Spectral-function determination of complex electroweak amplitudes with lattice QCD

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Outline of the talk

Introduction

- Sketch of the problem of the analytic continuation for hadronic amplitudes above kinematical thresholds.
- Smeared hadronic amplitudes as a way-out to the problem.

The HLT method to evaluate smeared hadronic amplitudes

- The noise problem.
- Brief description of the HLT method.

Testing the method in a physical QCD example

- Smeared amplitudes for $D_s \to \bar{l}' l' l \nu_l$ decays

Conclusions and future perspectives

- The rare kaon decay $K \to \bar{l}' l' l \nu_l$.

Introduction

General statement of the problem (I)

An hadronic amplitude $H(E)$ can be safely extracted on the lattice only if energy E smaller than the energies of all the intermediate states contributing to $H(E)$.

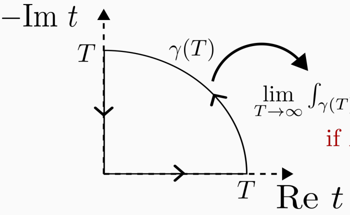
E.g. consider an hadronic amplitude of the form

$$H(E) = i\int_0^\infty dt\, e^{iEt}\, C(t), \quad C(t) \equiv \left\langle 0 \middle| T\left\{J_A(t)J_B(0)\right\} \middle| P\right\rangle \overset{t\geq 0}{=} \sum_{n=0}^{\infty} C_n\, e^{-iE_n t}$$

with J_A, J_B arbitrary currents and $\left|P\right\rangle$ an hadronic state.

If $E < E_n$ safe analytic continuation from Minkowskian to Euclidean space

$$H(E) = i\int_0^\infty dt\, e^{iEt}\, C(t) \overset{\tau = it}{=} \int_0^\infty d\tau\, e^{E\tau}\, C(-i\tau)$$



$$\lim_{T\to\infty} \int_{\gamma(T)} dt\, e^{Et} C(t) = 0$$

if $E < E_n$

General statement of the problem (II)

On a finite lattice, where non-analiticities are absent, we can access
$C_E(t) \equiv C(-it)$ for $0 \leq t \leq T$.

$$\boxed{H^T(E) = \int_0^T dt\, e^{Et}\, C_E(t) = \int_0^T dt\, e^{Et} \sum_{n=0}^{\infty} C_n e^{-E_n t} = \sum_{n=0}^{\infty} C_n \frac{1 - e^{-(E_n - E)T}}{E_n - E}}$$

if $E < E_n$:

$$H^T(E) \underset{T\to\infty}{\rightarrow} \sum_{n=0}^{\infty} \frac{C_n}{E_n - E}$$

✓

if $E_0 < E$:

$$H^T(E) \underset{T\to\infty}{\rightarrow} \sum_{n=0}^{\infty} \frac{C_n}{E_n - E} + \sum_n^{E_n < E} C_n \frac{e^{(E-E_n)T}}{E - E_n}$$

✗

- For $E_0 < E$ dominant $T-$divergent part of $H^T(E)$ must be subtracted $\implies$ difficult in presence of statistical errors, problem worsens when many states E_n below energy E.
- Above threshold hadronic amplitudes become complex (for $E = E_n$). How do we get imaginary parts?

Phenomenological relevance of the problem

Many phenomenologically relevant hadronic observables are affected by problems of analytic continuation, which hinder their lattice determination.

- Electromagnetic pion form factor $F_\pi(q^2)$ in the time-like region $q^2 > (2m_\pi)^2$ [Maiani & Testa, 1990].
- ... and generally hadronic scattering amplitudes above thresholds.

Many strategies have been put forward to circumvent the problem of analytic continuation

[Barata and Fredenhagen, 1991], [Bulava and Hansen, 2019] , [Bruno and Hansen, 2020]

In this seminar I will discuss the strategy proposed in [Frezzotti et. al, 2023] to tackle the problem of analytic continuation for observables which involve an hadron-to-vacuum QCD matrix element of the product of two currents.

Many interesting observables fall in this category, e.g. the P_{l_4} decays:

$$P \to \bar{l}' l' \bar{l} \nu_l \ , \qquad P = \text{flavoured and charged pseudoscalar meson.}$$

Hadronic amplitudes via the spectral representation (I)

The spectral density $\rho(E')$ of the correlator $C(t>0)$ is defined as

$$\boxed{\rho(E') = 2\pi \left\langle 0 \middle| J_A(0)\, \delta(\mathbb{H} - E')\, J_B(0) \middle| P \right\rangle}$$

- $\mathbb{H}$ is the QCD Hamiltonian.
- In $\rho(E')$ the delta function restrict the propagation to those states having energy E'.
- Its relation to the Minkowskian ($C(t)$) and Euclidean ($C_E(t)$) correlators can be easily worked out

$$C(t) \overset{t\geq 0}{=} \int_0^\infty \frac{dE'}{2\pi}\, \rho(E')\, e^{-iE't}, \qquad C_E(t) \overset{t\geq 0}{=} \int_0^\infty \frac{dE'}{2\pi}\, \rho(E')\, e^{-E't}$$

Spectral density $\rho(E')$ related to $C_E(t)$ through an inverse Laplace transform.

Hadronic amplitudes via the spectral representation (II)

- The hadronic amplitude $H(E)$ can be computed as

$$H(E) = \int_0^\infty dt\, e^{iEt}\, C(t) = \lim_{\epsilon\to 0^+} \int_0^\infty \frac{dE'}{2\pi}\, \rho(E') \int_0^\infty dt\, e^{-i(E'-E)t} f(\epsilon, t)$$

- $f(\epsilon, t)$ is any regulator for the time integral, with $f(0,t) = 1$.
- E.g. $f(\epsilon, t) = \exp(-\epsilon t)$, $\exp(-\epsilon^2 t^2/2)$. Using standard $\epsilon-$prescription:

$$\boxed{H(E) = \lim_{\epsilon\to 0^+} \int_0^\infty \frac{dE'}{2\pi}\, \frac{\rho(E')}{E' - E - i\epsilon}}$$

Note: the lower-end of integration is actually always positive since the support of the spectral density is $[E_0, \infty]$, with $E_0 > 0$.

$$\int_0^\infty dE' \to \int_{E_0}^\infty dE'$$

Hadronic amplitudes via the spectral representation (III)

From the knowledge of $\rho(E')$, the real and imaginary part of $H(E)$ can be computed:

$$\mathrm{Re}\,[H(E)] = \lim_{\epsilon\to 0^+}\int_0^\infty \frac{dE'}{2\pi}\,\rho(E')\,\frac{E'-E}{(E-E')^2+\epsilon^2} = \mathrm{P.V.}\int_0^\infty \frac{dE'}{2\pi}\,\frac{\rho(E')}{E'-E}$$

$$\mathrm{Im}\,[H(E)] = \lim_{\epsilon\to 0^+}\int_0^\infty \frac{dE'}{2\pi}\,\rho(E')\,\frac{\epsilon}{(E-E')^2+\epsilon^2} = \frac{\rho(E)}{2}$$

For $E < E_0$, since $\rho(E) = 0$, $\mathrm{Im}\,[H(E)] = 0$ and the $\mathrm{P.V.}$ can be dropped:

$$\mathrm{Re}\,[H(E)] = \int_{E_0}^\infty \frac{dE'}{2\pi}\rho(E')\,\underbrace{\int_0^\infty dt\,e^{-(E'-E)t}}_{=(E'-E)^{-1}\ \text{if}\ E'<E} = \int_0^\infty dt\,e^{Et}\,C_E(t)$$ ✓

For $E > E_0$, $\lim \epsilon \to 0^+$ can be taken only after evaluating the energy integral.

The smeared amplitude $H(E,\varepsilon)$

We propose to employ for $E > E_0$ the previous representation, evaluate the smeared amplitude $H(E,\epsilon)$ at finite ϵ, and then take $\lim \epsilon \to 0^+$.

$$H(E,\varepsilon) \equiv \int_{E_0}^{\infty} \frac{dE'}{2\pi} \rho(E')\, K(E'-E,\varepsilon)\ , \qquad K(E'-E,\varepsilon) \equiv \frac{1}{E'-E-i\varepsilon}$$

- Does the smeared $H(E,\varepsilon)$ have a physical interpretation? By using

$$\lim_{\eta\to 0^+} \frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega\ \frac{\varepsilon}{(E-\omega)^2+\varepsilon^2}\ \frac{1}{E'-\omega-i\eta} = \frac{1}{E'-E-i\varepsilon}$$

It follows that

$$H(E,\varepsilon) = \int_{-\infty}^{+\infty} d\omega\ \frac{1}{\pi} \frac{\varepsilon}{(E-\omega)^2+\varepsilon^2}\, H(\omega)$$

The smeared amplitude represents, in fact, an energy-smearing of size ε of the physical amplitude $H(E)$.

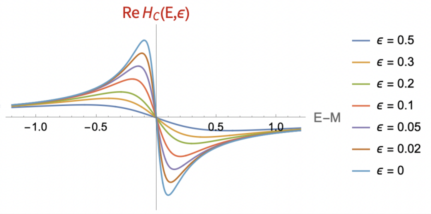
The smeared amplitude in a simple model

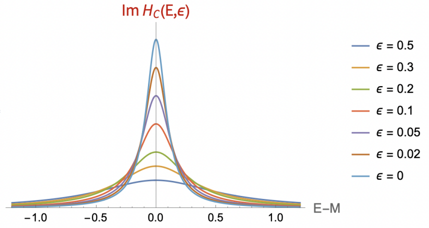
It is useful to look at the smeared amplitude in a simple one-resonance model for the spectral density $\rho(E')$

$$\boxed{\rho(E') = \frac{A\,\Gamma}{(E-M)^2 + (\frac{\Gamma}{2})^2}\,\theta(E) \quad \Longrightarrow \quad H(E) \simeq \frac{A}{M - E - i\,\frac{\Gamma}{2}}}$$

The effect of the smearing is to simply shift $\Gamma \to \Gamma + 2\varepsilon$

$$H(E,\varepsilon) \simeq \frac{A}{M - E - i\,(\frac{\Gamma}{2} + \varepsilon)}$$





The vanishing ε extrapolation (I)

As we will see later, $H(E,\varepsilon)$ can be computed from $C_E(t)$ only. The smallest ε that can be actually determined depends mainly on the stat. errors of $C_E(t)$.

Starting from $H(E,\varepsilon)$ at finite ε can we make contact with $H(E)$?

- It can be shown that in general

$$\mathrm{Re}[H(E,\varepsilon)] = \mathrm{Re}[H(E)] + \mathcal{O}(\varepsilon) \;, \qquad \mathrm{Im}[H(E,\varepsilon)] = \mathrm{Im}[H(E)] + \mathcal{O}(\varepsilon)$$

- For which values of ε does the linear regime set in? Answer strongly depends on $\rho(E')$ structure.

- Let's employ again the one-resonance model to get some understanding

$$H(E,\varepsilon) \simeq \frac{A}{M - E - i\left(\frac{\Gamma}{2} + \varepsilon\right)} \implies \frac{H(E,\varepsilon)}{H(E)} = \left[1 + \frac{i\varepsilon}{\Delta(E)} e^{-i\phi(E)}\right]^{-1}$$

$$\Delta(E) = \sqrt{(E-M)^2 + (\Gamma/2)^2} \quad , \quad \tan\phi(E) = \frac{\Gamma/2}{E-M}$$

The vanishing ε extrapolation (II)

The Breit-Wigner model shows that the condition for onset of linear regime is

$$\varepsilon \ll \Delta(E) = \sqrt{(E-M)^2 + (\Gamma/2)^2}$$

- For general $H(E)$, ε should be smaller than the typical size $\Delta(E)$ of the interval around E over which $H(E)$ is significantly varying

$$\frac{1}{\Delta(E)} \equiv \left| \frac{1}{H(E)} \frac{\partial H(E)}{\partial E} \right|$$

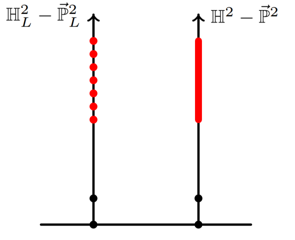
- In $E \pm \Delta(E)$, $H(E)$ varies by $\mathcal{O}(100\%)$.
- If E is close to a narrow-resonance peak, very small values of ε are needed to observe the onset of linear regime!.
- However, at the price of introducing some model dependence, useful information can be extracted also in this region. We will come back to this point later.

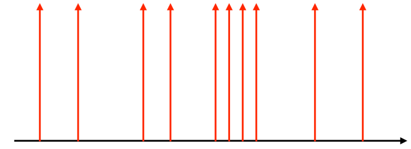
Finite volume effects

Lattice calculations are always performed in a finite volume $V = L^3$.

- Spectrum of the finite-volume Hamiltonian $\mathbb{H}_L$ is discrete.
- Finite-volume $\rho(E', L)$ is always a sum of isolated δ-peaks

$$\rho(E', L) = \sum_n c_n(L)\, \delta(E' - E_n(L))$$



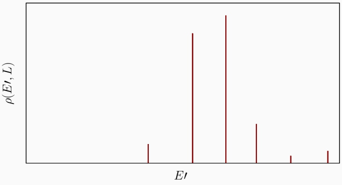


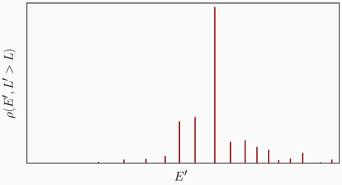
[Taken from N. Tantalo talk]

- $\rho(E', L)$ cannot be directly associated to any infinite-volume quantity.
- The ε-smeared amplitude $H(E, \varepsilon)$ has instead a well-defined infinite-volume limit...

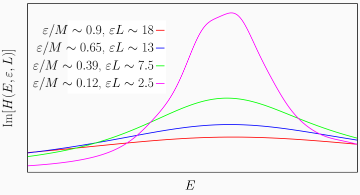
One-resonance model in finite volume

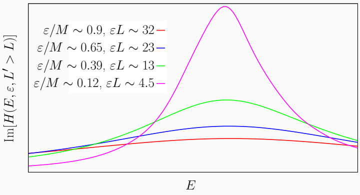
- In the finite volume, multi-particles part of the spectrum is always discrete.
- Multi-particles decays of a resonance, e.g. $\phi \to K^+K^-(k_n)$ allowed only if relative momentum k_n satisfies quantization conditions.
- Let's take a look to $\rho(E', L)$ for a 2-particle resonance decay:





- The smeared amplitude has instead small FVEs if $\varepsilon L \gg 1$ [Bulava et al, 2021].





Summary of the theoretical part

- We have seen that to overcome the problem of analytic continuation, the smeared amplitude $H(E,\varepsilon)$ must be introduced.
- $H(E,\varepsilon)$ admits a polynomial expansion of the form

$$H(E,\varepsilon) = H(E) + \sum_{n=1}^{\infty} a_n \left(\frac{\varepsilon}{\Delta(E)}\right)^n$$

- The condition for the onset of linear regime is $\varepsilon \ll \Delta(E)$.
- To evaluate $H(E)$ from finite-volume simulations the correct double-limit to be taken is

$$H(E) = \lim_{\varepsilon\to 0^+} \lim_{L\to\infty} H(E,\varepsilon,L)$$

- Finite-ε and finite-volume effects small if

$$\boxed{1/L \ll \varepsilon \ll \Delta(E)}$$

The HLT method to reconstruct smeared hadronic amplitudes

Evaluating $H(E,\varepsilon)$ from our lattice input, $C_E(t)$

- Let us try to evaluate $H(E,\varepsilon)$ from the knowledge of $C_E(t)$ at a discrete set of time $t = a, \ldots, Na$.

- To do so, it is sufficient to find an approximation of the kernel functions of the type $(\mathrm{I} = \{\mathrm{Re}, \mathrm{Im}\})$

$$K_{\mathrm{I}}(E'-E,\varepsilon) \simeq \sum_{n=1}^{N} g_n^{\mathrm{I}}(E) e^{-aE'n}$$

- Indeed, from the knowledge of $g_n^{\mathrm{I}}(E)$ one gets:

$$\boxed{\begin{aligned} \sum_{n=1}^{N} g_n^{\mathrm{Re}}(E)\, C_E(na) &= \int_0^\infty \frac{\mathrm{d}E'}{2\pi} \left(\sum_{n=1}^{N} g_n^{\mathrm{Re}}(E)\, e^{-naE'} \right) \rho(E') \simeq \mathrm{Re}[H(E,\varepsilon)] \\ \sum_{n=1}^{N} g_n^{\mathrm{Im}}(E)\, C_E(na) &= \int_0^\infty \frac{\mathrm{d}E'}{2\pi} \left(\sum_{n=1}^{N} g_n^{\mathrm{Im}}(E)\, e^{-naE'} \right) \rho(E') \simeq \mathrm{Im}[H(E,\varepsilon)] \end{aligned}}$$

Finding the coefficients g_n^{I}

To obtain the coefficients $g_n^{\mathrm{I}}(E)$ we can minimize the L^2-distance between target and reconstructed function

$$A^{\mathrm{I}}[\boldsymbol{g}] \equiv \int_0^\infty \frac{\mathrm{d}E'}{2\pi} \left| K_{\mathrm{I}}(E'-E,\varepsilon) - \sum_{n=1}^{N} g_n e^{-naE'} \right|^2$$

- Minimization of $A^{\mathrm{I}}[\boldsymbol{g}]$ gives the coefficients $g_n^{\mathrm{I}}(E)$ as

$$g_n^{\mathrm{I}}(E) = \sum_{m=1}^{N} \left(H_N^{-1}\right)_{nm} f_m^{\mathrm{I}}, \qquad f_n^{\mathrm{I}} \equiv \int_0^\infty \mathrm{d}E'\, K_{\mathrm{I}}(E'-E,\varepsilon)\, e^{-naE'}$$

- H_N is the $N \times N$ Hilbert matrix, textbook example of ill-conditioned matrix

$$(H_N)_{nm} = \frac{1}{n+m-1}, \qquad \det H_N \approx N^{-1/4}(2\pi)^N 4^{-N^2}$$

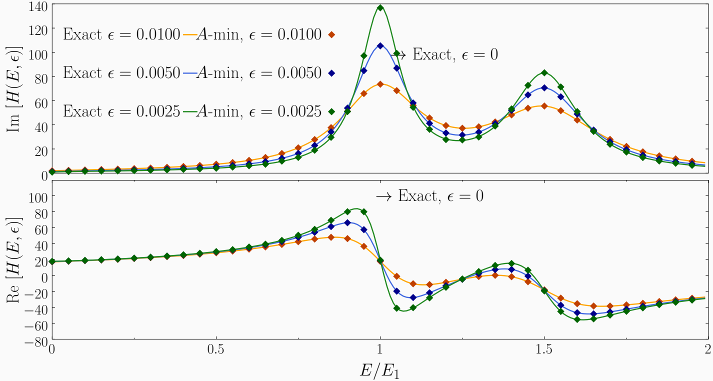
- E.g. $\det H_{10} \simeq \mathcal{O}(10^{-53})$, $\det H_{20} \simeq \mathcal{O}(10^{-226})$. Let's take a look at some results. . .

The reconstruction at work in a toy-model without errors

Two-resonances model:

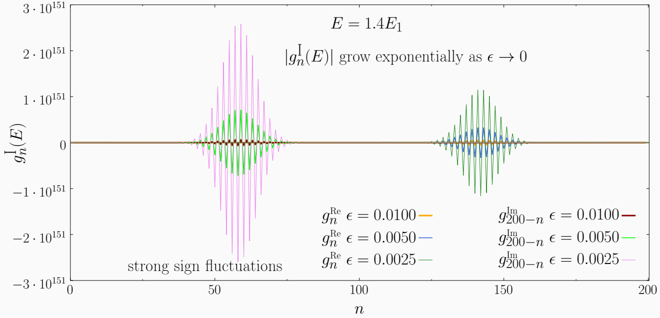
$$\rho(E) = \frac{1}{\pi} \sum_{n=1,2} \frac{\Gamma_n/2}{(E-E_n)^2 + (\Gamma_n/2)^2}\,, \qquad \begin{array}{l} E_1 = 0.10,\ \Gamma_1 = 10^{-2} \\ E_2 = 0.15,\ \Gamma_2 = 2\cdot 10^{-2} \end{array}$$

- We computed $C_E(t)$ with extended machine precision for $t = 1, \ldots, 200$.
- $H(E,\epsilon)$ reconstructed from $C_E(t)$ using $A[\boldsymbol{g}]$-minimization method.



The coefficients g_n^{I}

While the $A[\boldsymbol{g}]$-minimization leads to a perfect reconstruction of $H(E,\varepsilon)$, the resulting coefficients g_n^{I} are strongly oscillating



- If $C_E(t)$ is known with some uncertainty $\delta C_E(t)$, the resulting error on the smeared hadronic amplitude BLOWS UP!!

$$\Delta H(E,\varepsilon) = \sqrt{\sum_n \left[g_n^{\mathrm{Re}}(E)\,\delta C_E(na)\right]^2} + i\sqrt{\sum_n \left[g_n^{\mathrm{Im}}(E)\,\delta C_E(na)\right]^2}$$

The smeared amplitude from a Backus-Gilbert-like approach

We need a regularization mechanism to tame the oscillations of the $\boldsymbol{g}^{\mathrm{I}}$ coefficients.

The Hansen-Lupo-Tantalo (HLT) method provides the coefficients $\boldsymbol{g}^{\mathrm{I}}(E)$ minimizing a functional $W^{\mathrm{I}}[\boldsymbol{g}]$ which balances syst. and stat. errors of reconstructed $H(E, \varepsilon)$

$$\boxed{W^{\mathrm{I}}[\boldsymbol{g}] = \frac{A^{\mathrm{I}}[\boldsymbol{g}]}{A^{\mathrm{I}}[\mathbf{0}]} + \lambda B[\boldsymbol{g}] \;, \qquad \left.\frac{\partial W^{\mathrm{I}}[\boldsymbol{g}]}{\partial g}\right|_{\boldsymbol{g}=\boldsymbol{g}^{\mathrm{I}}} = 0}$$

$$A^{\mathrm{I}}[\boldsymbol{g}] = \int_{E_{\min}}^{\infty} \frac{\mathrm{d}E'}{2\pi} \left| K_{\mathrm{I}}(E'-E, \varepsilon) - \sum_{n=1}^{N} g_n e^{-naE'} \right|^2 \Longleftarrow \text{(syst.)}^2 \text{ error due to reconstruction}$$

$$B[\boldsymbol{g}] \propto \sum_{n_1, n_2 = 1}^{N} g_{n_1} g_{n_2} \operatorname{Cov}\left(C_E(an_1), C_E(an_2)\right) \Longleftarrow \text{(stat.)}^2 \text{ error of reconstructed } H(E, \varepsilon)$$

- λ is trade-off parameter $\Longrightarrow$ tuned for optimal balance of syst. and stat. errors.
- $E_{\min}$ should only satisfy the condition $E_{\min} < E_0$.

Testing the spectral reconstruction method in a physical QCD case: the $D_s \to \bar{l}' l' l \nu_l$ decay

The $P \to \bar{l}'l'l\nu_l$ decay

To test the effectiveness of the smeared-amplitude method, we considered a phenomenological interesting electroweak amplitude which features the problem of the analytic continuation.

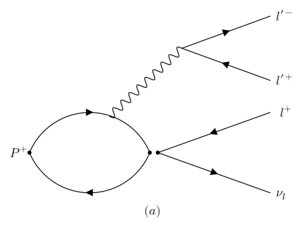
- This is the electroweak decay of a flavoured and charged pseudoscalar meson P into a dilepton $(\bar{l}'l')$ and a lepton pair $(l\nu_l)$.
- It proceeds via

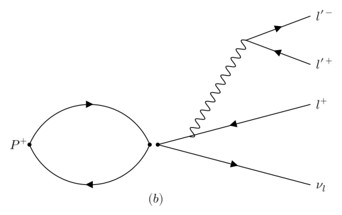
$$P \to \gamma^* l\nu_l \to \bar{l}'l'l\nu_l$$

- The intermediate virtual photon γ^* can be either emitted from the final-state lepton l (so-called Bremsstrahlung contribution) or from a quark (so-called structure-dependent contribution).
- They are rare decays with decay rates of order $\mathcal{O}(G_F^2\,\alpha_{\rm em}^2)$, which can thus be interesting probe of NP beyond the SM.

Relevant Feynman diagrams for the process

To lowest-order in $\alpha_{\rm em}$ and G_F the relevant Feynman diagrams are





- Diagram (b) is perturbative, only QCD input is decay constant f_P.
- Diagram (a) is non-perturbative. Virtual photon γ^* emitted from one of the two valence quarks.

Non-perturbative QCD contribution encoded in the hadronic tensor

$$H_W^{\mu\nu}(k,\boldsymbol{p}) = i\int dt\, e^{iEt}\, \mathrm{T}\left\langle 0\middle| J^{\mu}_{\rm em}(t,\boldsymbol{k}) J^{\nu}_{W}(0) \middle| P(\boldsymbol{p})\right\rangle, \quad W = V, A$$

- $k = (E,\boldsymbol{k})$ is photon 4-momentum, $\boldsymbol{p}$ is P-meson 3-momentum (from now on we set $\boldsymbol{p} = \boldsymbol{0}$).

First and second time-ordering contributions

To understand why/where there are problems of analytic continuation let us consider the two time-orderings separately

$$H_W^{\mu\nu}(k,\mathbf{0}) = i\int_{-\infty}^{\infty} dt\, e^{iEt}\, \mathrm{T}\left\langle 0 \middle| J^\mu_{\mathrm{em}}(t,\boldsymbol{k}) J^\nu_W(0) \middle| P(\mathbf{0}) \right\rangle =$$

$$= \underbrace{i\int_{-\infty}^{0} dt\, e^{iEt} \left\langle 0 \middle| J^\nu_W(0) J^\mu_{\mathrm{em}}(t,\boldsymbol{k}) \middle| P(\mathbf{0}) \right\rangle}_{H^{\mu\nu}_{W,1}(k)} + \underbrace{i\int_{0}^{\infty} dt\, e^{iEt} \left\langle 0 \middle| J^\mu_{\mathrm{em}}(t,\boldsymbol{k}) J^\nu_W(0) \middle| P(\mathbf{0}) \right\rangle}_{H^{\mu\nu}_{W,2}(k)}$$

We now make use of

$$J^\mu_{\mathrm{em}}(t,\boldsymbol{k}) = e^{i(\mathbb{H}-i\varepsilon)t}\, J^\mu_{\mathrm{em}}(0,\boldsymbol{k})\, e^{-i(\mathbb{H}-i\varepsilon)t}$$

and performing the time-integral one gets

$$\boxed{\begin{aligned} H^{\mu\nu}_{W,1}(k) &= \langle 0 | J^\nu_W(0) \frac{1}{\mathbb{H} + E - M_P - i\varepsilon} J^\mu_{\mathrm{em}}(0,\boldsymbol{k}) | P(\mathbf{0}) \rangle \\[2ex] H^{\mu\nu}_{W,2}(k) &= \langle 0 | J^\mu_{\mathrm{em}}(0,\boldsymbol{k}) \frac{1}{\mathbb{H} - E - i\varepsilon} J^\nu_W(0) | P(\mathbf{0}) \rangle \end{aligned}}$$

Threshold problems at large virtualities k^2

We now insert a complete set of states between the two currents:

$$H^{\mu\nu}_{W,1}(k) = \sum_r \frac{\langle 0|J^\nu_W(0)|r\rangle\langle r|J^\mu_{\mathrm{em}}(0,\boldsymbol{k})|P(\boldsymbol{0})\rangle}{E_r + E - M_P - i\epsilon}, \quad \boldsymbol{p}_r = -\boldsymbol{k} \ ,$$

$$H^{\mu\nu}_{W,2}(k) = \sum_n \frac{\langle 0|J^\mu_{\mathrm{em}}(0,\boldsymbol{k})|n\rangle\langle n|J^\nu_W(0)|P(\boldsymbol{0})\rangle}{E_n - E - i\epsilon}, \quad \boldsymbol{p}_n = +\boldsymbol{k} \ ,$$

- $|r\rangle$ states have same flavour content as P meson. Their masses (M_r) are larger than M_P which implies

$$E_r = \sqrt{M_r^2 + |\boldsymbol{k}|^2} > M_P - E \qquad ✓$$

- $|n\rangle$ states are unflavoured and have $J^P = 1^-$. Threshold at

$$E_n = \sqrt{M_n^2 + |\boldsymbol{k}|^2} = E \implies \sqrt{k^2} \equiv \sqrt{E^2 - |\boldsymbol{k}|^2} = M_n$$

- In the second TO we have problems of analytic continuation! For $P = D_s$ this occurs at $\sqrt{k^2_{\mathrm{th}}} = M_\phi$. ✗

Lattice calculation for $P = D_s$

We compute on the lattice the Euclidean correlation function

$$C_W^{\mu\nu}(t,\boldsymbol{k}) \equiv T\langle 0|J_{\rm em}^{\mu}(t,\boldsymbol{k})\, J_W^{\nu}(0)|D_s(\boldsymbol{0})\rangle \ ,$$

- For this proof-of-principle calculation we used a single $N_f = 2+1+1$ Wilson-clover twisted-mass ETMC gauge ensemble at the physical point

Ensemble	a [fm]	L/a	T/a	$N_{\rm confs}$	$N_{\rm sources}$
cB211.072.64	0.079	64	128	302	4

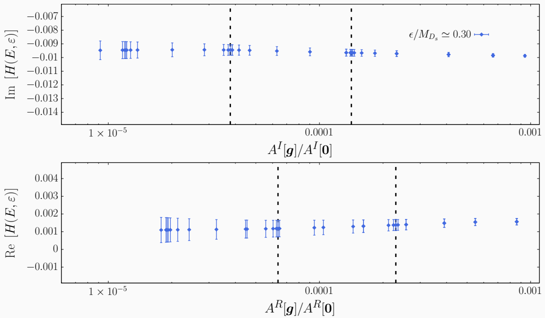
- Calculation restricted to a single momentum $\boldsymbol{k}$ along z-axis, with $x_\gamma = 2|\boldsymbol{k}|/M_{D_s} = 0.2$.

- Since problem of an. cont. occurs only in 2nd TO and for emission of γ^* from a strange quark, we concentrate only on this contribution.
- We show results for $W = V$ and from now on set $C_E(t) = C_V^{12}(t,\boldsymbol{k})$.

$$H(E,\varepsilon) = \int_0^\infty \frac{dE'}{2\pi}\,\rho(E')\,K(E'-E,\varepsilon)\ , \quad C_E(t) = \int_0^\infty \frac{dE'}{2\pi} e^{-E't}\,\rho(E')$$

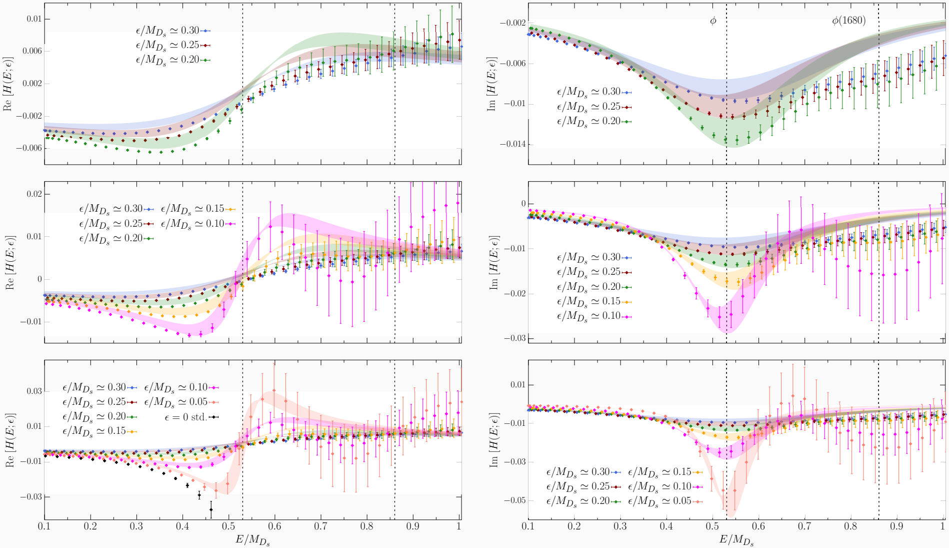
Stability analysis

To evaluate the smeared amplitude $H(E, \varepsilon)$ via the HLT method we perform the so-called stability analysis



- We monitor the evolution of $H(E, \varepsilon)$ as a function of the trade-off parameter λ (results plotted in terms of corresponding $A[\boldsymbol{g}]/A[\mathbf{0}]$).
- When λ sufficiently small, $H(E, \varepsilon)$ is stable under variations of λ.
- In this region, where syst.err. $<$ stat.err, we can determine $H(E, \varepsilon)$.

The smeared amplitude $H(E, \varepsilon)$



$\varepsilon \in [100 - 600]$ MeV. For $E > E_\phi = \sqrt{M_\phi^2 + |\boldsymbol{k}|^2}$ errors increase by decreasing ε.

Extrapolation of the results to vanishing ε

The condition for the onset of the linear regime and for small FVEs is

$$1/L \ll \varepsilon \ll \Delta(E)$$

We distinguish three energy regions:

- $E < E_\phi$: no problems of analytic continuation, we could directly set $\varepsilon = 0$ and evaluate $H(E)$ (which is purely real) as

$$H(E) = \int_0^\infty dt\, e^{Et}\, C_E(t) \qquad [1]$$

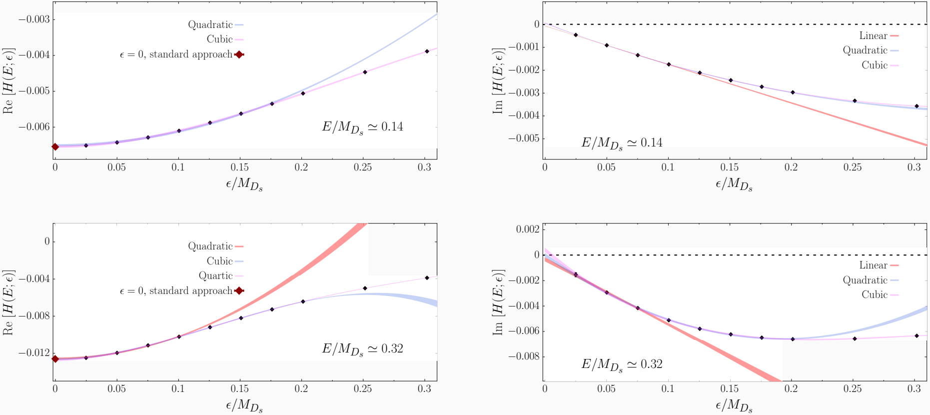
we however evaluate also $H(E, \varepsilon)$ to check that $\lim_{\varepsilon \to 0} H(E, \varepsilon)$ reproduces [1].

- $E \sim E_\phi$: $\Delta(E)$ very small due to narrow ϕ-resonance. Large FVEs expected, on currently available L, $\phi \to K^+ K^-$ is forbidden ($\Gamma_\phi = 0$). Not everything is lost, we employ in this region a Breit-Wigner model for $\phi-$resonance contribution (Γ_ϕ from PDG, A is a fit parameter)

$$\rho_\phi(E') = \frac{A\Gamma_\phi}{(E_\phi - E')^2 + (\frac{\Gamma_\phi}{2})^2} \implies H_\phi(E, \varepsilon) = \frac{A}{E_\phi - E - i(\frac{\Gamma_\phi}{2} + \varepsilon)}$$

- $E \gg E_\phi$: $\Delta(E)$ is larger, and in this region we attempt a polynomial extrapolation in ε. Noise of the data however increases (smallest ε for which errors are still under control, increases with E).

Extrapolation to vanishing ε for $E < E_\phi$

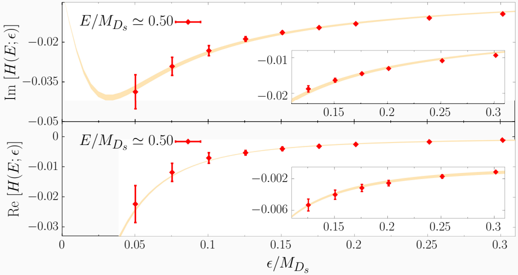


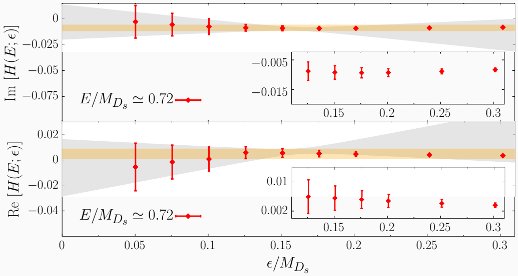
- As expected, through a polynomial extrapolation in ε we are able to recover in this energy region the results of the "standard approach"

$$H(E) = \int_0^\infty dt\, e^{Et}\, C_E(t)$$

- $\lim_{\varepsilon\to 0^+} \mathrm{Im}[H(E,\varepsilon)] = 0.$

Extrapolation to vanishing ε for $E \sim E_\phi$ and $E \gg E_\phi$



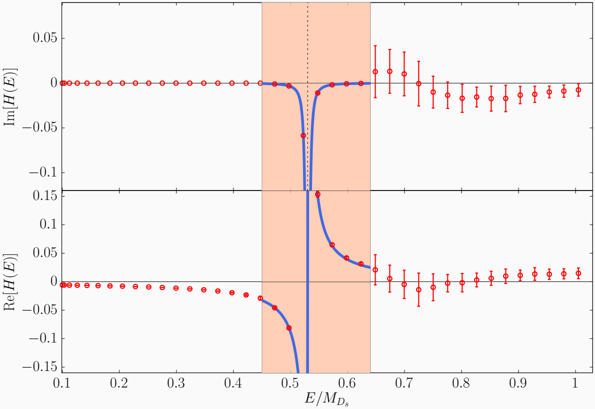


- $E \sim E_\phi$: extrapolation employing the BW model $H_\phi(E, \varepsilon)$.

- $E \gg E_\phi$: Smooth ε-dependence observed. constant and linear ε-extrapolation in the region $\varepsilon < \Delta_\phi(E) \equiv \sqrt{(E - E_\phi)^2 + (\Gamma_\phi/2)^2}$.

The amplitude $H(E)$

Combining the $\varepsilon \to 0^+$ extrapolations at the different values of E we get



- Orange band represents the area around E_ϕ where the $\varepsilon \to 0^+$ extrapolation has been carried using $H_\phi(E, \varepsilon)$.
- Data obtained through model-independent polynomial extrapolation are smoothly connected with the ones obtained using $H_\phi(E, \varepsilon)$.

Conclusions

- We propose a new method to extract complex electroweak amplitudes involving two EW-currents and an hadronic state or the vacuum in the external states.
- In our approach, the problem of analytic continuation which is present above hadronic threshold is bypassed by evaluating, via spectral reconstruction, hadronic amplitudes $H(E,\epsilon)$ smeared over a finite-energy interval ϵ around E, and then taking $\lim \epsilon \to 0^+$.
- We performed a pilot-study on a single ETMC ensemble, computing the hadronic amplitude relative to $D_s \to \bar{l}' l' l \nu_l$ decays (below and above threshold(s)) for $\epsilon \in [100 - 600]$ MeV, using the HLT method.
- Vanishing-ε extrapolation performed through a polynomial fit of $H(E,\varepsilon)$, except for the energy region $E \sim E_\phi$, where $H(E)$ is non-smooth due to the narrow ϕ-resonance. In this energy-region we employed a Breit-Wigner model to describe the $\varepsilon-$dependence of the lattice data.

Work in progress

- Full calculation for $K \to \bar{l}' l' l \nu_\ell$, where no narrow resonances are present.
- Application of the method to FCNC processes where similar problems arise, e.g. $B_s \to \mu^+ \mu^- \gamma$.

Thank you for your attention!