Towards the evolution of the Parton Distribution Functions to percent accuracy

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Synergies between the EIC and the LHC
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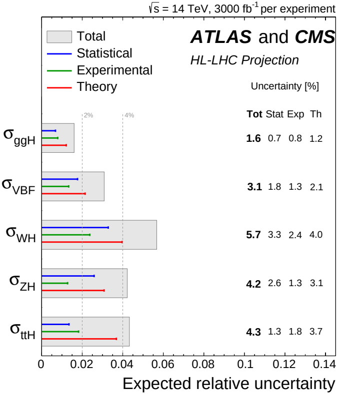
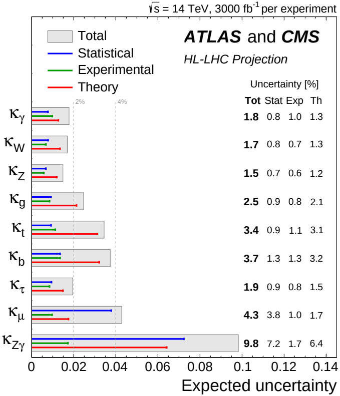
Based on

- *Renormalization of gluonic leading-twist operators in covariant gauges*, with **F. Herzog**, JHEP 05 (2022) 177
- *Four-loop splitting functions in QCD – The quark-quark case*, with **F. Herzog, S. Moch, A. Vogt**, PLB 842 (2023) 137944
- *Four-loop splitting functions in QCD – The gluon-to-quark case*, with **F. Herzog, S. Moch, A. Vogt**, PLB 846 (2023) 138215
- *The double fermionic contribution to the four-loop quark-to-gluon splitting function*, with **F. Herzog, S. Moch, J. Vermaseren, A. Vogt**, PLB 848 (2024) 138351

LHC physics to 1% accuracy

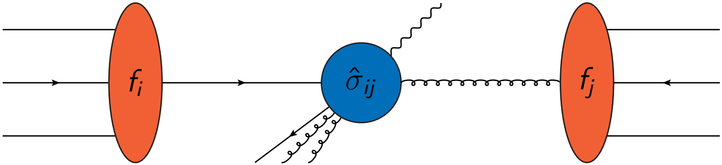
HL-LHC: theory dominates the error budget.





Vol. 7/2019 of CERN Yellow Reports: Monograph.

Theory framework



- $\hat{\sigma}_{ij}(s\, x_i x_j, \mu^2)$: perturbative cross section.
- $f_i(x_i, \mu^2)$: non-perturbative Parton Distribution Functions.
- μ: **factorisation scale** dependence controlled by the **DGLAP equation** (Gribov,Lipatov 1972; Lipatov 1975; Altarelli,Parisi 1977; Dokshitzer 1977).

How far shall we go in perturbation theory?

	Q [GeV]	$\delta\sigma^{\mathrm{N^3LO}}$	δ(scale)	δ(PDF-TH)
NCDY	100	-2.1%	$^{+0.66\%}_{-0.79\%}$	$\pm$2.5%
CCDY(W^+)	150	-2.0%	$^{+0.5\%}_{-0.5\%}$	$\pm$2.1%
CCDY(W^-)	150	-2.1%	$^{+0.6\%}_{-0.5\%}$	$\pm$2.13%

J. Baglio, C. Duhr, B. Mistlberger, R. Szafron 2022

- N^{3}LO corrections $O(\%)$: $\delta\sigma^{\mathrm{N^3LO}} = \frac{\sigma^{\mathrm{N^3LO}} - \sigma^{\mathrm{NNLO}}}{\sigma^{\mathrm{NNLO}}}$.
- δ(scale): renorm./factoris. scale uncertainty of the perturbative cross section.
- δ(PDF-TH): Additional error due missing N^{3}LO PDFs.

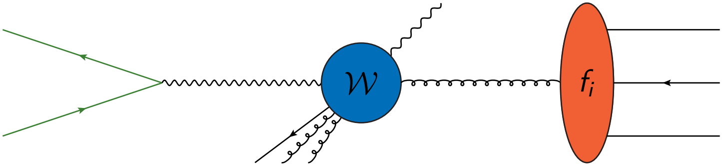
Ingredients for N^{3}LO phenomenology

- 3-loop partonic cross sections
  - Impressive progress in recent years (see e.g. Anastasiou et al. 2016; Mistlberger 2018; Duhr,Dulat,Mistlberger 2020; Baglio,Duhr,Mistlberger,Szafron 2022)
- **4-loop DGLAP kernels**.

$$\mu^2 \frac{d}{d\mu^2} f_i\left(x,\mu^2\right) = \int_x^1 \frac{dy}{y} P_{ij}(\alpha_s, y)\, f_j\left(\frac{x}{y},\mu^2\right),$$

$$P_{ij}(\alpha_s, x) = \underbrace{a\, P_{ij}^{(0)}}_{\text{LO}} + \underbrace{a^2\, P_{ij}^{(1)}}_{\text{NLO}} + \underbrace{a^3\, P_{ij}^{(2)}}_{\text{NNLO}} + \underbrace{a^4\, P_{ij}^{(3)}}_{\text{N}^3\text{LO}},\, a = \frac{\alpha_s}{4\pi}$$

Interplay with the EIC



- Wilson coefficients $\mathcal{W}$ to N^{3}LO (Moch,Vermaseren,Vogt 2005, 2009; Blümlein,Marquard,Schneider,Schönwald 2022).
- Progress in 3-loop **heavy flavour** contributions (Ablinger et al. 2022), see talk by **J. Blümlein**.
- EIC data expected to improve large-x PDFs (Armesto et al. 2023), see talks by **E. Nocera** and **N. Armesto**.

Towards the N^{3}LO DGLAP kernels

- Large-n_f limit (Gracey 1994, 1996; Davies,Vogt,Ruijl,Ueda,Vermaseren 2016)
- Flavour non-singlet quark combination

$$f^{\pm}_{\mathrm{NS},ik} = (f_i \pm f_{\bar{i}}) - (f_k \pm f_{\bar{k}}), \qquad i,k = u,d,s,\ldots$$

  Complete planar limit (Ruijl,Ueda,Vermaseren,Vogt 2017)
- n_f^2 term in $P^{(3)}_{qq}$ (Gehrmann,von Manteuffel,Sotnikov,Yang 2023)
- n_f^2 term in $P^{(3)}_{gq}$ (Falcioni,Herzog,Moch,Vermaseren,Vogt 2023)
- Flavour non-singlet: $n_f C_F^3$ term (Gehrmann,von Manteuffel,Sotnikov,Yang 2023)

Approximate results

- The **moments** of the **DGLAP kernerls** govern the evolution of the **moments** of the **PDFs**.

$$\int_0^1 dx\, x^{N-1}\, P_{ij}^{(N)}(x, \alpha_s) = -\gamma_{ij}^{(N)}.$$

- Moments up to $N = 16$ (**non-singlet PDFs**), $N = 12$ (γ_{qq}) and $N = 10$ (Moch,Ruijl,Ueda,Vermaseren,Vogt 2017, 2021, 2023)
- Approximate evolution from a **fixed** number of moments (Moch,Ruijl,Ueda,Vermaseren,Vogt **non-singlet 2017**; McGowan,Cridge,Harland Lang,Thorne 2022; Hekhorn,Magni 2023) See also talks by **T. Cridge** and **G. Magni**

This talk

- An efficient approach to compute **moments**.
- Evolution of the **quark (singlet)** density q_s:
  - Moments of the quark-to-quark splitting kernel P_{qq}.
  - Moments of the gluon-to-quark splitting kernel P_{qg}.
- Approximate evolution of q_S for $x \gtrsim 10^{-5}$ to % accuracy.
- Outlook

The OPE method

- Scale dependence in DIS controlled by anomlaous dimensions of local operators (Gross,Wilczek 1974; Politzer, Georgi 1974)

$$\mu^2 \frac{d}{d\mu^2} \mathcal{O}_i^{(N)} = -\gamma_{ij}^{(N)} \mathcal{O}_j^{(N)}$$

  - ...but $\mathcal{O}_j^{(N)}$ can **mix** with **unphysical operators**.
- Unphysical terms determined to **two loops** (Dixon,Taylor 1974) but overlooked.
- Two-loop calculation settled after 20 years (Hamberg,van Neerven 1993)!

Operator structure to four loops

- Systematic construction of **all** the operators for each N (Falcioni,Herzog 2022)

$$\widetilde{\mathcal{L}} = \underbrace{\mathcal{L}_0 + c_g\,\mathcal{O}_g^{(N)} + \mathcal{O}_{\mathrm{EOM}}^{(N)}}_{\mathcal{L}_{\mathrm{GGI}}} + \underbrace{\mathbf{s'}\left[\bar{c}^a\left(\partial^\mu A_\mu^a - \frac{\xi_L}{2} b^a\right)\right]}_{\text{Gauge fixing + ghost}}$$

$$\mathcal{O}_{\mathrm{EOM}}^{(N)} = (D^\mu F_\mu)^a \Big[\underbrace{\eta\,\partial^{N-2} A^a}_{\mathcal{O}_g^{I}} + g f^{a a_1 a_2} \sum_{i_1+i_2=N-3} \underbrace{\kappa_{i_1 i_2} (\partial^{i_1} A^{a_1})(\partial^{i_2} A^{a_2})}_{\mathcal{O}_g^{II}}$$

$$+ g^2 \sum_{\substack{i_1+i_2+i_3\\ N-4}} \underbrace{\left(\kappa^{(1)}_{i_1 i_2 i_3} f^{a a_1 z} f^{a_2 a_3 z} + \kappa^{(2)}_{i_1 i_2 i_3} d_4^{a a_1 a_2 a_3} + \kappa^{(3)}_{i_1 i_2 i_3} d_{\widehat{4ff}}^{a a_1 a_2 a_3}\right)(\partial^{i_1} A^{a_1})..(\partial^{i_3} A^{a_3})}_{\mathcal{O}_g^{III}} + \ldots$$

- Agreement with general theory (Joglekar,Lee 1974)
- Agreement with explicit results at **three loops** (Gehrmann,von Manteuffel,Yang 2023).

Moments of the quark-to-quark kernel

- Only $\mathcal{O}^{I}$ and $\mathcal{O}^{II}$ mix with **pure singlet** $\gamma_{\mathrm{ps}}(N)$

$$\gamma_{qq}(N) = \gamma_{\mathrm{ps}}(N) + \gamma_{\mathrm{ns}}^{+}(N)$$

- Results for the first ten moments ($N = 20$) (Falcioni,Herzog,Moch,Vogt 2023)

$$\begin{aligned}\gamma_{\mathrm{ps}}^{(3)}(N=2) &= -691.5937093\, n_f + 84.77398149\, n_f^2 + 4.466956849\, n_f^3\,, \\ &\dots \\ \gamma_{\mathrm{ps}}^{(3)}(N=20) &= -0.442681568\, n_f + 0.805745333\, n_f^2 - 0.020918264\, n_f^3\,.\end{aligned}$$

- Checks with moments up to $N = 12$ (Moch,Ruijl,Ueda,Vermaseren,Vogt 2023), large n_f limit (Davies,Moch,Ruijl,Ueda,Vermaseren,Vogt 2016), complete n_f^2 colour factor (Gehrmann,von Manteuffel,Sotnikov,Yang 2023).

Moments of the gluon-to-quark kernel

The same approach applied to $\gamma_{qg}(N)$

- Results for ten moments (Falcioni,Herzog,Moch,Vogt 2023)

$$
\begin{aligned}
\gamma_{\mathrm{qg}}^{(3)}(N\!=\!2) &= -654.4627782\, n_f + 245.6106197\, n_f^{\,2} - 0.924990969\, n_f^{\,3}\,, \\
\gamma_{\mathrm{qg}}^{(3)}(N\!=\!4) &= 290.3110686\, n_f - 76.51672403\, n_f^{\,2} - 4.911625629\, n_f^{\,3}\,, \\
&\;\;\ldots \\
\gamma_{\mathrm{qg}}^{(3)}(N\!=\!20) &= 52.24329555\, n_f - 109.3424891\, n_f^{\,2} - 2.153153725\, n_f^{\,3}\,.
\end{aligned}
$$

- Agreement with moments up to $N = 10$ (Moch,Ruijl,Ueda,Vermaseren,Vogt 2021,2023)
- Agreement with the large-n_f limit (Davies,Vogt,Ruijl,Ueda,Vermaseren 2016)

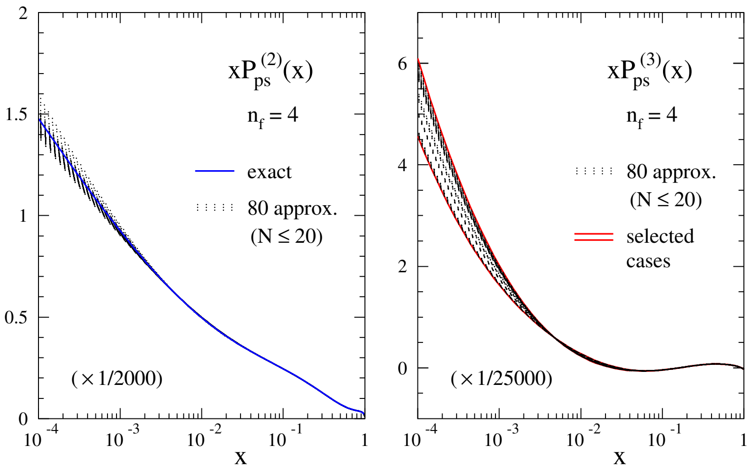
Approximations of $P_{qq}^{(3)}(x)$ (I)

Construction of 80 function matching $\gamma_{qq}(2)\ldots\gamma_{qq}(20)$ and known end-point behaviour (Falcioni,Herzog,Moch,Vogt 2023)

- Small-x limits
  - Coefficients of $\frac{\log^2 x}{x}$ (Catani,Hautmann 1994)
  - Coefficients of $\log^k x$ with $k = 6, 5, 4$ (Davies,Kom,Moch,Vogt 2022)

- Large-x limits
  - Coefficients of $(1-x)^j \log^k(1-x)$ with $k = 4, 3$ and $\forall j \geq 1$ (Soar,Moch,Vermaseren,Vogt 2010)

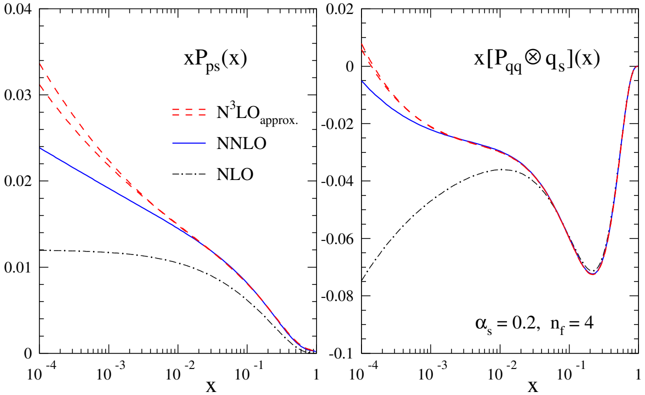
Approximations of $P_{qq}^{(3)}(x)$ (II)



Physical contribution to $q_S(x, \mu^2)$ evolution

$P_{qq}(x)$ to N^3LO using $\alpha_s = 0.2$ (left). $P_{qq} \otimes q_S$ (right)

$$x\, q_S(x) = 0.6\, x^{-0.3}\, (\mathbf{1} - \mathbf{x})^{\mathbf{3.5}}\, (1 + 5.0 x^{0.8})$$



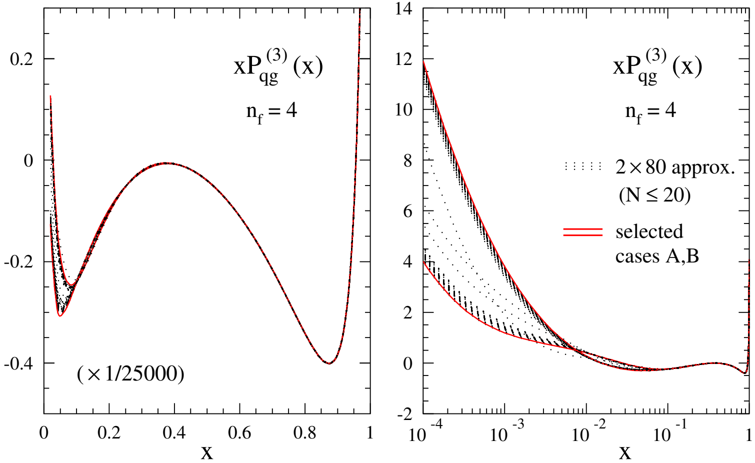
Approximations of $P_{qg}^{(3)}$ (I)

The trial functions for $P_{qg}^{(3)}$ are constrained by the limits at

- Small-x:
  - Coefficients of $\frac{\log^2 x}{x}$ (Catani,Hautmann 1994)
  - Coefficients of $\log^k x$ with $k = 6, 5, 4$ (Davies,Kom,Moch,Vogt 2022)
- Large-x:
  - Coefficients of $\log^k(1-x)$ with $k = 6, 5, 4$ (Soar,Moch,Vermaseren,Vogt 2010; Vogt 2010; Almasy,Soar,Vogt 2011)
  - Coefficients of $(1-x)\log^k(1-x)$ with $k = 6, 5, 4$ (Soar,Moch,Vermaseren,Vogt 2010)

The coefficients of $\log^k(1-x)$ with $k = 1, 2, 3$ are **unknown**

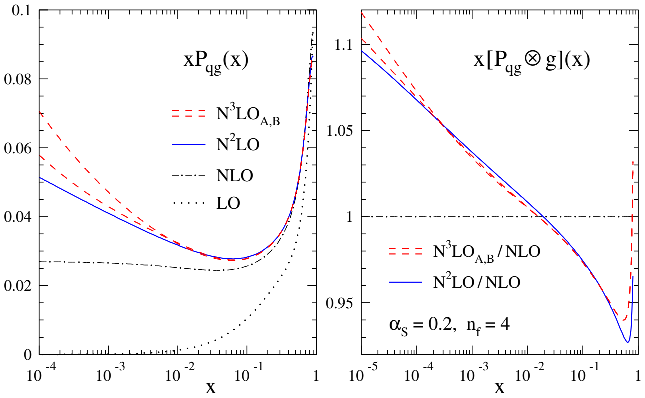
Approximations of $P_{qg}^{(3)}$ (II)



Physical contribution to $q_S(x, \mu^2)$ evolution

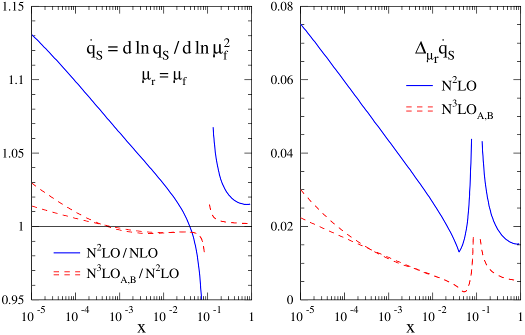
$P_{qg}(x)$ to N^{3}LO using $\alpha_s = 0.2$ (left). $P_{qg} \otimes g$ (right)

$$x\, g(x) = 1.6\, x^{-0.3}\, (\mathbf{1 - x})^{\mathbf{4.5}}\, (1 - 0.6 x^{0.3})$$



Scale evolution of the quark distribution

Physical evolution of the quark density $\dot{q}_S = P_{qq} \otimes q_S + P_{qg} \otimes g$



Stability under variations of the renormalisation scale

$$\Delta_{\mu_r} \dot{q}_S = \frac{1}{2} \frac{\max[\dot{q}_S(\mu_r^2 = \lambda \mu_f^2)] - \min[\dot{q}_S(\mu_r^2 = \lambda \mu_f^2)]}{\text{average}[\dot{q}_S(\mu_r^2 = \lambda \mu_f^2)]}, \qquad \lambda = \frac{1}{4} \ldots 4$$

Getting ready for %-level accuracy at LHC and EIC

- The OPE approach allows us to compute efficiently moments up to $N = 20$ of $P_{qq}(x)$ and $P_{qg}(x)$.
- We constructed approximate N^{3}LO kernels quantifying uncertainties
  - Negligible uncertainties for $P_{qq} \otimes q_S$.
  - Uncertainty of $O(1\%)$ for $P_{qg} \otimes g$ up to $x \gtrsim 10^{-5}$.
- N^{3}LO corrections to the scale derivativative of q_S amount to $(2 \pm 1)\%$ at $x = 10^{-5}$.
- Improved scale stability $O(\%)$ for $x \gtrsim 10^{-5}$

Outlook

- Ongoing work on the evolution of the **gluon density**. Higher technical complexity.
- Progress on the n_f^2 colour factor of $P_{gq}(x)$ (Falcioni,Herzog,Moch,Vermaseren,Vogt 2023)
  - Results for 30 moments ($N = 60$).
  - Analytic reconstruction in terms of ζ-values, harmonic sums and rational denominators with integer coefficients.
  - Full result for the n_f^2 colour factor!
- Stay tuned!

Thank you!