1\right)-\frac{128(5N^2+8N+10)}{9N}S_{-2}-\frac{64(5N^4+26N^3+47N^2+43N+20)}{9N(N+1)(N+2)}S_2\\
& -\frac{64(5N^4+20N^3+41N^2+49N+20)}{9N(N+1)(N+2)}S_1^2+\frac{64P_1(N)}{27N(N+1)^2(N+2)^2}S_1\\
& +\frac{16P_2(N)}{27(N-1)N^4(N+1)^3(N+2)^3}\Bigg]\\
& +C_F\Bigg[\frac{32}{9}\frac{N^2+N+2}{N}\{10S_3-S_1^3-3S_1S_2\}\\
& +\frac{32(5N^2+3N+2)}{3N^2}S_2+\frac{32(10N^3+13N^2+29N+6)}{9N^2}S_1^2\\
& -\frac{32(47N^4+145N^3+426N^2+412N+120)}{27N^2(N+1)}S_1+\frac{4P_3(N)}{27(N-1)N^5(N+1)^4(N+2)^3}\Bigg]\Bigg\}
\end{aligned}
$$

in agreement with [Moch, Vermaseren, Vogt 2004]

- furthermore the $N_F T_F^2 C_F$ and $N_F T_F^2 C_A$-contributions to the OMEs $A_{qq,Q}^{NS,(3)}$, $A_{Qq}^{PS,(3)}$, $A_{qg,Q}$ (complete), $A_{qq,Q}^{PS,(3)}$ (complete) and $A_{qq,Q}^{NS,Trans.,(3)}$ have been computed
- this holds also for contributions to the 3-loop anamolous dimensions γ_{qq}^{PS}, γ_{qq}^{NS} and $\gamma_{qq}^{NS,Trans}$

NNLO contributions $\propto T_F^2 C_{A,F}$, $m_1 = m_2$

- Feynman diagrams with two massive fermion lines contribute, two cases: $m_1 = m_2$ and $m_1 \neq m_2$
- Feynman integrals cannot be mapped directly onto higher functions.
- An analytic Mellin-Barnes representation is introduced.
- It yields representations in terms of Meijer G-function which can be expanded into generalized hypergeometric functions.

$$G_{p,q}^{m,n}\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle| x\right) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} d\sigma \times\Gamma\left[\begin{matrix} b_1+\sigma, \ldots, b_m+\sigma, 1-a_1-\sigma, \ldots, 1-a_n-\sigma \\ a_{n+1}, \ldots, a_p+\sigma, 1-b_{m+1}-\sigma, \ldots, 1-b_q-\sigma \end{matrix}\right] z^{-\sigma} .$$

NNLO contributions $\propto T_F^2 C_{A,F}$, $m_1 = m_2$

For the flavor pure-singlet contribution this yields

$$
\begin{aligned}
\hat{a}_{Qq}^{(3),PS} &= \frac{T_F^2 C_F}{(N-1)N^2(N+1)^2(2+N)}\Biggl\{(N^2+N+2)^2\Bigl(\frac{32}{27}S_1^3-\frac{512}{27}S_3+\frac{128}{3}S_{2,1}-\frac{1024}{9}\zeta_3-\frac{160}{9}S_2S_1 \\
&\quad +\frac{32}{3}\zeta_2 S_1\Bigr)-\frac{32P_1(N)}{9N(N+2)}\zeta_2+\frac{32P_2(N)}{27N(N+2)(N+3)(N+4)(N+5)}S_2-\frac{32P_3(N)}{27N(N+1)(N+2)(N+3)(N+4)(N+5)}S_1^2 \\
&\quad +\frac{64P_4(N)}{81N^2(N+1)^2(N+2)^2(N+3)(N+4)(N+5)}S_1-\frac{64P_5(N)}{243N^3(N+1)^2(N+2)^3(N+3)(N+4)(N+5)}\Biggr\}. \\
\hat{\gamma}_{qq}^{(3),PS} &= \frac{T_F^2 C_F}{(N-1)N^2(N+1)^2(2+N)}\Biggl\{-\frac{32}{3}(N^2+N+2)^2(S_1^2+S_2)+\frac{64P_6(N)}{9N(N+1)(N+2)}S_1-\frac{64P_7(N)}{27N^2(N+1)^2(N+2)^2}\Biggr\}, \\
&\text{with} \\
P_6(N) &= 8N^7+37N^6+68N^5-11N^4-86N^3-56N^2-104N-48\,, \\
P_7(N) &= 52N^{10}+392N^9+1200N^8+1353N^7-317N^6-1689N^5-2103N^4 \\
&\quad -2672N^3-1496N^2-48N+144\,.
\end{aligned}
$$

in agreement with [Moch, Vermaseren, Vogt 2004]

- Similiar results have been obtained for the NS & transversity OMEs.
- The computation of the T_F^2-contribution to the OME A_{Qg} is in progress.

$m_1 \neq m_2$

- At 3–loop order graphs containing both c- and b- quarks contribute.
- They do neither belong to the pure $c-$ or $b-$ contribution to the structure function.
- Note that:

$$\frac{m_c}{m_b} \simeq \frac{1.3 GeV}{4.2 GeV} \longrightarrow x^3 := \left(\frac{m_c}{m_b}\right)^6 \simeq 0.0001$$

$\rightarrow$ Expand in m_c/m_b

- for fixed values of N the diagrams can be mapped onto tadpole diagrams by projection operators [Bierenbaum, Blümlein, Klein 2009.]
- e.g. $N = 2$

$$\Pi_{\mu\nu} = \frac{1}{d-1}\left(\frac{-g_{\mu\nu}}{p^2} + d\frac{p(\mu)p(\nu)}{p^4}\right)$$

- more complex structures occur for higher Moments
- expansion in masses was performed using EXP [Harlander, Seidensticker, Steinhauser 1998, Seidensticker 1999]
- 120 out of 256 diagrams have been computed for general values of N

A_{Qg}, $N = 2$, $m_1 \neq m_2$

$$
\begin{aligned}
a_{Qg}^{(3)} = \; & T_F^2 C_A \Bigg\{ \frac{156458}{2187} - \frac{1696}{81}\zeta_3 - \frac{148}{81}\zeta_2 + \frac{512608}{10125}x + \frac{3130072}{496125}x^2 + \frac{112173472}{843908625}x^3 \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\left[-\frac{10}{3} + \frac{280}{9}\zeta_2 - \frac{14368}{675}x - \frac{12016}{4725}x^2 + \frac{328928}{2679075}x^3\right] \\
& + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\left[-\frac{70}{81} - \frac{16}{45}x - \frac{16}{45}x^2 - \frac{5104}{8505}x^3\right] + \ln^3\left(\frac{m_2^2}{\mu^2}\right)\frac{1192}{81} + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\frac{560}{27} \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\left[-\frac{304}{81} + \frac{32}{45}x + \frac{32}{45}x^2 + \frac{10208}{8505}x^3\right] + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln^2\left(\frac{m_1^2}{\mu^2}\right)\frac{208}{27} \\
& + \ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{18710}{243} + \frac{280}{9}\zeta_2 + \frac{14368}{675}x + \frac{12016}{4725}x^2 - \frac{328928}{2679075}x^3\right] \\
& + \ln^2\left(\frac{m_1^2}{\mu^2}\right)\left[-\frac{70}{81} - \frac{16}{45}x - \frac{16}{45}x^2 - \frac{5104}{8505}x^3\right] + \ln^3\left(\frac{m_1^2}{\mu^2}\right)\frac{1544}{81} \Bigg\} \\
& + T_F^2 C_F \Bigg\{ + \frac{128}{243} + \frac{3584}{81}\zeta_3 + \frac{640}{27}\zeta_2 - \frac{1517888}{30375}x + \frac{339785728}{10418625}x^2 + \frac{1653611968}{843908625}x^3 \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{13504}{243} - \frac{128}{9}\zeta_2 - \frac{45952}{2025}x - \frac{2056384}{99225}x^2 - \frac{11786368}{2679075}x^3\right] \\
& + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{1936}{81} + \frac{896}{135}x + \frac{9536}{945}x^2 + \frac{47744}{8505}x^3\right] - \ln^3\left(\frac{m_2^2}{\mu^2}\right)\frac{896}{81} - \ln^2\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\frac{256}{27} \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{1888}{81} - \frac{1792}{135}x - \frac{19072}{945}x^2 - \frac{95488}{8505}x^3\right] + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln^2\left(\frac{m_1^2}{\mu^2}\right)\frac{256}{27} \\
& + \ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{256}{81} - \frac{128}{9}\zeta_2 + \frac{45952}{2025}x + \frac{2056384}{99225}x^2 + \frac{11786368}{2679075}x^3\right] \\
& + \ln^2\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{1936}{81} + \frac{896}{135}x + \frac{9536}{945}x^2 + \frac{47744}{8505}x^3\right] - \ln^3\left(\frac{m_1^2}{\mu^2}\right)\frac{1408}{81} \Bigg\} + O\left(x^4 \ln(x)^3\right)
\end{aligned}
$$

$A_{Qg}, \; N=4, \; m_1 \neq m_2$

$$
\begin{aligned}
a_{Qg}^{(3)} = \; & T_F^2 C_A \Bigg\{ \frac{4887988511}{24300000} - \frac{47146}{2025}\zeta_3 + \frac{5807}{180}\zeta_2 + \frac{496855133}{7441875}x + \frac{2510388298}{468838125}x^2 + \frac{250077164867}{5616211899375}x^3 \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{47956573}{810000} + \frac{17963}{450}\zeta_2 - \frac{1877399}{70875}x - \frac{3269548}{1488375}x^2 + \frac{156082853}{1620840375}x^3\right] \\
& + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{532373}{16200} + \frac{707}{1350}x - \frac{284}{525}x^2 - \frac{744283}{935550}x^3\right] + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\frac{74657}{4050} + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\frac{17963}{675} \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{62893}{2025} - \frac{707}{675}x + \frac{568}{525}x^2 + \frac{744283}{467775}x^3\right] + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln^2\left(\frac{m_1^2}{\mu^2}\right)\frac{7579}{675} \\
& + \ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{384762007}{2430000} + \frac{17963}{450}\zeta_2 + \frac{1877399}{70875}x + \frac{3269548}{1488375}x^2 - \frac{156082853}{1620840375}x^3\right] \\
& + \ln^2\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{532373}{16200} + \frac{707}{1350}x - \frac{284}{525}x^2 - \frac{744283}{935550}x^3\right] + \ln^3\left(\frac{m_1^2}{\mu^2}\right)\frac{3817}{162}\Bigg\} \\
& + T_F^2 C_F \Bigg\{\left[-\frac{33406758667}{1093500000} + \frac{260414}{10125}\zeta_3 + \frac{1473641}{202500}\zeta_2 - \frac{119314474}{4134375}x + \frac{582667691}{37507050}x^2 + \frac{46049137562}{44929695195}x^3\right] \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{76621423}{4050000} - \frac{18601}{2250}\zeta_2 - \frac{368428}{39375}x - \frac{2876423}{297675}x^2 - \frac{570093292}{324168075}x^3\right] \\
& + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{530371}{81000} + \frac{4456}{1125}x + \frac{27101}{4725}x^2 + \frac{1759616}{467775}x^3\right] - \ln^3\left(\frac{m_2^2}{\mu^2}\right)\frac{130207}{20250} - \ln^2\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\frac{18601}{3375} \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{442267}{50625} - \frac{8912}{1125}x - \frac{54202}{4725}x^2 - \frac{3519232}{467775}x^3\right] + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln^2\left(\frac{m_1^2}{\mu^2}\right)\frac{18601}{3375} \\
& + \ln\left(\frac{m_1^2}{\mu^2}\right)\left[-\frac{37307959}{2430000} - \frac{18601}{2250}\zeta_2 + \frac{368428}{39375}x + \frac{2876423}{297675}x^2 + \frac{570093292}{324168075}x^3\right] \\
& + \ln^2\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{530371}{81000} + \frac{4456}{1125}x + \frac{27101}{4725}x^2 + \frac{1759616}{467775}x^3\right] - \ln^3\left(\frac{m_1^2}{\mu^2}\right)\frac{204611}{20250}\Bigg\} + O\left(x^4 \ln^3(x)\right)
\end{aligned}
$$

Conclusion

- We computed the $O\left(\alpha_s^3 N_F T_F^2 C_{A,F}\right)$ contributions to all the OMEs A_{ij} which contribute to the nucleonic structure function $F_2(x, Q^2)$ and transversity for general values of the Mellin variable N.
- All logarithmic contributions $O(\alpha_s^3 \ln^k(Q^2/m^2))$, $k = 1, 2, 3$ have been calculated.
- These calculations constitute first complete expressions for two color factors to the heavy flavor Wilson Coefficients for $F_2(x, Q^2)$ at $O(a_s^3)$. The Wilson Coefficients $L_{qq,Q}^{\rm PS}$ and $L_{qg,Q}^{\rm S}$ are known completely now.
- Along with the computation of the massive OMEs we obtained the corresponding parts of the 3-loop anomalous dimensions and confirmed results given in the literature analytically, partly for the first time.
- Results have been obtained for the $O(\alpha_s^3 T_F^2 C_{A,F})$ terms of the NS and PS OMEs resulting from the graphs with two massive lines with equal and non-equal masses.
- For the OME A_{Qg} fixed moments have been generated for the case of two non-equal masses. Many diagrams have already been computed for general values of N.