A non-perturbative determination of b_{g} (at small couplings)

Mattia Dalla Brida

work done in collaboration with

Roman Höllwieser, Francesco Knechtli, Tomasz Korzec,
Alberto Ramos, Stefan Sint, Rainer Sommer

ALPHA Collaboration

Lattice Field Theory Seminar

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Wilson-fermions and $\mathrm{O}(a)$ effects

Symanzik effective theory

(Symanzik '82; Lüscher et al. '96; ...)

$$S_{\mathrm{lat}} \approx S_{\mathrm{QCD}} + aS_1 + a^2 S_2 + \dots \qquad S_1 = \sum_{i=1}^{3} \int \mathrm{d}^4x\, \omega_i(g^2)\mathcal{O}_i(x)$$

$$\langle \mathcal{O}_R \rangle_{\mathrm{lat}} = \langle \mathcal{O}_R \rangle_{\mathrm{QCD}} - a\langle \mathcal{O}_R S_1 \rangle_{\mathrm{QCD}}^{\mathrm{conn}} + a\langle \delta\mathcal{O} \rangle_{\mathrm{QCD}} + \mathrm{O}(a^2)$$

$\mathrm{O}(a)$ action counterterms

(Sheikholeslami, Wohlert '85; Lüscher et al. '96)

$$\mathcal{O}_1 = \overline{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi \qquad \mathcal{O}_2 = \frac{m}{4g^2}F^a_{\mu\nu}F^a_{\mu\nu} \qquad \mathcal{O}_3 = m^2\overline{\psi}\psi$$

$\mathrm{O}(a)$ improvement

Add lattice representatives of $\mathcal{O}_i$ to S_{lat} in order to cancel S_1-contributions

$$S_{\mathrm{lat}} \to S_{\mathrm{lat}} + a\tfrac{i}{4}c_{\mathrm{sw}}(g_0^2)\sum_x \overline{\psi}(x)\sigma_{\mu\nu}F^{\mathrm{lat}}_{\mu\nu}(x)\psi(x)$$

$$\tilde{g}_0^2 = g_0^2(1 + b_{\mathrm{g}}(g_0^2)am_{\mathrm{q}}) \qquad \tilde{m}_{\mathrm{q}} = m_{\mathrm{q}}(1 + b_{\mathrm{m}}(g_0^2)am_{\mathrm{q}}) \qquad m_{\mathrm{q}} = m_0 - m_{\mathrm{cr}}$$

- Fixed a requires fixed $\tilde{g}_0^2$ while $\overline{m}_R(\mu) = Z_{\mathrm{m}}(\tilde{g}_0^2, a\mu)\tilde{m}_{\mathrm{q}}$ (Lüscher et al. '96)
- Strategies to compute c_{sw} and b_{m} are known (Lüscher et al. '97; Divitiis, Petronzio '98; Bhattacharya et al. '01; ...)
- b_{g} only known at 1-loop order in lattice PT (Sint, Sommer '96)

Heavy-quark decoupling and $\Lambda_{\overline{\mathrm{MS}}}^{(3)}$

Decoupling relation

(ALPHA Collab. '20, '23)

$$\frac{\Lambda_{\overline{\mathrm{MS}}}^{(3)}}{\mu_{\mathrm{dec}}} = \left(\frac{\Lambda_{\overline{\mathrm{MS}}}^{(0)}}{\Lambda_s^{(0)}}\right) \lim_{M/\mu_{\mathrm{dec}}\to\infty} \left[\frac{\varphi_s^{(0)}(\bar{g}_s^{(3)}(\mu_{\mathrm{dec}}, M))}{P_{0,3}\left(\frac{M}{\mu_{\mathrm{dec}}} \Big/ \frac{\Lambda_{\overline{\mathrm{MS}}}^{(3)}}{\mu_{\mathrm{dec}}}\right)}\right]$$

- $\Lambda_s^{(N_\mathrm{f})}$: Λ-parameter of N_f-flavour QCD in scheme s
- $\varphi_s^{(0)}(\bar{g}_s^{(0)}(\mu_{\mathrm{dec}})) = \Lambda_s^{(0)}/\mu_{\mathrm{dec}}$: $N_\mathrm{f} = 0$ running factor
- $\bar{g}_s^{(3)}(\mu_{\mathrm{dec}}, M)$: $N_\mathrm{f} = 3$ coupling in scheme s at the scale μ_{dec} and for a quark mass M
- $P_{0,3}(M/\Lambda_{\overline{\mathrm{MS}}}^{(3)}) = \Lambda_{\overline{\mathrm{MS}}}^{(0)}/\Lambda_{\overline{\mathrm{MS}}}^{(3)}$ at 5-loop order in PT

Challenge

Determine $\bar{g}_s^{(3)}(\mu_{\mathrm{dec}}, M)$ for $\mu_{\mathrm{dec}} = L^{-1} \ll M \ll a^{-1}$

$$\mu_{\mathrm{dec}} \approx 800\,\mathrm{MeV} \qquad M \approx 10\,\mathrm{GeV} \qquad L/a = 24 - 48 \quad \Rightarrow \quad aM \approx 0.25 - 0.5$$

Remark: $N_\mathrm{f} = 2+1$ hadronic simulations (Bruno et al. '15; Bali et al. '16)

- Can simulate at $\mathrm{tr}[M_\mathrm{q}] = \mathrm{const.} \Rightarrow \tilde{g}_0^2 = \mathrm{const.}$ if $g_0^2 = \mathrm{const.}$ (Bietenholz et al. '10)
- Moreover, $a \approx 0.035 - 0.075\,\mathrm{fm} \Rightarrow am_s \approx 0.02 - 0.04$

Heavy-quark decoupling and $\Lambda_{\overline{\rm MS}}^{(3)}$

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(ALPHA Collab. '20, '23)

$$\frac{\Lambda_{\overline{\rm MS}}^{(3)}}{\mu_{\rm dec}} P_{0,3}\left(M/\Lambda_{\overline{\rm MS}}^{(3)}\right) = \left(\frac{\Lambda_{\overline{\rm MS}}^{(0)}}{\Lambda_s^{(0)}}\right) \varphi_s^{(0)}\left(\bar{g}_s^{(3)}(\mu_{\rm dec}, M)\right) + \mathrm{O}\left(\alpha_{\overline{\rm MS}}^{8}(M)\right) + \mathrm{O}\left(\frac{\mu_{\rm dec}^2}{M^2}\right)$$

- $\Lambda_s^{(N_{\rm f})}$: Λ-parameter of $N_{\rm f}$-flavour QCD in scheme s
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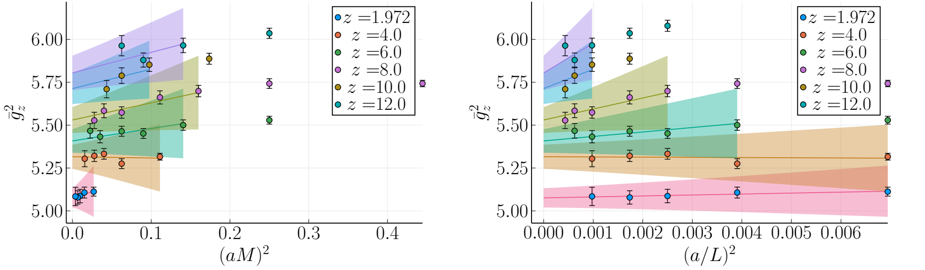
Continuum limit of the massive coupling



Fit ansatz

$$[\bar{g}^{(3)}_{\rm GFT}(\mu_{\rm dec}, M)]^2 = C(z) + p_1[\alpha^{(3)}_{\overline{\rm MS}}(a^{-1})]^{\hat{\Gamma}}(aM)^2 + p_2[\alpha^{(3)}_{\overline{\rm MS}}(a^{-1})]^{\hat{\Gamma}'}(a/L)^2 + b_{\rm g}\text{-error}$$

Line of constant physics

(ALPHA Collab. '17, '18)

- $\left[\bar{g}^{(3)}_{\rm GF}(\mu_{\rm dec})|_{M=0}\right]^2 = 3.95 \quad \Rightarrow \quad (\tilde{g}_0^2, L/a) \quad \Rightarrow \quad \mu_{\rm dec} = L^{-1} = 789(15)\,\mathrm{MeV}$
- $z = M/\mu_{\rm dec} = 2, \ldots, 12 \quad \Rightarrow \quad a\tilde{m}_{\rm q} \quad \Rightarrow \quad M \approx 1.5, \ldots, 9.5\,\mathrm{GeV}$

Simulation parameters $\left[m_{\rm q} = m_0 - m_{\rm cr}(g_0^2)\right]$

$z = (L/a)Z_{\rm M}(\tilde{g}_0^2)a\tilde{m}_{\rm q}$ $\qquad$ $g_0^2 = \tilde{g}_0^2/(1 + b_{\rm g}(\tilde{g}_0^2)am_{\rm q})$

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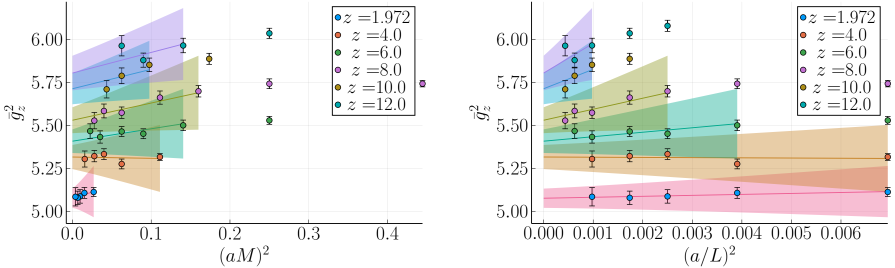

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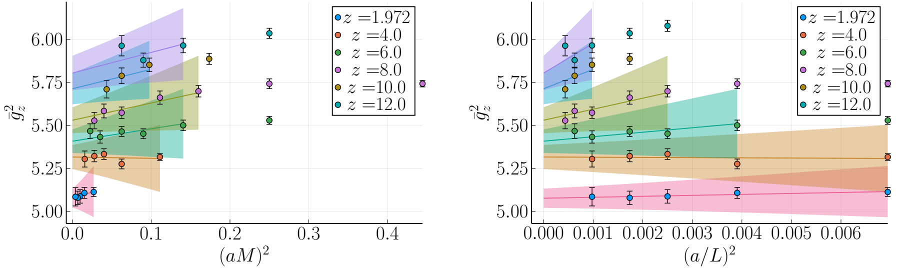
Continuum limit of the massive coupling



Fit ansatz

$$\left[\bar{g}^{(3)}_{\mathrm{GFT}}(\mu_{\mathrm{dec}}, M)\right]^2 = C(z) + p_1[\alpha^{(3)}_{\overline{\mathrm{MS}}}(a^{-1})]^{\hat{\Gamma}}(aM)^2 + p_2[\alpha^{(3)}_{\overline{\mathrm{MS}}}(a^{-1})]^{\hat{\Gamma}'}(a/L)^2 + b_{\mathrm{g}}\text{-error}$$

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$$z = (L/a)Z_{\mathrm{M}}(\tilde{g}_0^2)am_{\mathrm{q}}(1 + b_{\mathrm{m}}(\tilde{g}_0^2)am_{\mathrm{q}}) \qquad g_0^2 = \tilde{g}_0^2/(1 + [b_{\mathrm{g}}^{\text{1-loop}}(\tilde{g}_0^2) \pm \overbrace{\delta b_{\mathrm{g}}}^{b_{\mathrm{g}}^{\text{1-loop}}}]am_{\mathrm{q}})$$

Large-mass extrapolation of $\Lambda^{(3)}_{\overline{\rm MS}}$

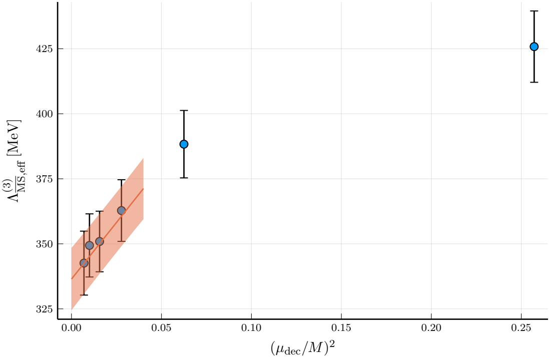
Pure-gauge running (MDB, Ramos '19)

- $\bar{g}^{(0)}_{\rm GFT}(\mu_{\rm dec}) \overset{\rm def.}{=} \bar{g}^{(3)}_{\rm GFT}(\mu_{\rm dec}, M)$
- $\Lambda^{(0)}_{\overline{\rm MS}}/\mu_{\rm dec} = (\Lambda^{(0)}_{\overline{\rm MS}}/\Lambda^{(0)}_{\rm GF})\varphi^{(0)}_{\rm GF}(\chi(\bar{g}^{(0)}_{\rm GFT}(\mu_{\rm dec})))$

Example

$$\left[\bar{g}^{(3)}_{\rm GFT}(\mu_{\rm dec}, 3.2\,{\rm GeV})\right]^2 = 5.316(70)[(62)_{b_{\rm g}}]$$

$$\Rightarrow \Lambda^{(0)}_{\overline{\rm MS}}/\mu_{\rm dec} = 0.719(16)$$



Master formula

$$\rho\, P^{(5\text{-loop})}_{0,3}(z/\rho) = \Lambda^{(0)}_{\overline{\rm MS}}/\mu_{\rm dec}$$

$$\rho = \Lambda^{(3)}_{\overline{\rm MS},\rm eff}/\mu_{\rm dec}$$

$$z = M/\mu_{\rm dec}$$

$$\mu_{\rm dec} = 789(15)\,{\rm MeV}$$

Fit ansatz

$$\Lambda^{(3)}_{\overline{\rm MS},\rm eff} = A + \frac{B}{z^2}\alpha_{\overline{\rm MS}}(m_\star)^{\hat{\Gamma}_m}$$

Result $[\, z \geq 6\, ;\, \hat{\Gamma}_m = 0\,]$

$$\Lambda^{(3)}_{\overline{\rm MS}} = 336(10)(6)_{b_{\rm g}}(3)_{\hat{\Gamma}_m}\,{\rm MeV}$$

Large-mass extrapolation of $\Lambda_{\overline{\mathrm{MS}}}^{(3)}$

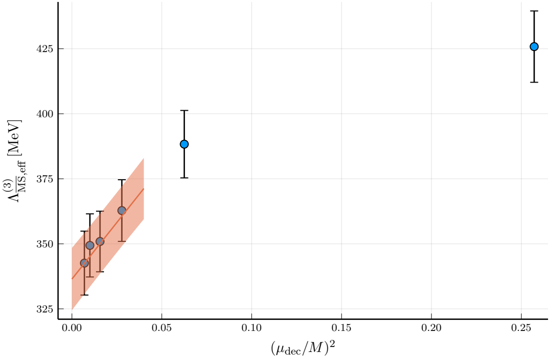
Pure-gauge running (MDB, Ramos '19)

- $\bar{g}_{\mathrm{GFT}}^{(0)}(\mu_{\mathrm{dec}}) \stackrel{\mathrm{def.}}{=} \bar{g}_{\mathrm{GFT}}^{(3)}(\mu_{\mathrm{dec}}, M)$
- $\Lambda_{\overline{\mathrm{MS}}}^{(0)}/\mu_{\mathrm{dec}} = (\Lambda_{\overline{\mathrm{MS}}}^{(0)}/\Lambda_{\mathrm{GF}}^{(0)})\varphi_{\mathrm{GF}}^{(0)}(\chi(\bar{g}_{\mathrm{GFT}}^{(0)}(\mu_{\mathrm{dec}})))$

Example

$$\left[\bar{g}_{\mathrm{GFT}}^{(3)}(\mu_{\mathrm{dec}}, 9.5\,\mathrm{GeV})\right]^2 = 5.80(10)[(8)_{b_{\mathrm{g}}}]$$

$$\Rightarrow \Lambda_{\overline{\mathrm{MS}}}^{(0)}/\mu_{\mathrm{dec}} = 0.797(21)$$



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$$z = M/\mu_{\mathrm{dec}}$$

$$\mu_{\mathrm{dec}} = 789(15)\,\mathrm{MeV}$$

Fit ansatz

$$\Lambda_{\overline{\mathrm{MS}},\mathrm{eff}}^{(3)} = A + \frac{B}{z^2}\alpha_{\overline{\mathrm{MS}}}(m_\star)^{\hat{\Gamma}_m}$$

Result $[\, z \geq 6\, ;\, \hat{\Gamma}_m = 0 \,]$

$$\Lambda_{\overline{\mathrm{MS}}}^{(3)} = 336(10)(6)_{b_{\mathrm{g}}}(3)_{\hat{\Gamma}_m}\ \mathrm{MeV}$$

The coupling from decoupling

More decoupling

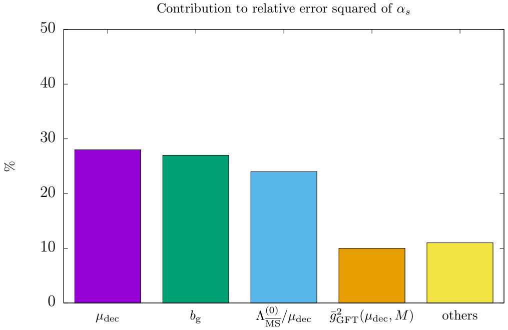
$$\Lambda^{(3)}_{\overline{\rm MS}} \xrightarrow{P^{(5\text{-loop})}_{3,4}(M_c/\Lambda^{(4)}_{\overline{\rm MS}})} \Lambda^{(4)}_{\overline{\rm MS}} \xrightarrow{P^{(5\text{-loop})}_{4,5}(M_b/\Lambda^{(5)}_{\overline{\rm MS}})} \Lambda^{(5)}_{\overline{\rm MS}} \xrightarrow{\beta^{(5\text{-loop})}_{\overline{\rm MS}}} \alpha^{(5)}_{\overline{\rm MS}}(m_Z)$$

Final result

(ALPHA Collab. '22)

$$\alpha^{(5)}_{\overline{\rm MS}}(m_Z) = 0.11823(69)(42)_{b_{\rm g}}(20)_{\hat{\Gamma}_m}(9)_{3\to 5} = 0.1182(8)$$

FLAG 21: $\alpha^{(5)}_{\overline{\rm MS}}(m_Z) = 0.1184(8)$ **PDG 21:** $\alpha^{(5)}_{\overline{\rm MS}}(m_Z) = 0.1179(9)$ (FLAG '21; PDG '21)



$\mu_{\rm dec}$ **uncertainty**^{2}

- 55% comes from $\sqrt{8t_0^\star}$ [MeV]
  (Bruno et al. '17)
- Correlated w/ **ALPHA 17:** $\alpha^{(5)}_{\overline{\rm MS}}(m_Z) = 0.1185(8)$
  (ALPHA Collab. '17)

Non-perturbative improvement condition for b_g

Basic idea

Physical small-mass dependence of a gluonic obs. $\mathcal{O}_\mathrm{g}$ in **finite-volume** is $\mathrm{O}(M^2)$ whereas b_g-term $\propto \mathrm{O}(aM)$

Mass-dependence

- N_f**-even:** $\psi \to \gamma_5\psi \quad \overline{\psi} \to -\overline{\psi}\gamma_5$

$$\Rightarrow \langle\mathcal{O}_\mathrm{g}\rangle^{(M)}_\mathrm{QCD} = \langle\mathcal{O}_\mathrm{g}\rangle^{(-M)}_\mathrm{QCD}$$

- N_f**-odd:** $\psi \to e^{i\frac{\pi}{N_\mathrm{f}}\gamma_5}\,\psi \quad \overline{\psi} \to \overline{\psi}\,e^{i\frac{\pi}{N_\mathrm{f}}\gamma_5}$ (MDB, Giusti, Pepe '20)

$$\Rightarrow \langle\mathcal{O}_\mathrm{g}\rangle^{(M)}_\mathrm{QCD} \overset{M\to 0}{=} \langle\mathcal{O}_\mathrm{g}\rangle^{(0)}_\mathrm{QCD} + \mathrm{O}(M^2)$$

Remark: True if **NO** SSB and bcs. compatible w/ chiral symmetry

Master formula $\left[\ z = LM = LZ_\mathrm{M}\tilde{m}_\mathrm{q}\ \right]$ (Sint '13; ALPHA Collab. in preparation)

$$\frac{\partial\langle\mathcal{O}_\mathrm{g}^R\rangle}{\partial z}\bigg|_{z,\tilde{g}_0^2,L/a} = \frac{1}{LZ_\mathrm{M}}(1-2b_\mathrm{m}am_\mathrm{q})\left(\frac{\partial\langle\mathcal{O}_\mathrm{g}^R\rangle}{\partial m_\mathrm{q}}\bigg|_{g_0^2} - ag_0^2 b_\mathrm{g}\frac{\partial\langle\mathcal{O}_\mathrm{g}^R\rangle}{\partial g_0^2}\bigg|_{m_\mathrm{q}}\right) + \mathrm{O}(a^2)$$

Remark: Different $\mathcal{O}_\mathrm{g}$'s result in b_g's that differ by $\mathrm{O}(a)$-effects

Non-perturbative improvement condition for $b_{\rm g}$

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- $N_{\rm f}$**-even:** $\quad \psi \to \gamma_5 \psi \quad \overline{\psi} \to -\overline{\psi}\gamma_5$

$$\Rightarrow \langle \mathcal{O}_{\rm g} \rangle_{\rm QCD}^{(M)} = \langle \mathcal{O}_{\rm g} \rangle_{\rm QCD}^{(-M)}$$

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$$\frac{\partial \langle \mathcal{O}_{\rm g}^R \rangle}{\partial z}\bigg|_{z=0,\tilde{g}_0^2,L/a} \overset{!}{=} 0 \quad \Rightarrow \quad b_{\rm g} = \frac{\partial \langle \mathcal{O}_{\rm g}^R \rangle}{\partial a m_{\rm q}}\bigg|_{g_0^2, m_{\rm q}=0} \times \left[g_0^2 \frac{\partial \langle \mathcal{O}_{\rm g}^R \rangle}{\partial g_0^2}\bigg|_{m_{\rm q}=0} \right]^{-1} + \mathrm{O}(a)$$

Remark: Different $\mathcal{O}_{\rm g}$'s result in $b_{\rm g}$'s that differ by $\mathrm{O}(a)$-effects

Lattice set-up and parameters

Action and bcs.

- $N_\mathrm{f} = 3$ $\mathrm{O}(a)$-improved Wilson quarks and LW gauge action (Lüscher, Weisz '84; Bulava, Schaefer '13)
- **Periodic** for gluons and **anti-periodic** for quarks in all 4d

  $\Rightarrow$ Simulations @ $M = 0$ feasible!

Observable(s) (Lüscher '10; Fodor et al. '12; Fritzsch et al. '13)

$$\sigma(c) \equiv \frac{\langle t^2 E(t) \delta_{Q(t),0} \rangle_\mathrm{pbc}}{\langle \delta_{Q(t),0} \rangle_\mathrm{pbc}} \Bigg|^{c=\sqrt{8t}/L}_{T=L} \qquad E(t,x) = \mathrm{tr}\{G_{\mu\nu}(t,x) G_{\mu\nu}(t,x)\}$$

Line of constant physics (LCP) (ALPHA Collab. '17, '18)

$$\bar{g}^2_\mathrm{GF}(\mu_\mathrm{dec}) = 3.95 \qquad \Rightarrow \qquad \beta \approx 4.3 - 5.2 \qquad L/a = 12 - 48 \qquad M = \mathrm{O}(a^2)$$

Simulation params.

$$\beta \in [4.3, 5.2] + 6, 8, 16 \qquad L/a = 12 - 24 \qquad L/a = 24 \quad \text{for} \quad \beta \geq 4.7$$

Remark: L-dependence of b_g is an $\mathrm{O}(a)$-effect!

Numerical derivatives

$$\sigma(c, am_\mathrm{q})|_{g_0^2, L/a} = \sum_{k=0}^{n_m} c_k (am_\mathrm{q})^k \quad \Rightarrow \quad \frac{\partial \sigma(c)}{\partial am_\mathrm{q}}\Bigg|_{g_0^2, m_\mathrm{q}=0}$$

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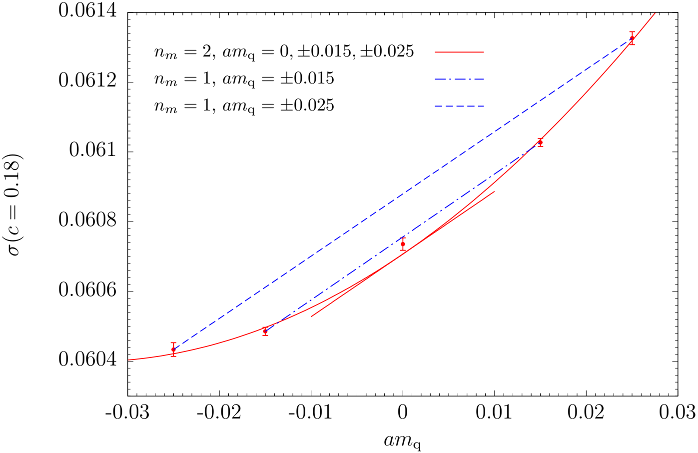
Numerical derivatives

$$\sigma(c, g_0^2)|_{m_\mathrm{q}=0, L/a} = \sum_{k=1}^{n_g} d_k g_0^{2k} \qquad \Rightarrow \qquad g_0^2 \left. \frac{\partial \sigma(c)}{\partial g_0^2} \right|_{g_0^2, m_\mathrm{q}=0}$$

am_q-derivatives ($\beta = 4.7$, $L/a = 24$)



Remarks

$$\frac{\partial^2_{am_\mathrm{q}}\sigma(c)}{\partial_{am_\mathrm{q}}\sigma(c)} = Z_\mathrm{m}\frac{L}{a}\underbrace{\overbrace{\frac{\partial^2_z\sigma(c)}{\partial_z\sigma(c)}}^{\mathrm{O}(1)}}_{\mathrm{O}(a/L)} \overset{a\to 0}{\propto} \left(\frac{L}{a}\right)^2$$

Fit type	$\partial\sigma(c)/\partial am_\mathrm{q}$	b_g
$n_m = 1,\ am_\mathrm{q} = \pm 0.025$	0.01786(54)	0.1080(33)
$n_m = 1,\ am_\mathrm{q} = \pm 0.015$	0.01806(55)	0.1092(33)
$n_m = 2$, 5 pts.	0.01796(38)	0.1086(23)

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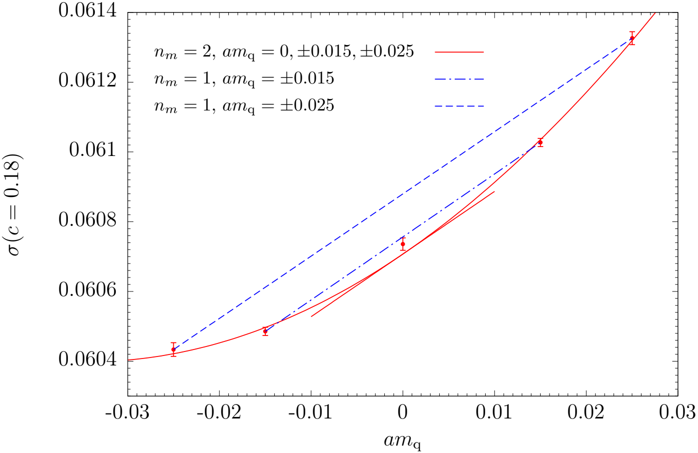

Remarks

$$\frac{\sigma(c, am_\mathrm{q}) - \sigma(c, -am_\mathrm{q})}{2(am_\mathrm{q})} \approx$$

$$\partial_{am_\mathrm{q}}\sigma(c) + \partial^3_{am_\mathrm{q}}\sigma(c)(am_\mathrm{q})^2 + \ldots$$

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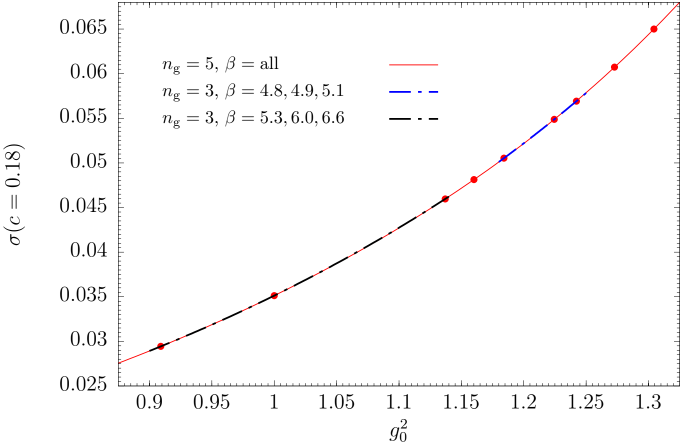
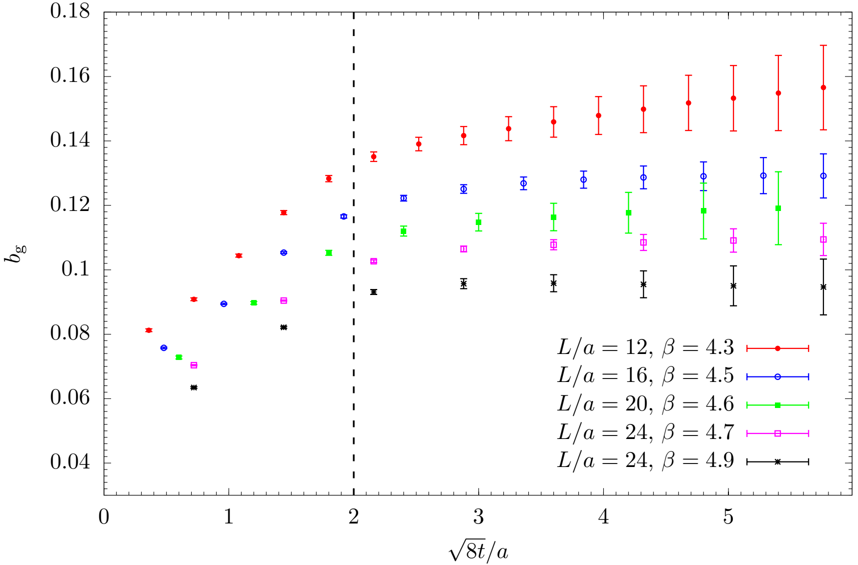
g_0^2-derivatives ($L/a = 24$)

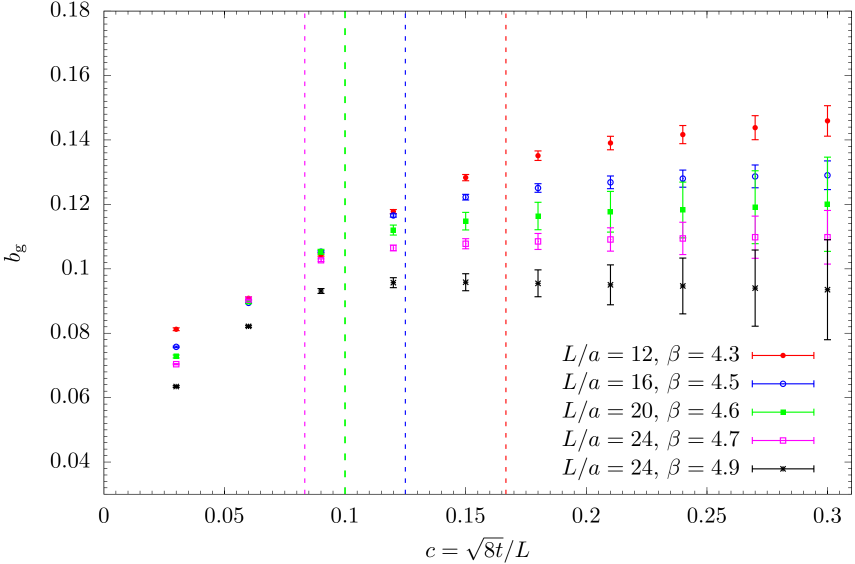


L/a	β_{LCP}	$g_0^2 \partial\sigma(c)/\partial g_0^2$ ($n_g = 5$)	$g_0^2 \partial\sigma(c)/\partial g_0^2$ ($n_g = 3$)	b_g ($n_g = 5$)	b_g ($n_g = 3$)
24	4.7	0.16474(46)	0.16531(58)	0.1090(23)	0.1086(23)
24	4.9	0.13907(20)	0.1379(11)	0.0957(40)	0.0965(41)
24	5.1	0.12132(23)	0.12195(62)	0.0961(40)	0.0956(40)
24	5.2	0.11211(20)	0.11246(54)	0.0831(39)	0.0829(39)
24	6.0	0.06835(16)	0.068681(87)	0.0627(43)	0.0624(43)

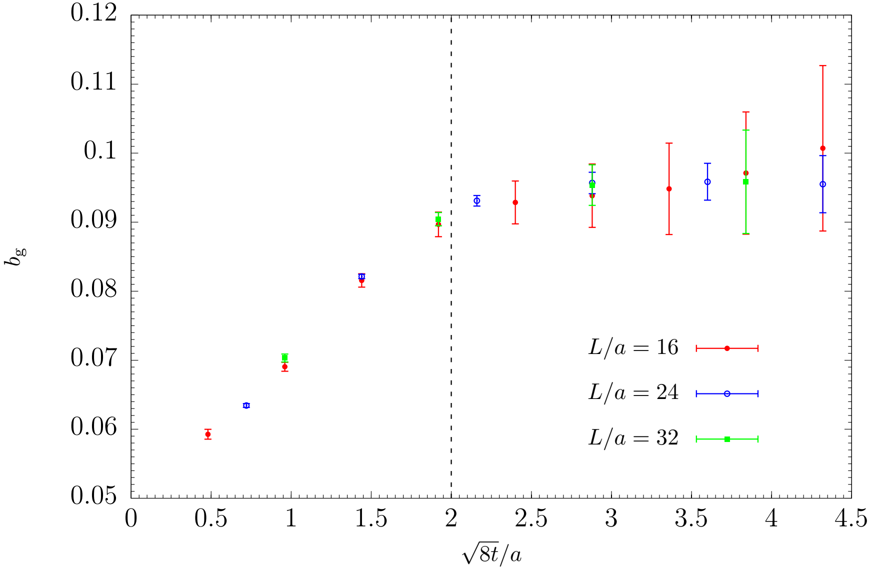
Choosing a flow time



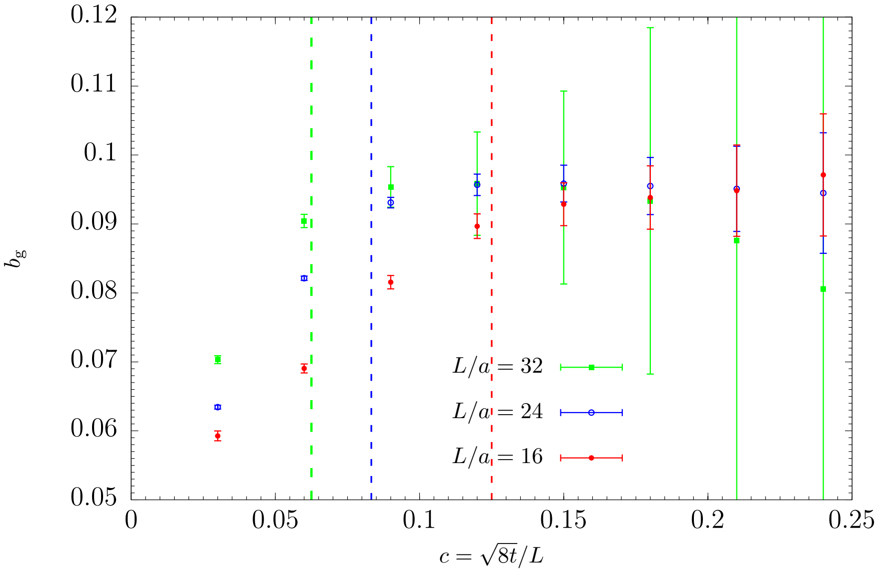
Choosing a flow time



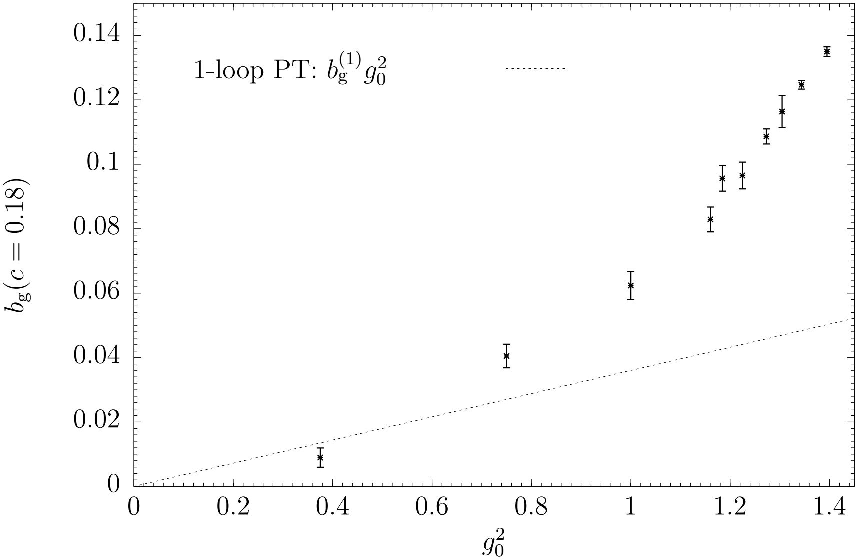
L-dependence of b_{g} ($\beta = 4.9$)



L-dependence of $b_{\rm g}$ ($\beta = 4.9$)

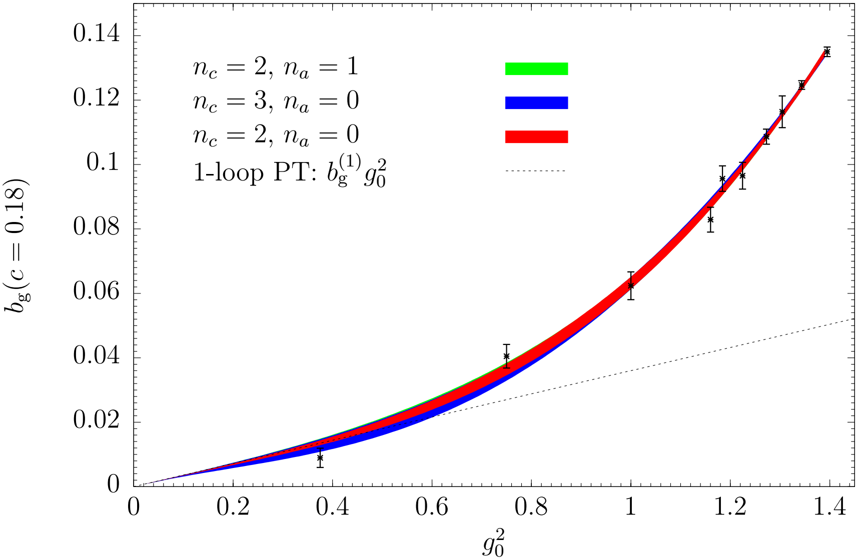


Results for b_{g} from $t^2 E(t)$



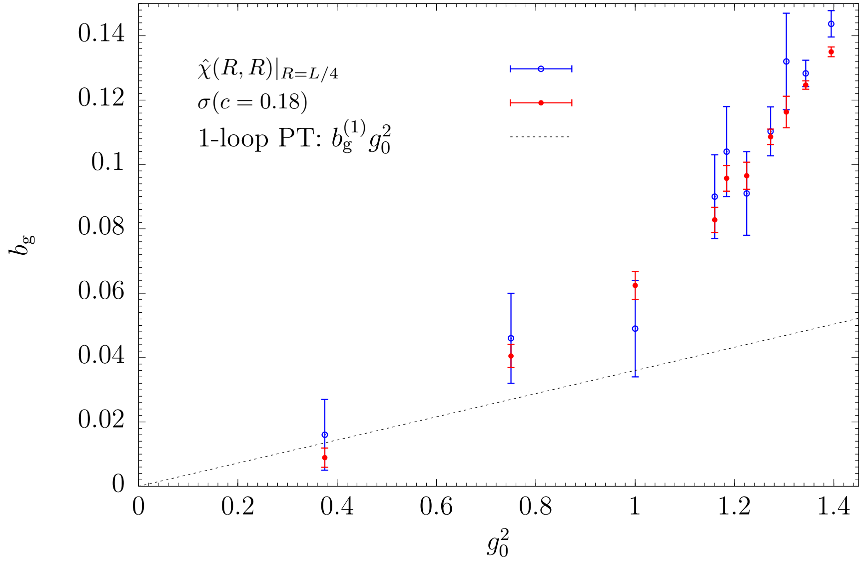
$$b_{\mathrm{g}}(x) = b_{\mathrm{g}}^{(1)} x + x \sum_{i=1}^{n_c} c_i x^i + \left(e^{-\frac{1}{2b_0 x}} (b_0 x)^{-\frac{b_1}{2b_0^2}} \right) \sum_{j=0}^{n_a - 1} a_j x^j$$

Results for $b_{\rm g}$ from $t^2E(t)$



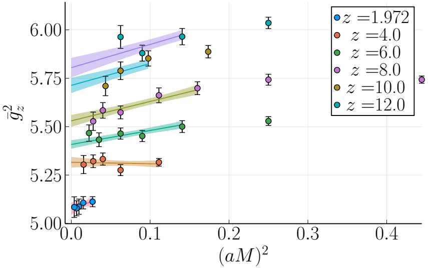
$$b_{\rm g}(x) = b_{\rm g}^{(1)} x + x \sum_{i=1}^{n_c} c_i x^i + \left(e^{-\frac{1}{2b_0 x}} (b_0 x)^{-\frac{b_1}{2b_0^2}} \right) \sum_{j=0}^{n_a - 1} a_j x^j$$

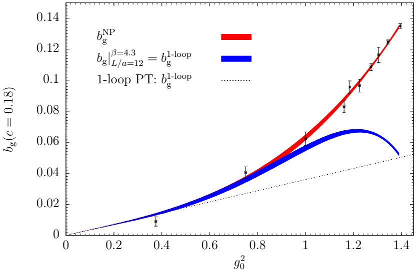
Results for $b_{\rm g}$ from HYP-smeared Creutz ratios



$$\hat{\chi}(R,T) = -RT\tilde{\partial}_R\tilde{\partial}_T \log(W(R,T)) \quad W(R,T) \equiv R \times T \text{ Wilson loop}$$

How do we plan to correct for b_g?





Fit ansatz

$$\left[\bar{g}^{(3)}_\mathrm{GFT}(\mu_\mathrm{dec}, M)\right]^2 = C(z) + p_1[\alpha^{(3)}_{\overline{\mathrm{MS}}}(a^{-1})]^{\hat{\Gamma}}(aM)^2 + p_2[\alpha^{(3)}_{\overline{\mathrm{MS}}}(a^{-1})]^{\hat{\Gamma}'}(a\mu_\mathrm{dec})^2$$

Bare coupling

$$g_0^2|_{b_\mathrm{g}^\text{1-loop}} = \tilde{g}_0^2/\left(1 + b_\mathrm{g}^\text{1-loop}(\tilde{g}_0^2)am_\mathrm{q}\right) \quad \Rightarrow \quad g_0^2|_{b_\mathrm{g}^\mathrm{NP}} = \tilde{g}_0^2/\left(1 + b_\mathrm{g}^\mathrm{NP}(\tilde{g}_0^2)am_\mathrm{q}\right)$$

Massive coupling

$$\bar{g}^2_\mathrm{GFT}(\mu_\mathrm{dec}, M) = \bar{g}^2_\mathrm{GFT}(\mu_\mathrm{dec}, M)|_{b_\mathrm{g}^\text{1-loop}} + \frac{\partial \bar{g}^2_\mathrm{GFT}}{\partial g_0^2} \times \left(g_0^2|_{b_\mathrm{g}^\mathrm{NP}} - g_0^2|_{b_\mathrm{g}^\text{1-loop}}\right)$$

$\mathrm{O}(a)$**-ambiguous** b_g

$$b_\mathrm{g}(g_0^2) \equiv b_\mathrm{g}^\mathrm{NP}(g_0^2) - c_0(a\mu_\mathrm{dec}) \qquad b_\mathrm{g}(g_0^2)|^{\beta=4.3}_{L/a=12} = f b_\mathrm{g}^\text{1-loop}(g_0^2) \quad f = \mathrm{O}(1)$$

Conclusions & Outlook

Conclusions

- Devised a viable strategy for a non-perturbative determination of b_{g} (at least for $\beta \geq 4.3$)
- Large deviation from 1-loop PT for $4.3 \lesssim \beta \lesssim 5.2$ (do not extrapolate conclusions to smaller β's!)

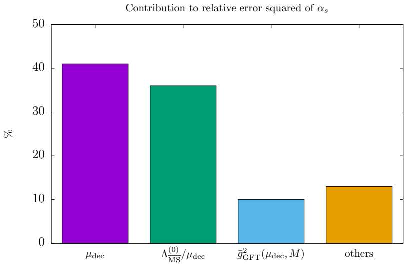
Outlook

- Reanalysis of $\Lambda^{(3)}_{\overline{\mathrm{MS}}}$ using the non-perturbative b_{g}
- Tackle the remaining sources of uncertainty in α_s

Expectation

$$\alpha^{(5)}_{\overline{\mathrm{MS}}}(m_Z) = 0.11823(72)(42)_{b_{\mathrm{g}}}$$

$$\Rightarrow \alpha^{(5)}_{\overline{\mathrm{MS}}}(m_Z) = 0.118??(72)$$




Thanks

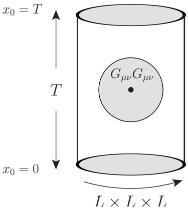
BACKUP

The Schrödinger functional and gradient flow

Schrödinger functional (SF) bcs.

Gauge fields (Lüscher et al. '92)

$$A_k(x)|_{x_0=0} = C_k \qquad A_k(x)|_{x_0=T} = C'_k$$



Quark fields $\left[\ P_\pm \equiv \frac{1}{2}(1 \pm \gamma_0)\ \right]$ (Sint '94)

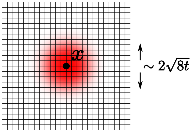
$$P_+\psi|_{x_0=0} = P_-\psi|_{x_0=T} = 0$$

$$\overline{\psi}P_-|_{x_0=0} = \overline{\psi}P_+|_{x_0=T} = 0$$

Gradient flow (GF)

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) \qquad B_\mu(0,x) = A_\mu(x)$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] \qquad D_\mu = \partial_\mu + [B_\mu, \cdot\,]$$



(Lüscher, Weisz '11)

- ✓ Composite (gauge-invariant) fields are automatically finite
- ✓ Simple to evaluate in Monte Carlo simulations
- ✗ PT is quite involved

Effective theory of decoupling and PT matching

Fundamental theory

$$\mathcal{L}_{\mathrm{QCD}_{\mathrm{N_f}}} = \frac{1}{4g^2}F^2 + \sum_{f=1}^{N_\ell} \overline{\psi}_f \not{D} \psi_f + \sum_{f=N_\ell+1}^{N_\mathrm{f}} \overline{\psi}_f (\not{D} + M)\psi_f$$

Effective theory

(Weinberg '80; ...)

$$\mathcal{L}_{\mathrm{eff}} = \mathcal{L}_{\mathrm{QCD}_{\mathrm{N}_\ell}} + \frac{1}{M^2}\sum_i \omega_i \Phi_i + \ldots \quad \Rightarrow \quad \mathbf{LO}: \quad \mathcal{L}_{\mathrm{eff}} = \mathcal{L}_{\mathrm{QCD}_{\mathrm{N}_\ell}}$$

Matching couplings in PT

(Bernreuther, Wetzel '82; ...; Chetyrkin, Kühn, Sturm '06; Schröder, Steinhauser '06)

EFT is matched at LO once the effective and fundamental couplings are matched

$$\alpha^{(N_\ell)}(\mu/\Lambda^{(N_\ell)}) = F_{\mathcal{O}}\big(\alpha^{(N_\mathrm{f})}(\mu/\Lambda^{(N_\mathrm{f})}), M/\mu\big) \qquad \mathcal{O} \equiv \text{matching obs.}$$

Matching Λ-parameters in PT

$$\Lambda_{\overline{\mathrm{MS}}}^{(N_\ell)}(M, \Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}) = P_{\ell,\mathrm{f}}(M/\Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})})\, \Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})} \quad \Rightarrow \quad P_{\ell,\mathrm{f}}(M/\Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}) = \frac{\varphi_{\overline{\mathrm{MS}}}^{(N_\ell)}\big(\alpha_\star \xi(\alpha_\star)\big)}{\varphi_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}(\alpha_\star)}$$

where

$$\Lambda_\mathrm{X}^{(N_\mathrm{f})} = \mu\, \varphi_\mathrm{X}^{(N_\mathrm{f})}(\alpha_\mathrm{X}(\mu)) \qquad \varphi_\mathrm{X}^{(N_\mathrm{f})}(\alpha) = \ldots \exp\left\{ -\int_0^\alpha \frac{\mathrm{d}y}{\beta_\mathrm{X}^{(N_\mathrm{f})}(y)} + \ldots \right\}$$

$$M = \overline{m}_\mathrm{X}(\mu)\, \varepsilon_\mathrm{X}^{(N_\mathrm{f})}(\alpha_\mathrm{X}(\mu)) \qquad \varepsilon_\mathrm{X}^{(N_\mathrm{f})}(\alpha) = \ldots \exp\left\{ -\int_0^\alpha \mathrm{d}y \frac{\tau_\mathrm{X}^{(N_\mathrm{f})}(y)}{\beta_\mathrm{X}^{(N_\mathrm{f})}(y)} + \ldots \right\}$$

Effective theory of decoupling and PT matching

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EFT is matched at LO once the effective and fundamental couplings are matched

$$\alpha_{\overline{\mathrm{MS}}}^{(N_\ell)}(\mu) = \alpha_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}(\mu)\,\xi\big(\alpha_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}(\mu), \bar{z}(\mu)\big) + \mathrm{O}(c_{\mathcal{O}}/\bar{z}^2) \qquad \bar{z}(\mu) = \overline{m}_{\overline{\mathrm{MS}}}(\mu)/\mu$$

Matching Λ-parameters in PT

$$\Lambda_{\overline{\mathrm{MS}}}^{(N_\ell)}(M, \Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}) = P_{\ell,\mathrm{f}}(M/\Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})})\,\Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})} \quad \Rightarrow \quad P_{\ell,\mathrm{f}}(M/\Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}) = \frac{\varphi_{\overline{\mathrm{MS}}}^{(N_\ell)}\big(\alpha_\star \xi(\alpha_\star)\big)}{\varphi_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}(\alpha_\star)}$$

where

$$\Lambda_\mathrm{X}^{(N_\mathrm{f})} = \mu\,\varphi_\mathrm{X}^{(N_\mathrm{f})}(\alpha_\mathrm{X}(\mu)) \qquad \varphi_\mathrm{X}^{(N_\mathrm{f})}(\alpha) = \ldots \exp\left\{-\int_0^\alpha \frac{\mathrm{d}y}{\beta_\mathrm{X}^{(N_\mathrm{f})}(y)} + \ldots\right\}$$

$$M = \overline{m}_\mathrm{X}(\mu)\,\varepsilon_\mathrm{X}^{(N_\mathrm{f})}(\alpha_\mathrm{X}(\mu)) \qquad \varepsilon_\mathrm{X}^{(N_\mathrm{f})}(\alpha) = \ldots \exp\left\{-\int_0^\alpha \mathrm{d}y\,\frac{\tau_\mathrm{X}^{(N_\mathrm{f})}(y)}{\beta_\mathrm{X}^{(N_\mathrm{f})}(y)} + \ldots\right\}$$

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(Bernreuther, Wetzel '82; ...; Chetyrkin, Kühn, Sturm '06; Schröder, Steinhauser '06)

EFT is matched at LO once the effective and fundamental couplings are matched

$$\alpha_{\overline{\mathrm{MS}}}^{(N_\ell)}(m_\star) \equiv \alpha_\star \, \xi(\alpha_\star) \qquad \alpha_\star \equiv \alpha_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}(m_\star) \qquad m_\star = \overline{m}_{\overline{\mathrm{MS}}}(m_\star)$$

Matching Λ-parameters in PT

$$\Lambda_{\overline{\mathrm{MS}}}^{(N_\ell)}(M, \Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}) = P_{\ell,\mathrm{f}}(M/\Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}) \, \Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})} \quad \Rightarrow \quad P_{\ell,\mathrm{f}}(M/\Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}) = \frac{\varphi_{\overline{\mathrm{MS}}}^{(N_\ell)}\big(\alpha_\star \xi(\alpha_\star)\big)}{\varphi_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}(\alpha_\star)}$$

where

$$\Lambda_\mathrm{X}^{(N_\mathrm{f})} = \mu \, \varphi_\mathrm{X}^{(N_\mathrm{f})}(\alpha_\mathrm{X}(\mu)) \qquad \varphi_\mathrm{X}^{(N_\mathrm{f})}(\alpha) = \ldots \exp\left\{ - \int_0^\alpha \frac{\mathrm{d}y}{\beta_\mathrm{X}^{(N_\mathrm{f})}(y)} + \ldots \right\}$$

$$M = \overline{m}_\mathrm{X}(\mu) \, \varepsilon_\mathrm{X}^{(N_\mathrm{f})}(\alpha_\mathrm{X}(\mu)) \qquad \varepsilon_\mathrm{X}^{(N_\mathrm{f})}(\alpha) = \ldots \exp\left\{ - \int_0^\alpha \mathrm{d}y \frac{\tau_\mathrm{X}^{(N_\mathrm{f})}(y)}{\beta_\mathrm{X}^{(N_\mathrm{f})}(y)} + \ldots \right\}$$

Non-perturbative renormalization by decoupling

Current situation

- $\delta\Lambda^{(3)}_{\overline{\rm MS}} \sim 3.5\% \Rightarrow$ room for **improvement!**
- $\delta\Lambda^{(3)}_{\overline{\rm MS}}$ dominated by NP running $0.2 - 70\,\mathrm{GeV}$
- Halving $\delta\Lambda^{(3)}_{\overline{\rm MS}}$ by brute force is CPU expensive

Key observations

- $P_{\ell,\mathrm{f}}(M/\Lambda)$ has **small** PT and NP corrections for $M/\Lambda \gtrsim 5$
- $\Lambda^{(N_\mathrm{f})}_{\overline{\rm MS}}$ is M-independent $\Rightarrow$ same for $\mathrm{QCD}_{N_\mathrm{f}}$ with any M
- LQCD can **access** $\mathrm{QCD}_{N_\mathrm{f}}$ with any M

Master equation 1.0

(ALPHA Collab. '20, '22)

$$\frac{\Lambda^{(N_\ell)}}{\mathcal{S}^{(N_\ell)}_{\rm had}} = P^{\rm had}_{\ell,\mathrm{f}}\left(M/\Lambda^{(N_\mathrm{f})}\right) \frac{\Lambda^{(N_\mathrm{f})}}{\mathcal{S}^{(N_\mathrm{f})}_{\rm had}(M)}$$

- Compute $\Lambda^{(0)}_{\overline{\rm MS}}/\mathcal{S}^{(0)}_{\rm had}$ in **pure Yang-Mills**
- Determine $\mathcal{S}^{(3)}_{\rm had}(M)/\mathcal{S}^{(3)}_{\rm had}(m^{\rm phys}_{u,d,s})$ and set $\mathcal{S}^{(3)}_{\rm had}(m^{\rm phys}_{u,d,s}) \equiv \mathcal{S}^{\rm exp}_{\rm had}$ [MeV]
- Extrapolate for $M \to \infty$

Non-perturbative renormalization by decoupling

Current situation

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- $\delta\Lambda_{\overline{\mathrm{MS}}}^{(3)}$ dominated by NP running $0.2 - 70\,\mathrm{GeV}$
- Halving $\delta\Lambda_{\overline{\mathrm{MS}}}^{(3)}$ by brute force is CPU expensive

Key observations

- $P_{\ell,\mathrm{f}}(M/\Lambda)$ has **small** PT and NP corrections for $M/\Lambda \gtrsim 5$
- $\Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}$ is M-independent $\Rightarrow$ same for $\mathrm{QCD}_{N_\mathrm{f}}$ with any M
- LQCD can **access** $\mathrm{QCD}_{N_\mathrm{f}}$ with any M

Master equation 1.0

(ALPHA Collab. '20, '22)

$$\frac{\Lambda_{\overline{\mathrm{MS}}}^{(0)}}{\mathcal{S}_{\mathrm{had}}^{(0)}} = P_{0,3}^{(n\text{-loop})}\left(M/\Lambda_{\overline{\mathrm{MS}}}^{(3)}\right)\frac{\Lambda_{\overline{\mathrm{MS}}}^{(3)}}{\mathcal{S}_{\mathrm{had}}^{(3)}(M)} + \mathrm{O}(\alpha_\star^{n-1}) + \mathrm{O}\left(\frac{\Lambda^2}{M^2}\right)$$

- Compute $\Lambda_{\overline{\mathrm{MS}}}^{(0)}/\mathcal{S}_{\mathrm{had}}^{(0)}$ in **pure Yang-Mills**
- Determine $\mathcal{S}_{\mathrm{had}}^{(3)}(M)/\mathcal{S}_{\mathrm{had}}^{(3)}(m_{u,d,s}^{\mathrm{phys}})$ and set $\mathcal{S}_{\mathrm{had}}^{(3)}(m_{u,d,s}^{\mathrm{phys}}) \equiv \mathcal{S}_{\mathrm{had}}^{\mathrm{exp}}$ [MeV]
- Extrapolate for $M \to \infty$

Non-perturbative renormalization by decoupling

Is this feasible?

$$L^{-1} \ll \mathcal{S}^{(3)}_{\rm had}(m^{\rm phys}_{u,d,s}) \sim \Lambda_{\rm QCD} \ll M \ll a^{-1}$$

Example

$$L/a = 100 \qquad m_\pi L \sim 4 \quad \Rightarrow \; a^{-1} \sim 3\,{\rm GeV} \;\Rightarrow\; M \sim {\rm O}(1)\,{\rm GeV}$$

Decoupling in a finite volume

- Decoupling scale

$$\alpha^{(3)}_{\mathcal{O}}(\mu^{(3)}_{\rm dec}) \overset{\rm e.g.}{=} 0.3 \quad \Rightarrow \quad \mu^{(3)}_{\rm dec} = L^{-1}_{\rm dec} \sim 1\,{\rm GeV}$$

- Massive coupling

$$\alpha^{(0)}_{\mathcal{O}}(\mu^{(0)}_{\rm dec}) \overset{\rm def.}{=} \alpha^{(3)}_{\mathcal{O}}(\mu^{(3)}_{\rm dec}, M) \quad \Rightarrow \quad \mu^{(0)}_{\rm dec} = \mu^{(3)}_{\rm dec} + {\rm O}(\mu^2_{\rm dec}/M^2)$$

Master formula 2.0 (ALPHA Collab. '18, '22)

$$\frac{\Lambda^{(0)}_{\overline{\rm MS}}}{\mu^{(0)}_{\rm dec}} = P^{(n\text{-loop})}_{0,3}(M/\Lambda^{(3)}_{\overline{\rm MS}})\,\frac{\Lambda^{(3)}_{\overline{\rm MS}}}{\mu^{(3)}_{\rm dec}} + {\rm O}(\alpha_\star^{n-1}) + {\rm O}\left(\frac{\mu^2_{\rm dec}}{M^2}\right)$$

- Determine $\alpha^{(3)}_{\mathcal{O}}(\mu^{(3)}_{\rm dec}, M)$ with $L^{-1}_{\rm dec} = \mu_{\rm dec} \ll M \ll a^{-1}$

$$L_{\rm dec}/a = 50 \qquad \mu_{\rm dec} \sim 1\,{\rm GeV} \quad \Rightarrow \; M \sim {\rm O}(10)\,{\rm GeV}$$

- Compute $\Lambda^{(0)}_{\overline{\rm MS}}/\mu^{(0)}_{\rm dec} = (\Lambda^{(0)}_{\overline{\rm MS}}/\Lambda^{(0)}_{\mathcal{O}})\varphi^{(0)}_{\mathcal{O}}(\alpha^{(0)}_{\mathcal{O}}(\mu^{(0)}_{\rm dec}))$

Lattice set-up and parameters

Action

$N_\mathrm{f} = 3$ $\mathrm{O}(a)$-improved Wilson quarks and LW gauge action

Finite-volume coupling(s)

(Lüscher '10; Fodor et al. '12; Fritzsch, Ramos '13; ALPHA Collab. '16, '22)

$$\bar{g}^2_\mathrm{GF}(\mu) \propto \frac{\langle t^2 E_\mathrm{sp}(t,x)\delta_{Q,0}\rangle_\mathrm{SF}}{\langle \delta_{Q,0}\rangle_\mathrm{SF}}\Bigg|^{x_0=T/2\,,\,c=\sqrt{8t}/L}_{\mu=L^{-1},T=L,M=0} \qquad E_\mathrm{sp}(t,x) = \mathrm{tr}\{G_{kl}(t,x)G_{kl}(t,x)\}$$

RGI quark mass

(ALPHA Collab. '18)

$$M = Z_\mathrm{M}(\tilde{g}_0^2)\tilde{m}_\mathrm{q} = \left(\frac{M}{\overline{m}_\mathrm{SF}(\mu_\mathrm{dec})}\right) Z_\mathrm{m}^\mathrm{SF}(\tilde{g}_0^2, a\mu_\mathrm{dec})\tilde{m}_\mathrm{q} \qquad z = M/\mu_\mathrm{dec}$$

Line of constant physics (LCP)

(ALPHA Collab. '17, '18)

- $[\bar{g}^{(3)}_\mathrm{GF}(\mu_\mathrm{dec})]^2 = 3.95 \quad \Rightarrow \quad (\tilde{g}_0^2, L/a) \quad \Rightarrow \quad \mu_\mathrm{dec} = 789(15)\,\mathrm{MeV}$
- $z = 2, 4, \ldots, 12 \quad \Rightarrow \quad \tilde{m}_\mathrm{q} \quad \Rightarrow \quad M \approx 1.5, \ldots, 9.5\,\mathrm{GeV}$

$\mathrm{O}(a)$-improved parameters

(Lüscher et al. '96)

$$\tilde{g}_0^2 = g_0^2(1 + b_\mathrm{g}(g_0^2)am_\mathrm{q}) \qquad \tilde{g}_0^2 = \mathrm{const.} \quad \Rightarrow \quad a = \mathrm{const.}$$

$$\tilde{m}_\mathrm{q} = m_\mathrm{q}(1 + b_\mathrm{m}(g_0^2)am_\mathrm{q}) \qquad m_\mathrm{q} = m_0 - m_\mathrm{cr}$$

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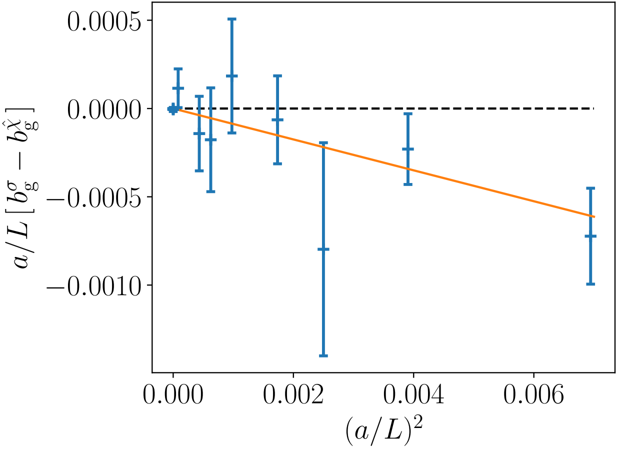
$$\tilde{m}_\mathrm{q} = m_\mathrm{q}(1 + b_\mathrm{m}(g_0^2)am_\mathrm{q}) \qquad m_\mathrm{q} = m_0 - m_\mathrm{cr}$$

Results for b_{g} from HYP-smeared Creutz ratios



$\sigma = \langle t^2 E(t) \rangle \qquad \hat{\chi} = -R^2 \tilde{\partial}_R \tilde{\partial}_T \log(W(R,T))|_{T=R=L/4}$