Pseudoscalar transition form factors and the muon $g-2$

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Based on 2305.04570 [hep-lat]

DESY - Lattice Seminar
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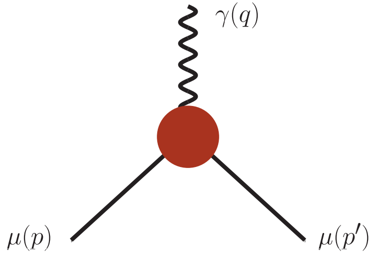
CPT Centre de Physique Théorique

$$\vec{\mu} = g \left(\frac{Qe}{2m} \right) \vec{S}$$

- Corrections to the vertex function : Dirac and Pauli form factors

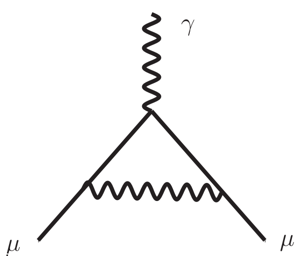


$$= -ie\,\overline{u}(p',\sigma')\Gamma_\mu(p',p)u(p,\sigma)$$

$$= -ie\,\overline{u}(p',\sigma')\left[\gamma_\mu F_1(q^2) + \frac{i\sigma_{\mu\nu}q_\nu}{2m}F_2(q^2)\right]u(p,\sigma)$$

$$F_2(0) = a_\mu = \tfrac{g-2}{2}$$

- Classical result : $g = 2$ for elementary fermions (Dirac equation)



Quantum field theory : $a_\mu = \frac{g-2}{2} \neq 0$

$\hookrightarrow$ generated by quantum effects

$$a_\mu^{(1)} = \frac{\alpha}{2\pi}$$ [Schwinger '48]

"*The anomalous magnetic moment of the muon in the Standard Model*" [Phys.Rept. 887 (2020) 1-166]

Contribution	$a_\mu \times 10^{11}$
- **QED** (10^{th} order)	$116\,584\,718.931 \pm 0.104$
- **Electroweak**	153.6 ± 1.0
- **Strong interaction**	
HVP (LO)	$6\,931 \pm 40$
HVP (NLO)	-98.3 ± 0.7
HVP (NNLO)	12.4 ± 0.1
HLbL	92 ± 18
Total (Standard Model)	$116\,591\,810 \pm 43$
Experiment	$116\,592\,059 \pm 22$

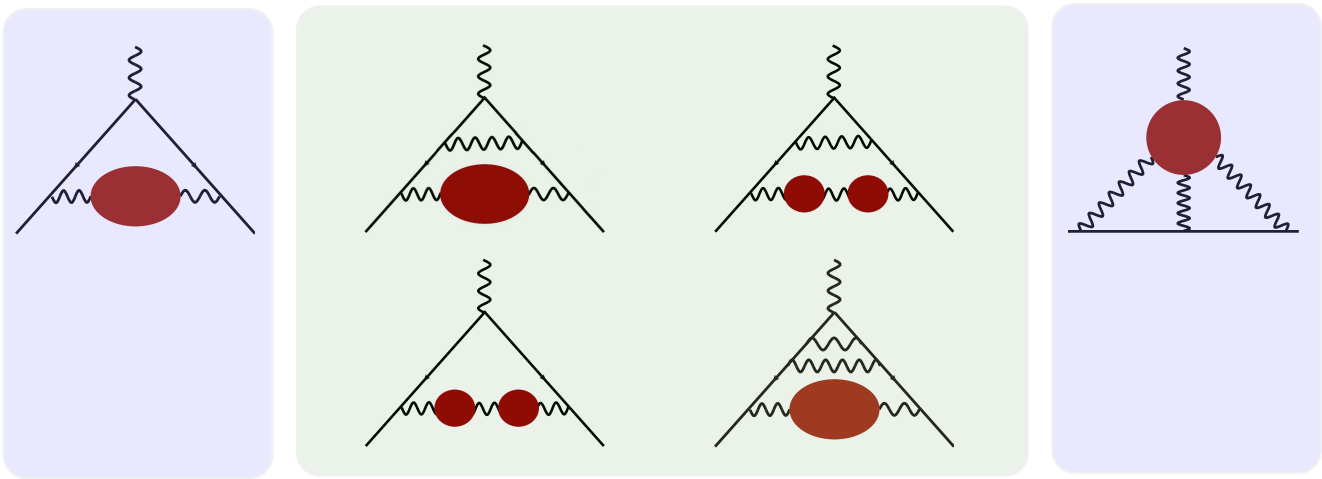
- Theory error larger than current experimental precisions
- Error budget **dominated by hadronic contributions : LO-HVP and HLbL**

$\rightarrow$ HVP : precision of a few permil is needed

$\rightarrow$ HLbL : precision of 10% would be sufficient

$\rightarrow$ dominated by low-energy physics where QCD is non-perturbative

LO HVP (α^2) **NLO HVP** (α^3) **NNLO HVP** (α^4) **HLbL** (α^3)

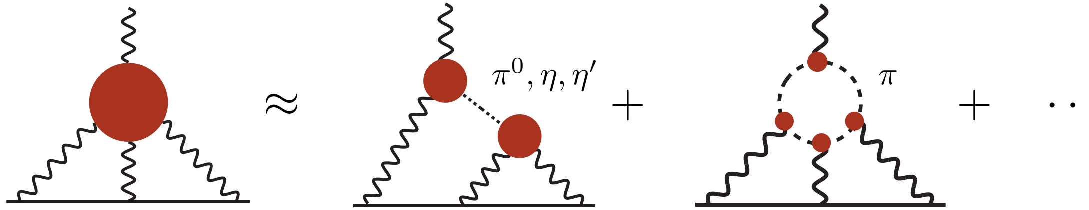


- LO-HVP
- NLO HVP and NNLO HVP differ by the QED kernel functions

  $\longrightarrow$ NLO HVP : same order as HLbL

  $\longrightarrow$ not negligible, but error under control

$a_\mu^{\mathrm{HLbL}} =$



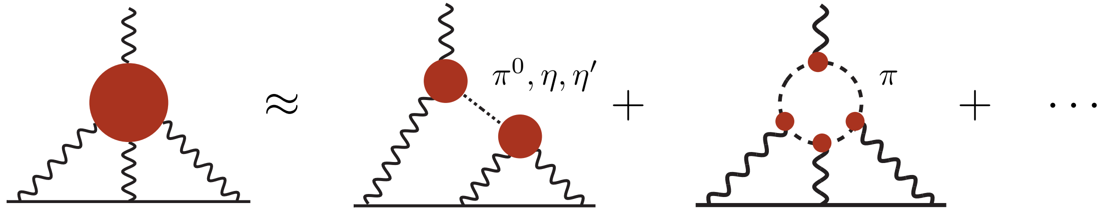
Dispersive framework ('21)	$a_\mu \times 10^{11}$
π^0, η, η'	93.8 ± 4
pion/kaon loops	-16.4 ± 0.2
S-wave $\pi\pi$	-8 ± 1
axial vector	6 ± 6
scalar + tensor	-1 ± 3
q-loops / short. dist. cstr	15 ± 10
charm + heavy q	3 ± 1
sum	92 ± 19
Mainz '22	109.6 ± 15.9
RBC/UKQCD '23	124.7 ± 15.2

Two approaches on the lattice :

π^0, η, η' : accessible on the lattice

direct lattice calculation

- both approaches are complementary
- target precision for HLbL : <10%

$a_\mu^{\mathrm{HLbL}} =$ 

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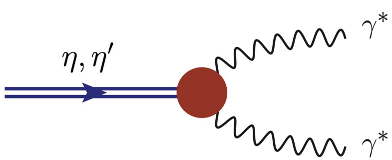

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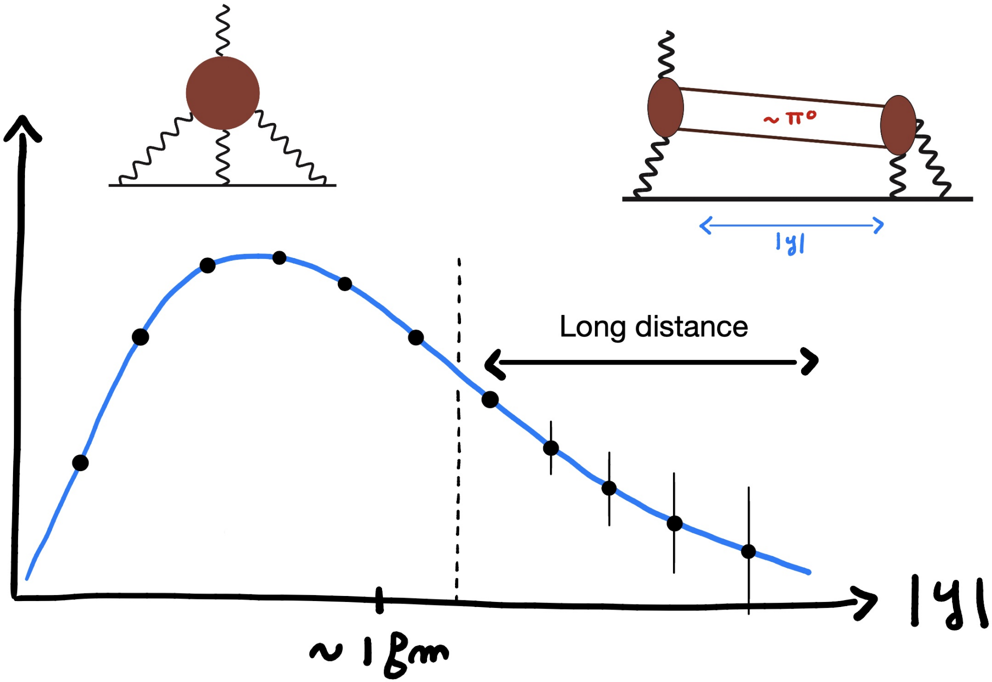
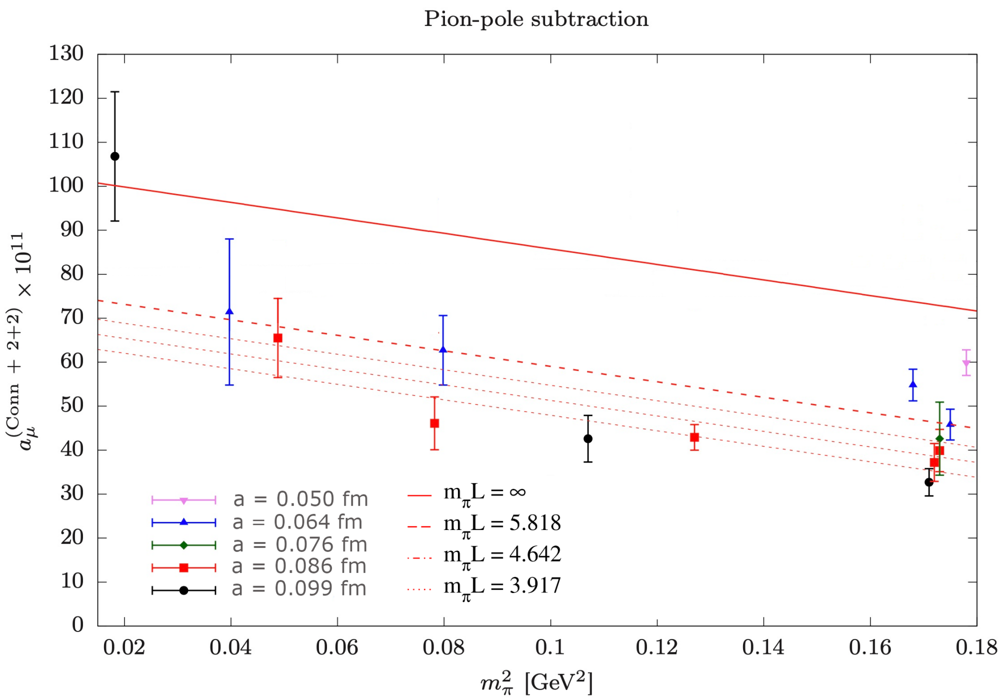
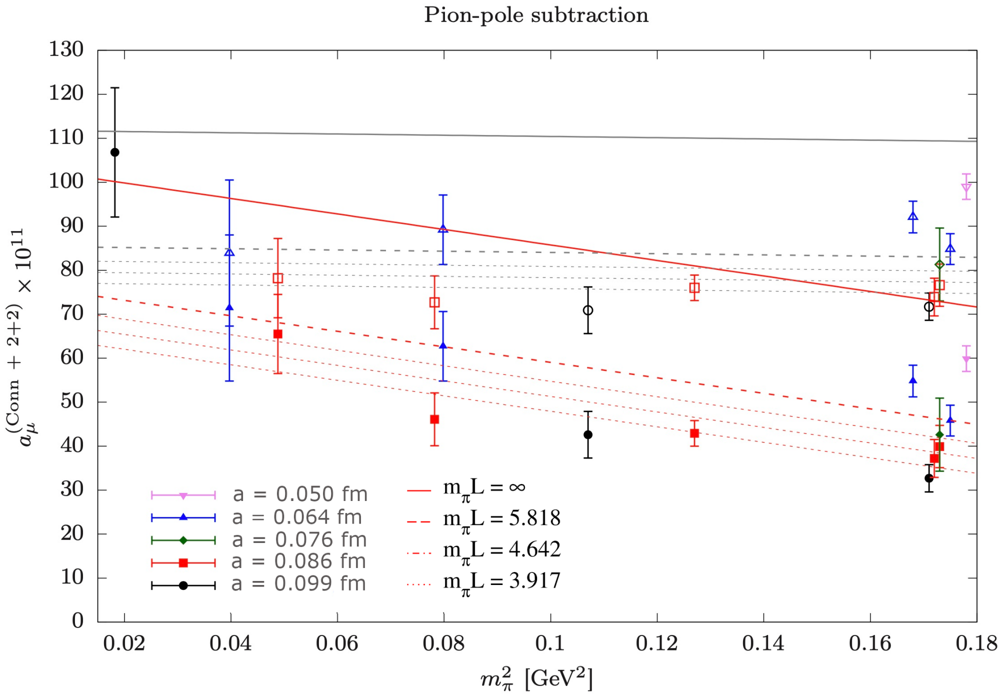


- Noisy long distance dominated by pion-pole
- Finite-volume effects → long-distance physics : pion-pole

Pseudoscalar-pole plays an important role in the direct calculation

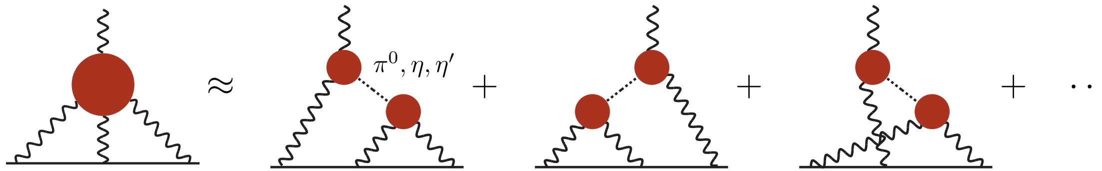


- Results obtained with the Mainz group [Eur.Phys.J.C 81 (2021) 7, 651]
- Statistical precision deteriorates rapidly at low pion masses
- Correction : $a_\mu^{\mathrm{hlbl,cor}}(a, m_\pi) = a_\mu^{\mathrm{hlbl,data}}(a, m_\pi) + \left(a_\mu^{\pi^0,\mathrm{phys}(a,m_\pi)} - a_\mu^{\pi^0}(a, m_\pi) \right)$



- Open symbols : $a_\mu^{\mathrm{hlbl,cor}}(a,m_\pi) = a_\mu^{\mathrm{hlbl,data}}(a,m_\pi) + \left(a_\mu^{\pi^0,\mathrm{phys}(a,m_\pi)} - a_\mu^{\pi^0}(a,m_\pi) \right)$

Chiral extrapolation significantly improved !



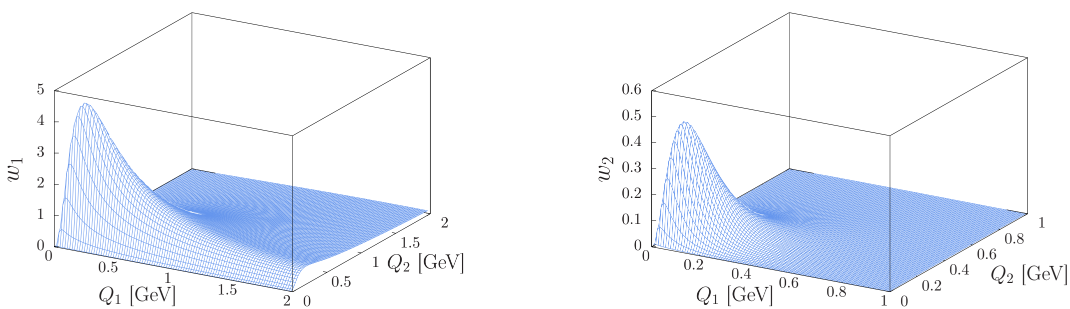
[Jegerlehner & Nyffeler '09]

$$a_\mu^{\mathrm{HLbL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \; w_1(Q_1,Q_2,\tau)\, \mathcal{F}_{P\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2)\, \mathcal{F}_{P\gamma^*\gamma^*}(-Q_2^2, 0) + w_2(Q_1,Q_2,\tau)\, \mathcal{F}_{P\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)\, \mathcal{F}_{P\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$

→ Product of one single-virtual and one double-virtual transition form factors

→ $w_{1,2}(Q_1,Q_2,\tau)$ are model-independent weight functions

→ The weight functions are **concentrated at small momenta below 1** GeV



↪ Hadronic input : TFFs for arbitrary space-like virtualities in the momentum range $[0-3]\ \mathrm{GeV}^2$

- Relative contribution of the pseudoscalar-pole contributions $a_\mu^{\rm hlbl,P}$ as a function of the momentum cutoff $Q_{\rm cut}$

$Q_{\rm cut}$ [GeV]	π^0	η	η'
0.50	71%	50%	38%
1.00	84%	68%	60%
2.00	92%	81%	78%
3.00	94%	86%	85%
5.00	97%	91%	92%

- **pion transition form factor**

→ first lattice calculation done in collaboration with the Mainz group [Phys.Rev.D 100 (2019) 3]

$$a_\mu^{\mathrm{HLbL};\pi^0} = (59.7 \pm 3.6) \times 10^{-11} \quad [6\%]$$

$$\Gamma(\pi^0 \to \gamma\gamma) = 7.17 \pm 0.49 \text{ eV}$$

→ recently : new result by ETM [2308.12458 [hep-lat]]

→ this talk : new independent calculation

→ warm-up with staggered quarks before moving to η and η' TFFs

- **η and η' transition form factors**

→ calculation at the physical point + continuum limit

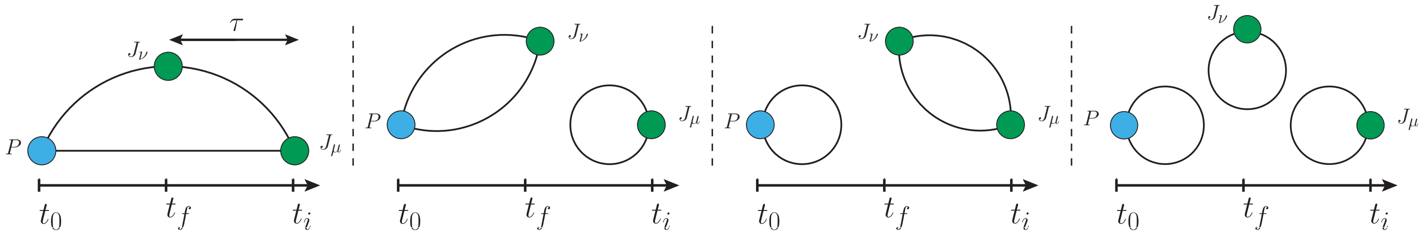
→ estimates from Canterbury approximants [PRD 95, 054026 (2017)] and Dyson-Schwinger equations [Phys. Lett. B 797, 134855 (2019)] [PRD 101, 074021 (2020)]

$$a_\mu^{\mathrm{HLbL};\eta} \approx 15 \times 10^{-11}$$

$$a_\mu^{\mathrm{HLbL};\eta'} \approx 14 \times 10^{-11}$$

→ much more difficult : $\eta - \eta'$ mixing, noisy quark-disconnected contributions

→ 25-30% on $a_\mu^{\mathrm{HLbL};\eta+\eta'}$ precision would suffice to reach 10%



$$C^{(3)}_{\mu\nu}(\tau, t_\pi) = \sum_{\vec{x},\vec{z}} \langle T\left\{ J_\nu(\vec{0}, t_f) J_\mu(\vec{z}, t_i) P(\vec{x}, t_0) \right\} \rangle \, e^{i\vec{p}\vec{x}} \, e^{-i\vec{q}_1\vec{z}}$$

- **Diagram 1** : contributes to all meson TFFs
- **Diagram 2** : contributes to all meson TFFs (only disconnected diagram for π^0)
  → we work in the isospin limit of QCD ⇒ single pseudoscalar loop vanishes
  → numerically small : O(1%) [Mainz '19, BMW '23]
- **Diagram 3 and 4** : contribute to the η and η'
  → third diagram is large, noisy and enters with an opposite sign
  → fourth diagram expected to be small : SU(3)-suppressed vector loops

Calculation based on Budapest-Marseille-Wuppertal (BMW) gauge ensembles

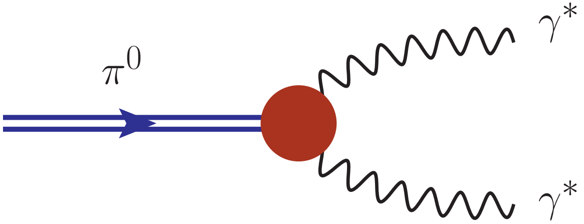
- Goldstone pion/kaon are tuned to their physical pion/kaon masses
  → $N_f = 2 + 1 + 1$ dynamical staggered fermions with four steps of stout smearing

- up to 6 lattice spacings from 0.13 fm down to 0.064 fm

- different physical volumes from 3 to 6 fm
  → pion TFF : based on large 6 fm ensembles only
  → smaller boxes also used for the η and η' to boost statistics

More details are given in 2305.04570 [hep-lat]

The pion transition form factor



In Minkowski space-time :

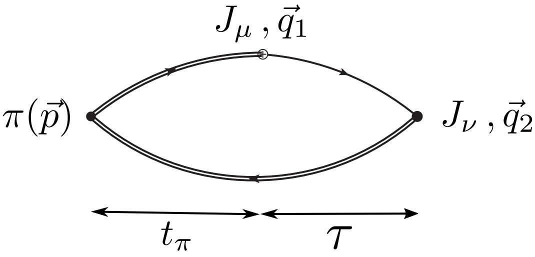
$$M_{\mu\nu}(q_1^2, q_2^2) = i \int \mathrm{d}^4x \, e^{iq_1 x} \, \langle 0 | T\{J_\mu(x) J_\nu(0)\} | \pi^0(p) \rangle = \epsilon_{\mu\nu\alpha\beta} \, q_1^\alpha \, q_2^\beta \; \mathcal{F}_{\pi^0\gamma\gamma}(q_1^2, q_2^2)$$

- $J_\mu(x)$ hadronic component of the electromagnetic current : $J_\mu(x) = \frac{2}{3}\overline{u}(x)\gamma_\mu u(x) - \frac{1}{3}\overline{d}(x)\gamma_\mu d(x) + \dots$

In Euclidean space-time : [Ji & Jung '01] [Cohen et al. '08] [Feng et al. '12]

$$M^E_{\mu\nu}(q_1^2, q_2^2) = -\int \mathrm{d}\tau \, e^{\omega_1 \tau} \int \mathrm{d}^3x \, e^{-i\vec{q}_1 \vec{x}} \, \langle 0 | T\left\{ J_\mu(\vec{x}, \tau) J_\nu(\vec{0}, 0) \right\} | \pi^0(p) \rangle = \int_{-\infty}^{\infty} \mathrm{d}\tau \, \widetilde{A}^{(\pi^0)}_{\mu\nu}(\tau) \, e^{\omega_1 \tau}$$

- Kinematics : $q_1 = (\omega_1, \vec{q}_1)$
- $|\vec{q}_1|^2 = (2\pi/L)^2 |\vec{n}|^2 \quad , \quad |\vec{n}|^2 = 1, 2, 3, 4, 5, \dots$
- valid below hadronic threshold



Euclidean three-point correlation function :

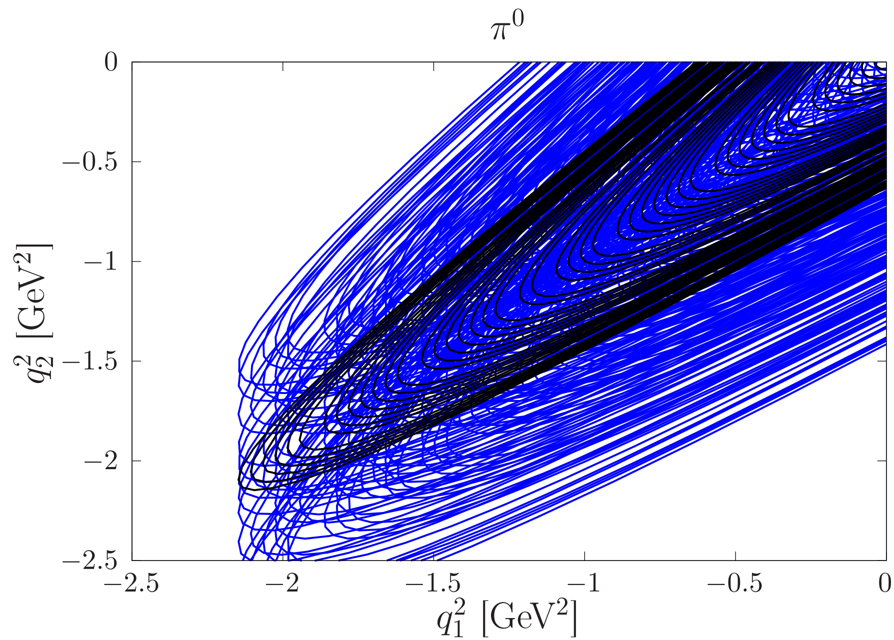
$$C^{(3)}_{\mu\nu}(\tau, t_\pi; \vec{p}, \vec{q}_1, \vec{q}_2) = \sum_{\vec{x}, \vec{z}} \left\langle T\left\{ J_\nu(\vec{0}, t_f) J_\mu(\vec{z}, t_i) P(\vec{x}, t_0) \right\} \right\rangle e^{i\vec{p}\vec{x}} \, e^{-i\vec{q}_1 \vec{z}}$$

Photons virtualities (pion rest frame) :

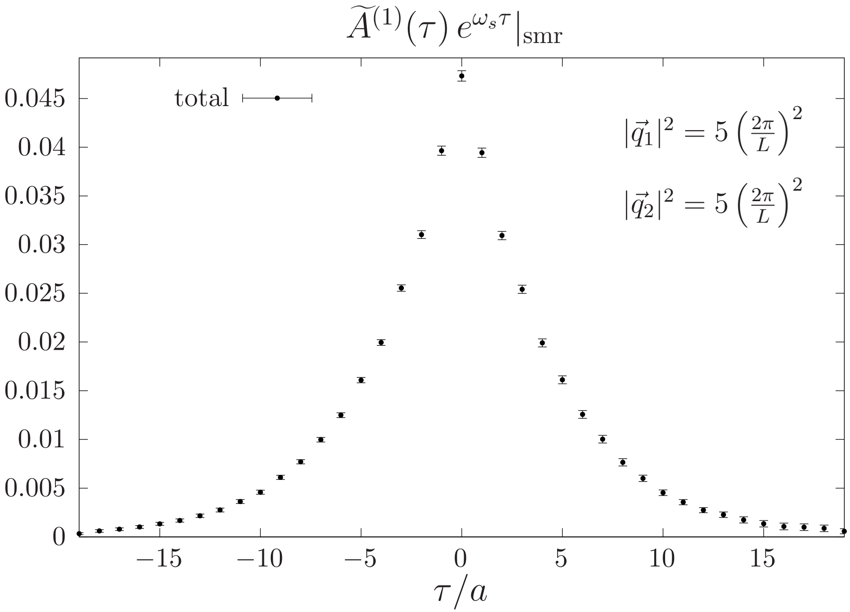
$$q_1^2 = \omega_1{}^2 - |\vec{q}_1|^2$$
$$q_2^2 = (m_\pi - \omega_1)^2 - |\vec{q}_1|^2$$

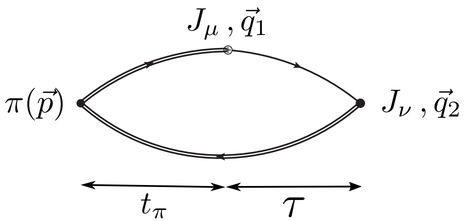
$\Rightarrow |\vec{q}_1|^2 = (2\pi/L)^2 |\vec{n}|^2 \quad , \quad |\vec{n}|^2 = 1, 2, 3, 4, 5, \ldots$

$\Rightarrow \omega_1$ is a (real) free parameter



$$\mathcal{F}_{\pi^0\gamma\gamma}(q_1^2, q_2^2) \propto \int_{-\infty}^{\infty} \mathrm{d}\tau\, \widetilde{A}_{\mu\nu}(\tau)\, e^{\omega_1 \tau} \qquad \widetilde{A}^{(\pi)}_{\mu\nu}(\tau) = \lim_{t_P \to +\infty} \frac{2E_\pi}{Z_\pi} e^{E_\pi (t_f - t_0)} C^{(3)}_{\mu\nu}(\tau, t_P)$$





On the lattice :

- Discrete sum over lattice points :

$$\int \mathrm{d}\tau \to a \sum_\tau$$

- Finite size of the box :

$$|\tau| \leq \tau_{\max} \neq \infty$$

- The small contribution $\tau > \tau_{\max}$ is evaluated assuming a VMD (or LMD) parametrization

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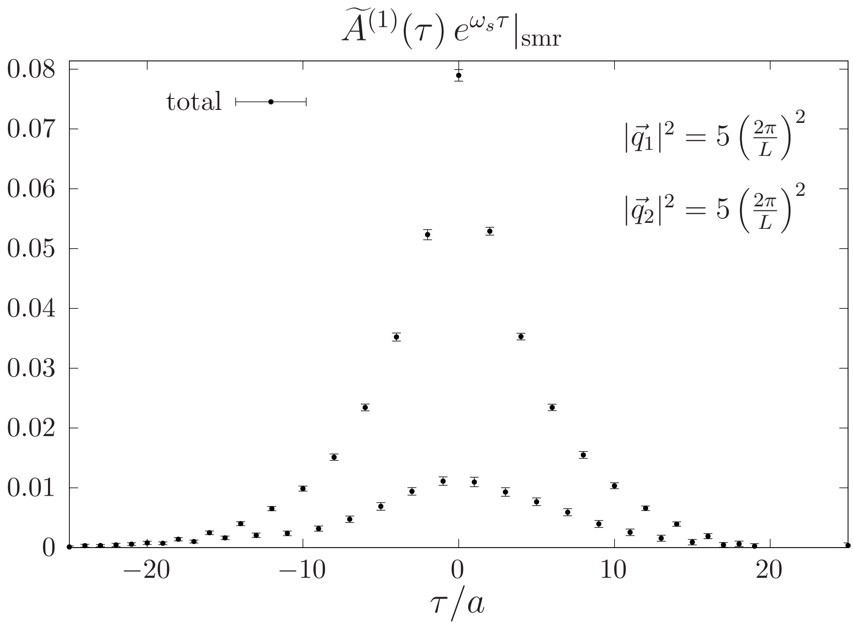

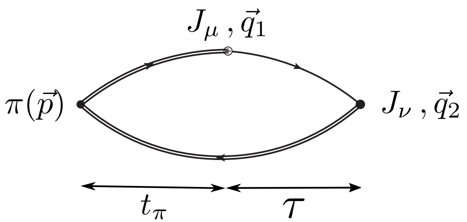

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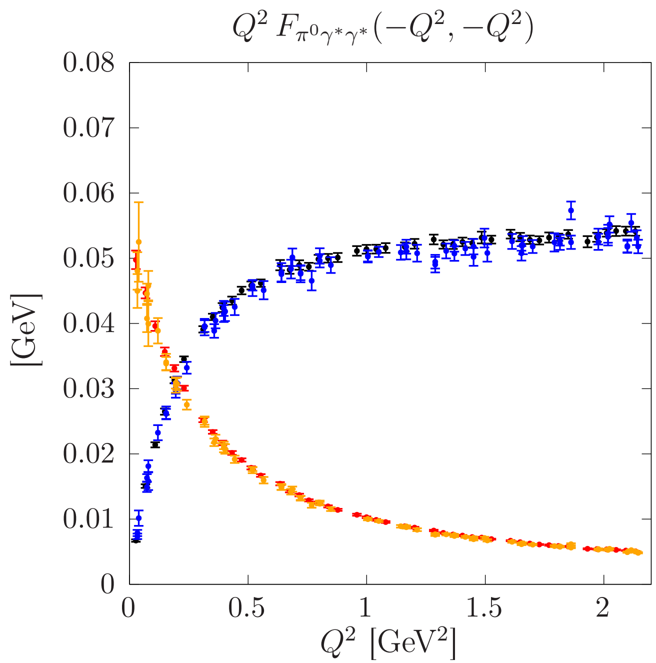

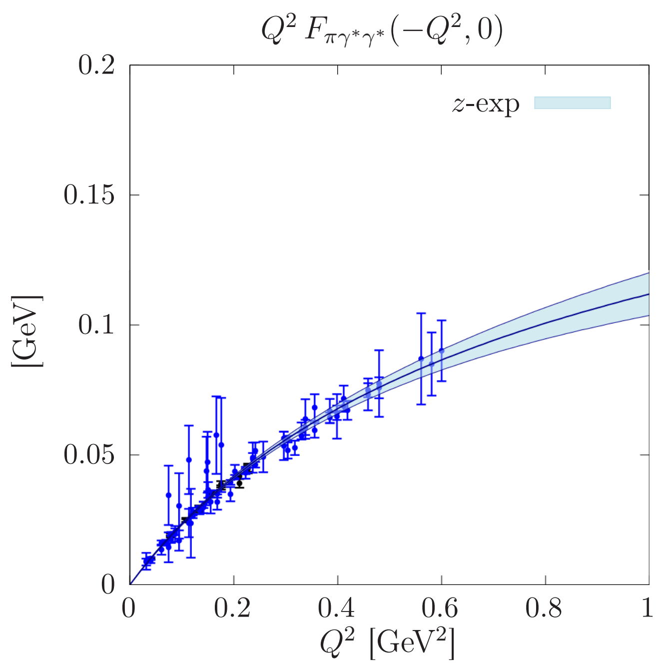

→ Black/blue points : two different pion frames

→ Reminder : lattice data are not restricted to these kinematics

→ OPE / Brodsky Lepage : TFF decays as $1/Q^2$ for large virtualities

- Model independent double z-expansion for space-like momenta [Mainz '19]

$$\mathcal{F}_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2) = \sum_{n,m=0}^{N} c_{nm}\, z_1^n\, z_2^n \quad , \qquad z_i = \frac{\sqrt{t_c + Q_i^2} - \sqrt{t_c - t_0}}{\sqrt{t_c + Q_i^2} + \sqrt{t_c - t_0}}$$

$\rightarrow$ $t_c = 4m_\pi^2$

$\rightarrow$ t_0 chosen to reduce the maximum value of $|z_i|$ in the range $[0, Q^2_{\mathrm{max}}]$

$\rightarrow$ truncation of the sum (finite N)

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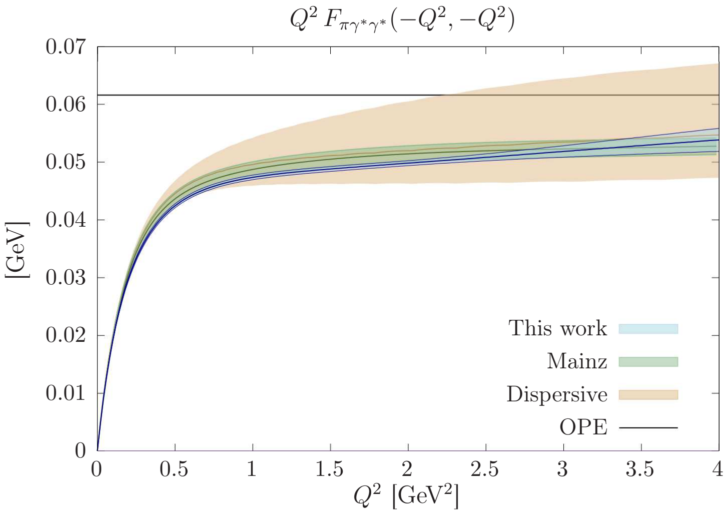
- Analytical + short-distance constraints (Brodsky-Lepage behavior and OPE) [Mainz '19]

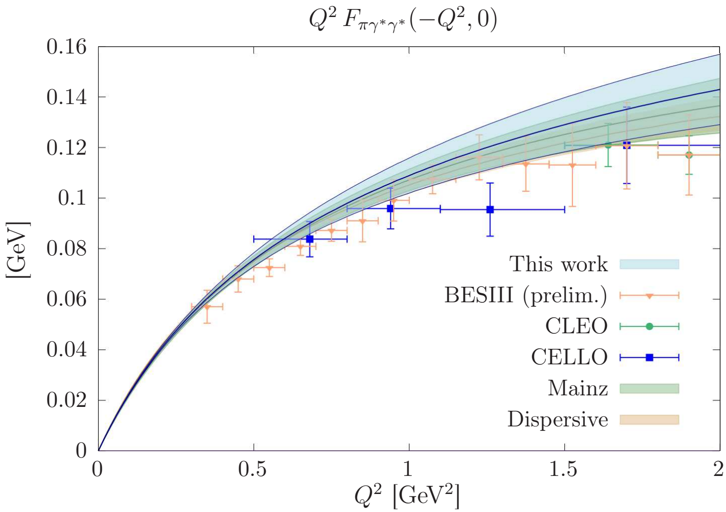
$$P(Q_1^2, Q_2^2)\, \mathcal{F}_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2) = \sum_{n,m=0}^{N-1} c_{nm} \left(z_1^n - \frac{(-1)^{N+n}\, n\, z_1^N}{N} \right) \left(z_2^m - \frac{(-1)^{N+m}\, m\, z_2^N}{N} \right)$$

with
$$P(Q_1^2, Q_2^2) = 1 + \frac{Q_1^2 + Q_2^2}{M_V^2} \qquad \Rightarrow \qquad \mathcal{F}_{\pi^0\gamma^*\gamma} \sim \frac{1}{Q^2}$$

- Discretization effects : $\tilde{c}_{mn}(a) = c_{mn} + \gamma_{mn}^{(1)}\, a^2 + \gamma_{mn}^{(2)}\, a^4$

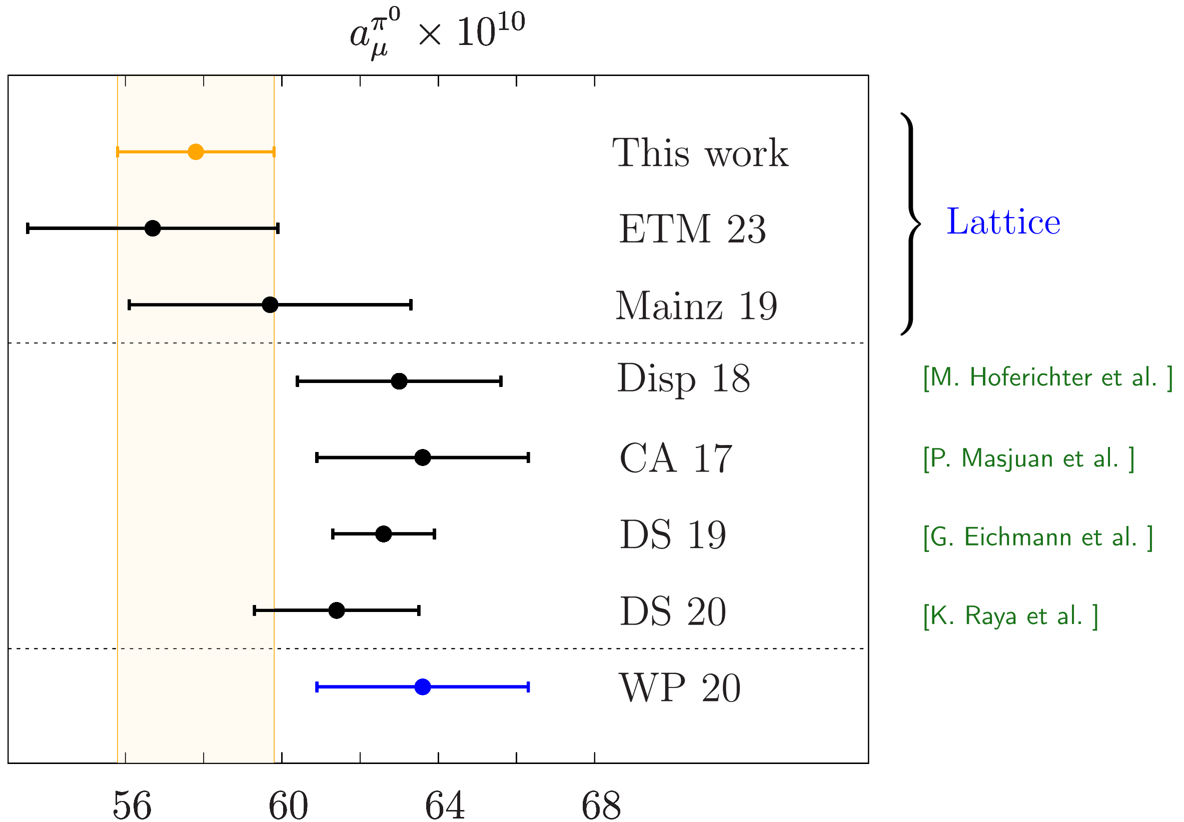
- Uncorrelated fits. Systematic error by varying N





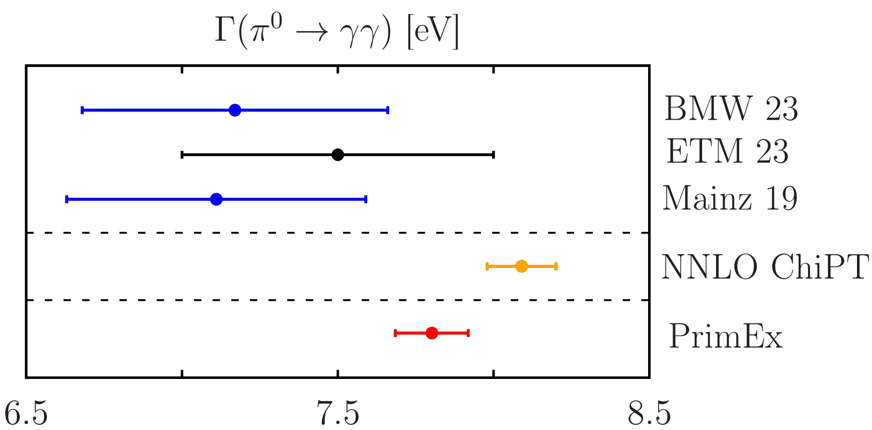
$$\Gamma(\pi^0 \to \gamma\gamma) = 7.11(0.44)_{\rm stat}(0.21)_{\rm syst}\ \mathrm{eV}$$

$$a_\mu^{\mathrm{HLbL};\pi^0} = 57.8(1.8)_{\rm stat}(0.9)_{\rm syst} \times 10^{-11}$$


$a_\mu^{\pi^0} \times 10^{10}$
This work
ETM 23
Mainz 19
Disp 18
CA 17
DS 19
DS 20
WP 20
56
60
64
68
Lattice
[M. Hoferichter et al.]
[P. Masjuan et al.]
[G. Eichmann et al.]
[K. Raya et al.]

▶ two-photon decay width :

$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{\pi\alpha^2 m_{\pi^0}^3}{4} \mathcal{F}^2_{\pi^0\gamma\gamma}(0,0)$$

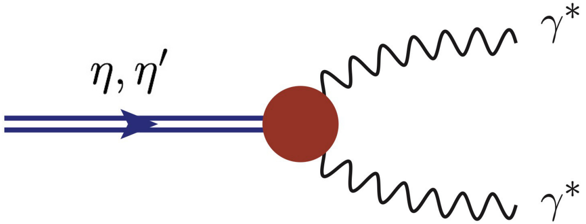


- lattice results tend to be smaller than exp. value ... but compatible within 1 - 1.5 σ

  $\rightarrow$ error dominated by statistics

- $\Gamma(\pi^0 \to \gamma\gamma)$ and $a_\mu^{\pi^0}$ are correlated : low-Q^2 region dominates in $a_\mu^{\pi^0}$

The η and η' transition form factors



- In principle, the same techniques work for η

$$C^{(i)}_{\mu\nu}(\tau, t_P) = \sum_{\vec{x},\vec{z}} \langle J_\mu(\vec{z}, t_i) J_\nu(\vec{0}, t_f) \mathcal{O}_i(\vec{x}, t_0) \rangle \, e^{i\vec{p}\cdot\vec{x}} e^{-i\vec{q}_1\cdot\vec{z}}$$

with the η_8 and η_0 interpolating operators :

$$\mathcal{O}_8(x) = \frac{1}{\sqrt{6}} \left(\overline{u}\gamma_5 u(x) + \overline{d}\gamma_5 d(x) - 2\overline{s}\gamma_5 s(x) \right) ,$$
$$\mathcal{O}_0(x) = \frac{1}{\sqrt{3}} \left(\overline{u}\gamma_5 u(x) + \overline{d}\gamma_5 d(x) + \overline{s}\gamma_5 s(x) \right) .$$

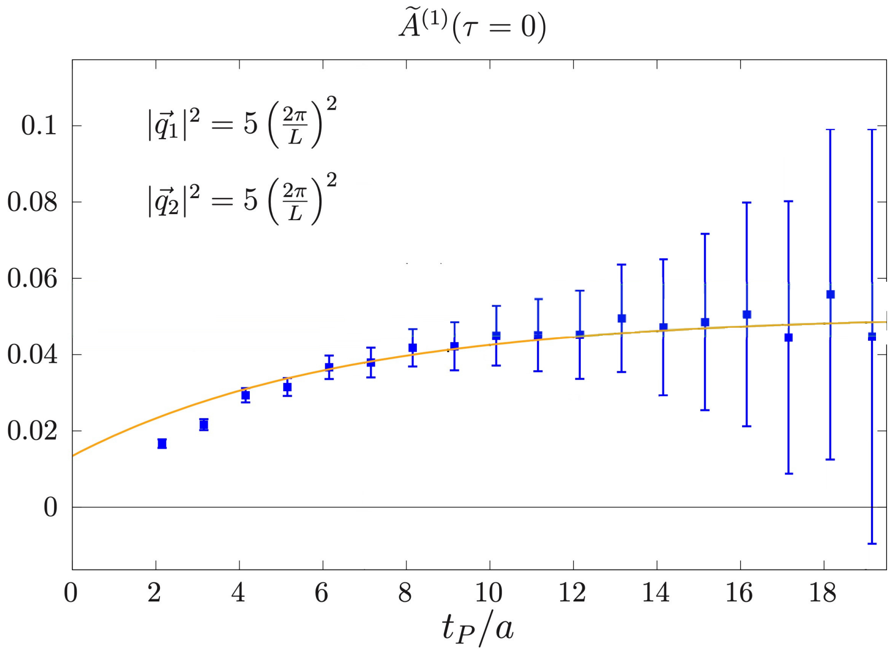
For the η, the spectral decomposition reads

$$C^{(i)}_{\mu\nu}(\tau, t_P) = \sum_{\vec{z}} \langle 0 | J_\mu(\vec{z}, \tau) J_\nu(\vec{0}, 0) | \eta(\vec{p}) \rangle e^{-i\vec{q}_1\cdot\vec{z}} \times \frac{1}{2E_\eta} \langle \eta(\vec{p}) | \mathcal{O}_i | 0 \rangle e^{E_\eta (t_0 - t_f)}$$
$$+ \sum_{\vec{z}} \langle 0 | J_\mu(\vec{z}, \tau) J_\nu(\vec{0}, 0) | \eta'(\vec{p}) \rangle e^{-i\vec{q}_1\cdot\vec{z}} \times \frac{1}{2E_{\eta'}} \langle \eta'(\vec{p}) | \mathcal{O}_i | 0 \rangle e^{E'_\eta (t_0 - t_f)}$$
$$+ \text{ higher excited states}$$

$\rightarrow$ η' is just an excited state : contribution vanishes exponentially with t_P

$\rightarrow$ but the mass gap $\Delta E = E_{\eta'} - E_\eta \approx 400$ MeV is not so large

- $\Delta E = E_{\eta'} - E_{\eta} \approx 400$ MeV not so large
- Large excited state contribution : large statistical error at large t_P



- Large time separations t_P are needed
- Does not work for the $\eta' \rightarrow$ excited state

Solution : Generalized eigenvalue problem to deal with excited states.

Spectral decomposition :

$$C^{(i)}_{\mu\nu}(\tau,t_P) = \sum_{\vec z} \langle 0|J_\mu(\vec z,\tau)J_\nu(\vec 0,0)|\eta(\vec p)\rangle e^{-i\vec q_1\cdot\vec z} \times \frac{1}{2E_\eta}\langle \eta(\vec p)|\mathcal{O}_i|0\rangle e^{E_\eta(t_0-t_f)}$$
$$+\sum_{\vec z} \langle 0|J_\mu(\vec z,\tau)J_\nu(\vec 0,0)|\eta'(\vec p)\rangle e^{-i\vec q_1\cdot\vec z} \times \frac{1}{2E_{\eta'}}\langle \eta'(\vec p)|\mathcal{O}_i|0\rangle e^{E'_\eta(t_0-t_f)} + \cdots$$

Matrix notation :

$$\begin{pmatrix} C^{(8)}_{\mu\nu} \\ C^{(0)}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} T^{(8)}_\eta & T^{(8)}_{\eta'} \\ T^{(0)}_\eta & T^{(0)}_{\eta'} \end{pmatrix} \begin{pmatrix} \widetilde{A}^{(\eta)}_{\mu\nu} \\ \widetilde{A}^{(\eta')}_{\mu\nu} \end{pmatrix} ,$$

with

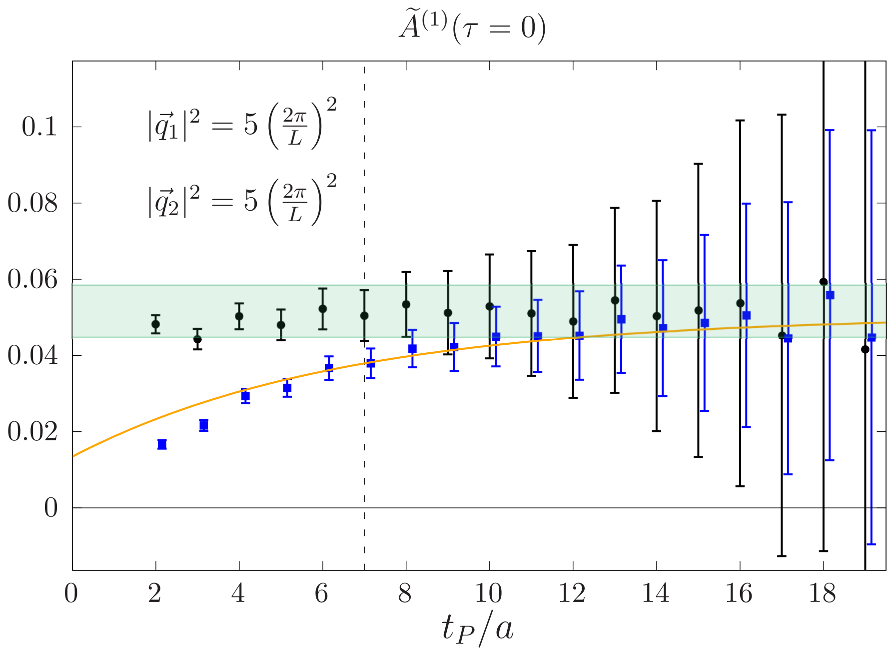
$$T^{(i)}_n = \frac{Z^{(i)}_n}{2E_n} e^{-E_n(t_f-t_0)} , \quad \widetilde{A}^{(n)}_{\mu\nu}(\tau) = \sum_{\vec z} \langle 0|J_\mu(\vec z,\tau)J_\nu(\vec 0,0)|n(\vec p)\rangle\, e^{-i\vec q_1\cdot\vec z}$$

Inverting the system :

$$\widetilde{A}^{(\eta)}_{\mu\nu} = \cos^2\phi_I \, \frac{C^{(8)}_{\mu\nu}}{T^{(8)}_\eta} + \sin^2\phi_I \, \frac{C^{(0)}_{\mu\nu}}{T^{(0)}_\eta}$$
$$\widetilde{A}^{(\eta')}_{\mu\nu} = \sin^2\phi_I \, \frac{C^{(8)}_{\mu\nu}}{T^{(8)}_{\eta'}} + \cos^2\phi_I \, \frac{C^{(0)}_{\mu\nu}}{T^{(0)}_{\eta'}}$$

with $\tan^2\phi_I = -(Z^{(8)}_{\eta'} Z^{(0)}_\eta)/(Z^{(8)}_\eta Z^{(0)}_{\eta'})$ (mixing angle)

- $\Delta E = E_{\eta'} - E_{\eta} \approx 400$ MeV not so large
- Large excited state contribution : large statistical error at large t_P



- Black point : new estimator

We obtain a plateau for short time separation t_P

Two staggered operators with correct quantum numbers (taste-singlet pseudoscalar) [Altmeyer et al. '93]

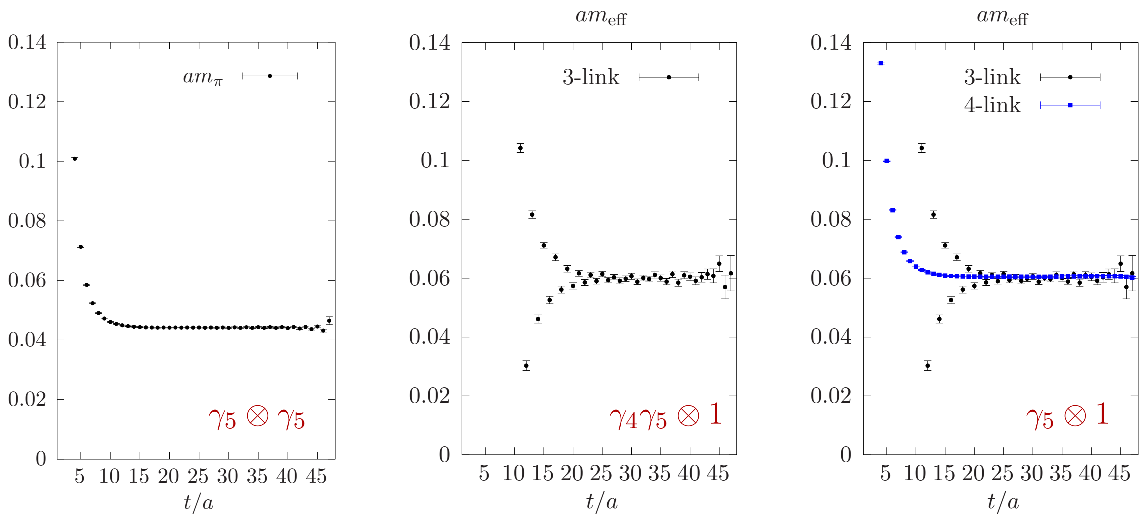
$$O_3(x) = \frac{1}{6} \sum_{i,j,k} \epsilon_{ijk} \, \overline{\chi}(x) [\eta_i \Delta_i [\eta_j \Delta_j [\eta_k \Delta_k]]] \chi(x) \qquad (\gamma_4 \gamma_5 \otimes 1)$$

$$O_4(x) = \frac{1}{2} \eta_4(x) \left[\overline{\chi}(x) \hat{O}_3 \chi_+(x) + \overline{\chi}_+(x) \hat{O}_3 \chi(x) \right] \qquad (\gamma_5 \otimes 1)$$

- Symmetric shift operator : $\Delta_\mu \chi(x) = \frac{1}{2} [U_\mu(x) \chi(x + a\hat{\mu}) + U_\mu^\dagger(x - a\hat{\mu}) \chi(x - a\hat{\mu})]$
- Time shifted fields : $\chi_+(x) = U_4(x) \chi(x + a\hat{t})$ and $\overline{\chi}_+(x) = \overline{\chi}(x + a\hat{t}) U_4^\dagger(x)$
- Staggered quark operators : parity is not a "good quantum number " :

$$C(t; \vec{p}) = A \cosh\left(E_1(\vec{p})[T/2 - t]\right) + (-1)^{t/a} B \cosh\left(E_2(\vec{p})[T/2 - t]\right) ,$$

Pion :



Masses of the η ans η' meson : taste-singlet operators

Two staggered operators with correct quantum numbers (taste-singlet pseudoscalar) [Altmeyer et al. '93]

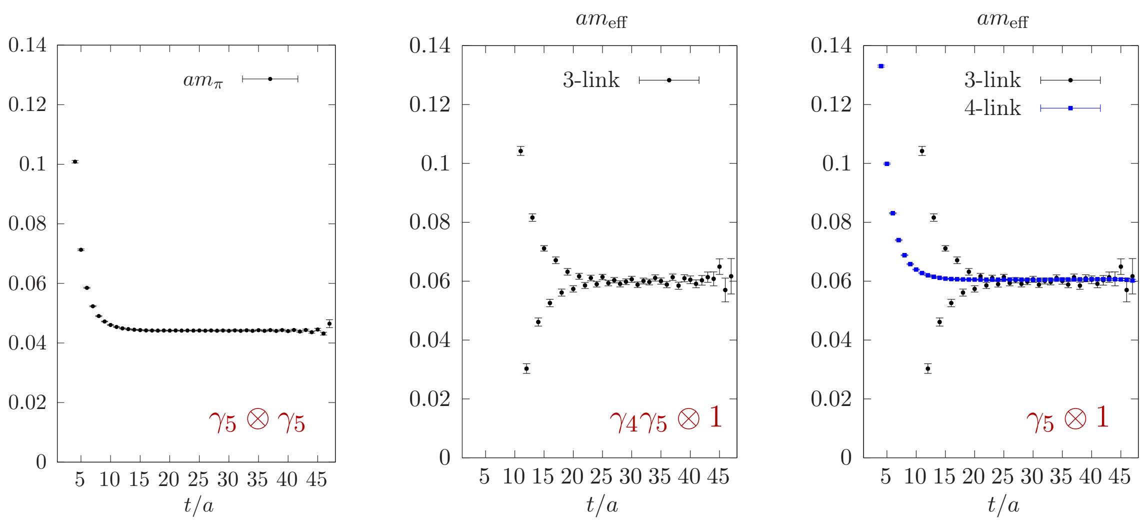
$$O_3(x) = \frac{1}{6} \sum_{i,j,k} \epsilon_{ijk} \, \overline{\chi}(x) [\eta_i \Delta_i [\eta_j \Delta_j [\eta_k \Delta_k]]] \chi(x) \qquad (\gamma_4 \gamma_5 \otimes 1)$$

$$O_4(x) = \frac{1}{2} \eta_4(x) \left[\overline{\chi}(x) \hat{O}_3 \chi_+(x) + \overline{\chi}_+(x) \hat{O}_3 \chi(x) \right] \qquad (\gamma_5 \otimes 1)$$

- Staggered quark operators : parity is not a "good quantum number " :

$$C(t;\vec{p}) = A \cosh\left(E_1(\vec{p})[T/2 - t]\right) + (-1)^{t/a} B \cosh\left(E_2(\vec{p})[T/2 - t]\right) \; ,$$

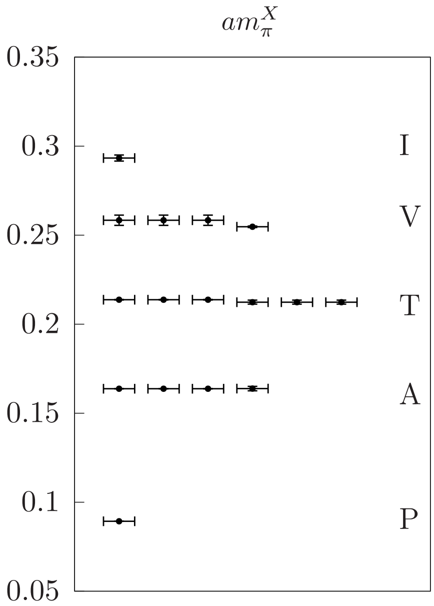
Pion :



→ 4-link operators : efficient methods to compute the the disconnected loops [Venkataraman & Kilcup '97]

→ we only use the 4-link operator for the TFFs

- Pion spectrum with staggered quarks (16 tastes)
- Staggered ChiPT : $(m_\pi^X)^2 = \mu\,(m_u + m_d) + a^2\Delta_X$ [Lee & Sharpe '99]



- from rSχPT is [Bernard 2007] :

$$m^2_{\eta_8,I} = \frac{1}{3}m^2_{uu,I} + \frac{2}{3}m^2_{ss,I}$$

- expect discretization effects ~ 140 MeV for the η meson mass at our coarsest lattice spacing

Consider two operators that interpolate the η and η' mesons :

$$\mathcal{O}_8(x) = \frac{1}{\sqrt{6}} \left(\overline{u}\gamma_5 u(x) + \overline{d}\gamma_5 d(x) - 2\overline{s}\gamma_5 s(x) \right)$$
$$\mathcal{O}_0(x) = \frac{1}{\sqrt{3}} \left(\overline{u}\gamma_5 u(x) + \overline{d}\gamma_5 d(x) + \overline{s}\gamma_5 s(x) \right)$$

and build the correlation matrix

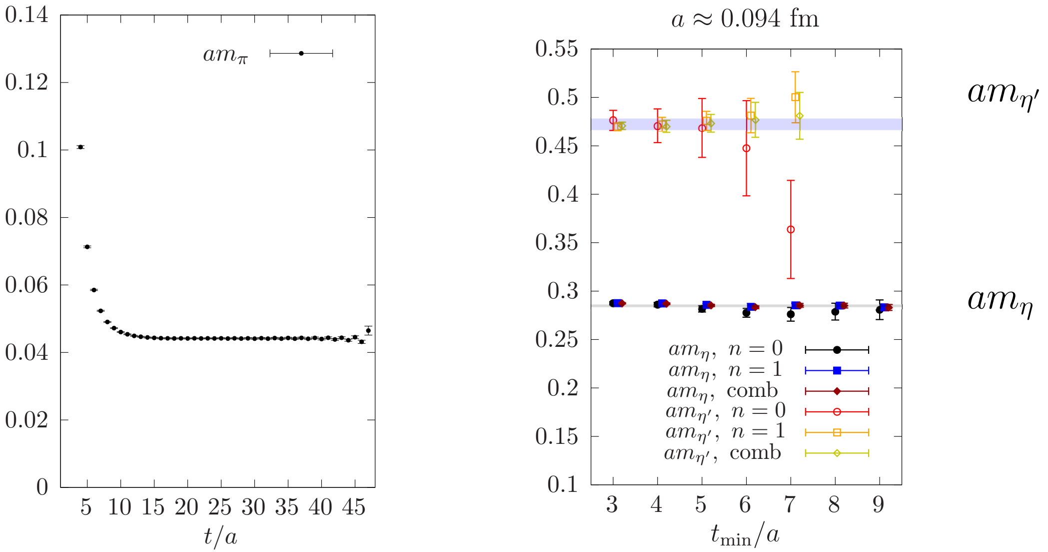
$$C_I(t) = \begin{pmatrix} \langle O_8(t) O_8^\dagger(0) \rangle & \langle O_8(x) O_0^\dagger(0) \rangle \\ \langle O_0(t) O_8^\dagger(0) \rangle & \langle O_0(x) O_0^\dagger(0) \rangle \end{pmatrix}$$

In this isospin basis, the correlators are given by

$$\langle O_8(t) O_8^\dagger(0) \rangle = \frac{1}{3} \left(\mathcal{C}_l + 2\mathcal{C}_s + 2\mathcal{D}_{ll} + 2\mathcal{D}_{ss} - 2\mathcal{D}_{ls} - 2\mathcal{D}_{sl} \right)$$
$$\langle O_8(t) O_0^\dagger(0) \rangle = \frac{\sqrt{2}}{3} \left(\mathcal{C}_l - \mathcal{C}_s + 2\mathcal{D}_{ll} - \mathcal{D}_{ss} + \mathcal{D}_{ls} - 2\mathcal{D}_{sl} \right)$$
$$\langle O_0(t) O_8^\dagger(0) \rangle = \frac{\sqrt{2}}{3} \left(\mathcal{C}_l - \mathcal{C}_s + 2\mathcal{D}_{ll} - \mathcal{D}_{ss} + \mathcal{D}_{sl} - 2\mathcal{D}_{ls} \right)$$
$$\langle O_0(x) O_0^\dagger(0) \rangle = \frac{1}{3} \left(2\mathcal{C}_l + \mathcal{C}_s + 4\mathcal{D}_{ll} + \mathcal{D}_{ss} + 2\mathcal{D}_{ls} + 2\mathcal{D}_{sl} \right)$$

$$\langle O_a(t) O_b^\dagger(0) \rangle = \frac{Z_\eta^{(a)} Z_\eta^{(b)*}}{2E_\eta} e^{-E_\eta t} + \frac{Z_{\eta'}^{(a)} Z_{\eta'}^{(b)*}}{2E_{\eta'}} e^{-E_{\eta'} t} + \cdots$$

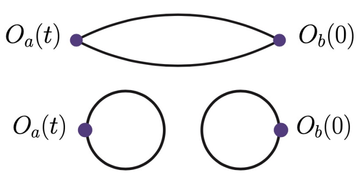
- Extraction of the π^0, η and η' masses from two-point correlation functions



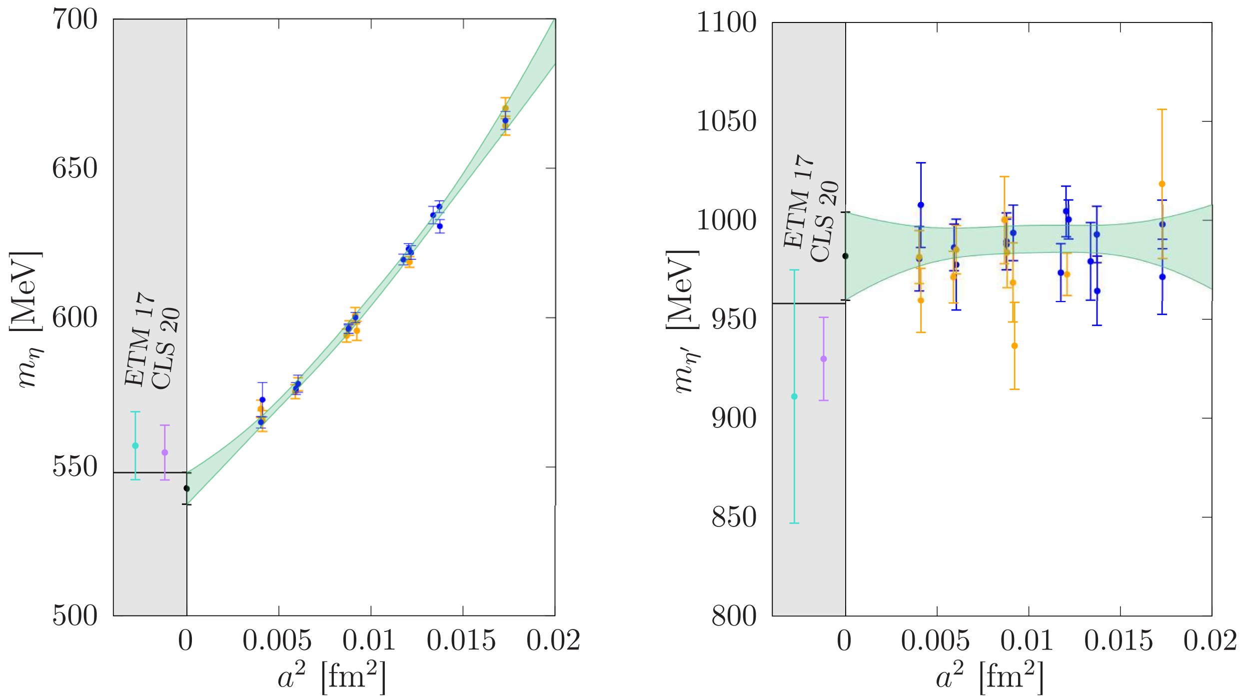
→ signal deteriorates rapidly for the η and η'

→ need to study the mixing between the η and η'

→ noisy disconnected diagrams



- Preliminary - no systematic error



→ We reproduce the experimental values in the continuum limit

→ Blue points are obtained using large volume ensembles ($L = 6$fm)

→ Orange points correspond to smaller volumes ($L = 3/4$fm) (not included in the fit)

- 3-point correlation functions for the η and η'

$$C^{(i)}_{\mu\nu}(\tau, t_P) = \sum_{\vec{x},\vec{z}} \langle J_\mu(\vec{z}, t_i) J_\nu(\vec{0}, t_f) \mathcal{O}_i(\vec{x}, t_0) \rangle \, e^{i\vec{p}\cdot\vec{x}} e^{-i\vec{q}_1\cdot\vec{z}}$$

with the η_8 and η_0 interpolating operators :

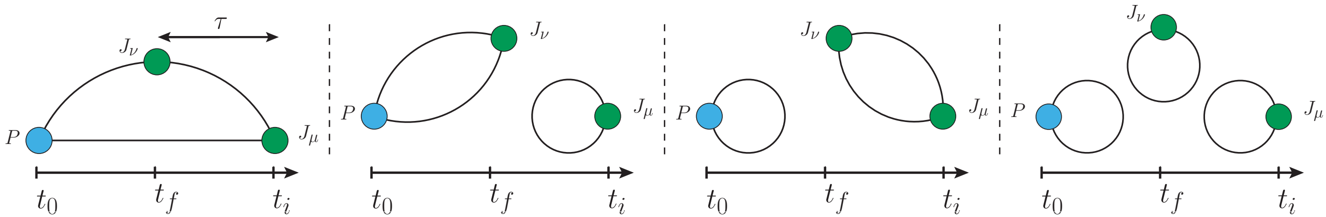
$$\mathcal{O}_8(x) = \frac{1}{\sqrt{6}} \left(\overline{u}\gamma_5 u(x) + \overline{d}\gamma_5 d(x) - 2\overline{s}\gamma_5 s(x) \right),$$
$$\mathcal{O}_0(x) = \frac{1}{\sqrt{3}} \left(\overline{u}\gamma_5 u(x) + \overline{d}\gamma_5 d(x) + \overline{s}\gamma_5 s(x) \right).$$

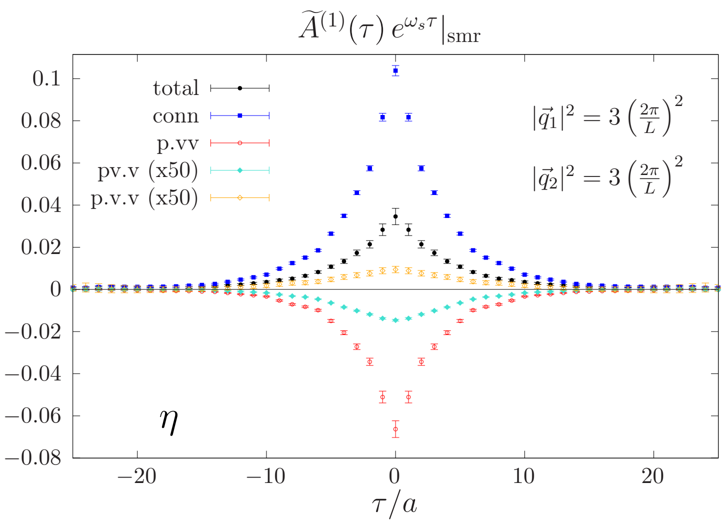
For the η, the spectral decomposition reads

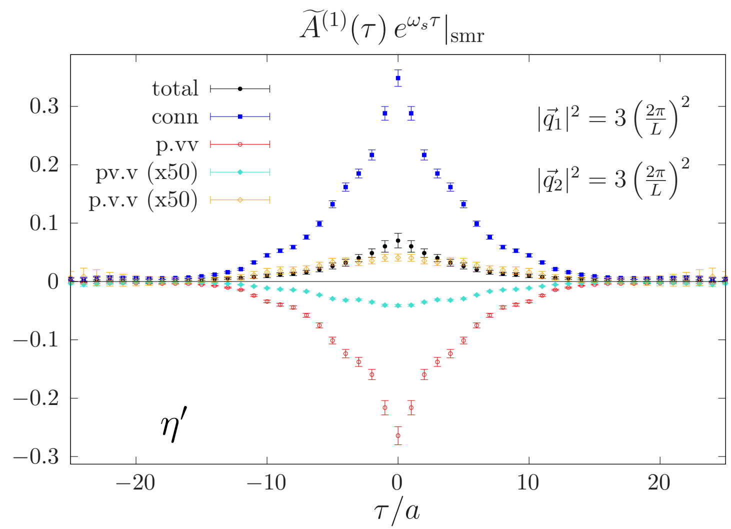
$$\begin{aligned} C^{(i)}_{\mu\nu}(\tau, t_P) = & \sum_{\vec{z}} \langle 0 | J_\mu(\vec{z}, \tau) J_\nu(\vec{0}, 0) | \eta(\vec{p}) \rangle e^{-i\vec{q}_1\cdot\vec{z}} \times \frac{1}{2E_\eta} \langle \eta(\vec{p}) | \mathcal{O}_i | 0 \rangle e^{E_\eta (t_0 - t_f)} \\ & + \sum_{\vec{z}} \langle 0 | J_\mu(\vec{z}, \tau) J_\nu(\vec{0}, 0) | \eta'(\vec{p}) \rangle e^{-i\vec{q}_1\cdot\vec{z}} \times \frac{1}{2E_{\eta'}} \langle \eta'(\vec{p}) | \mathcal{O}_i | 0 \rangle e^{E'_\eta (t_0 - t_f)} \\ & + \text{higher excited states} \end{aligned}$$

$\rightarrow$ η' is just an excited states, its contribution vanishes exponentially with t_P

$\rightarrow$ but the mass gap $\Delta E = E_{\eta'} - E_\eta \approx 400$ MeV is not so large

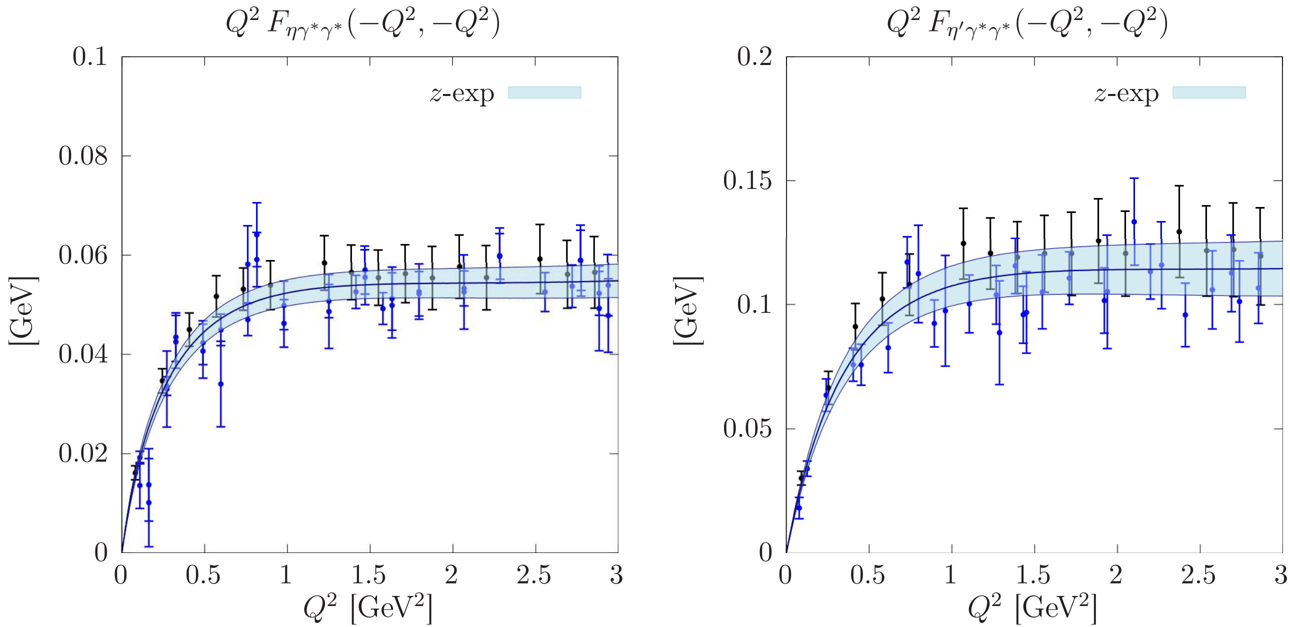






$$\mathcal{F}_{P\gamma\gamma}(q_1^2, q_2^2) \;\propto\; \int_{-\infty}^{\infty} \mathrm{d}\tau\, \widetilde{A}_{\mu\nu}(\tau)\, e^{\omega_1 \tau}$$

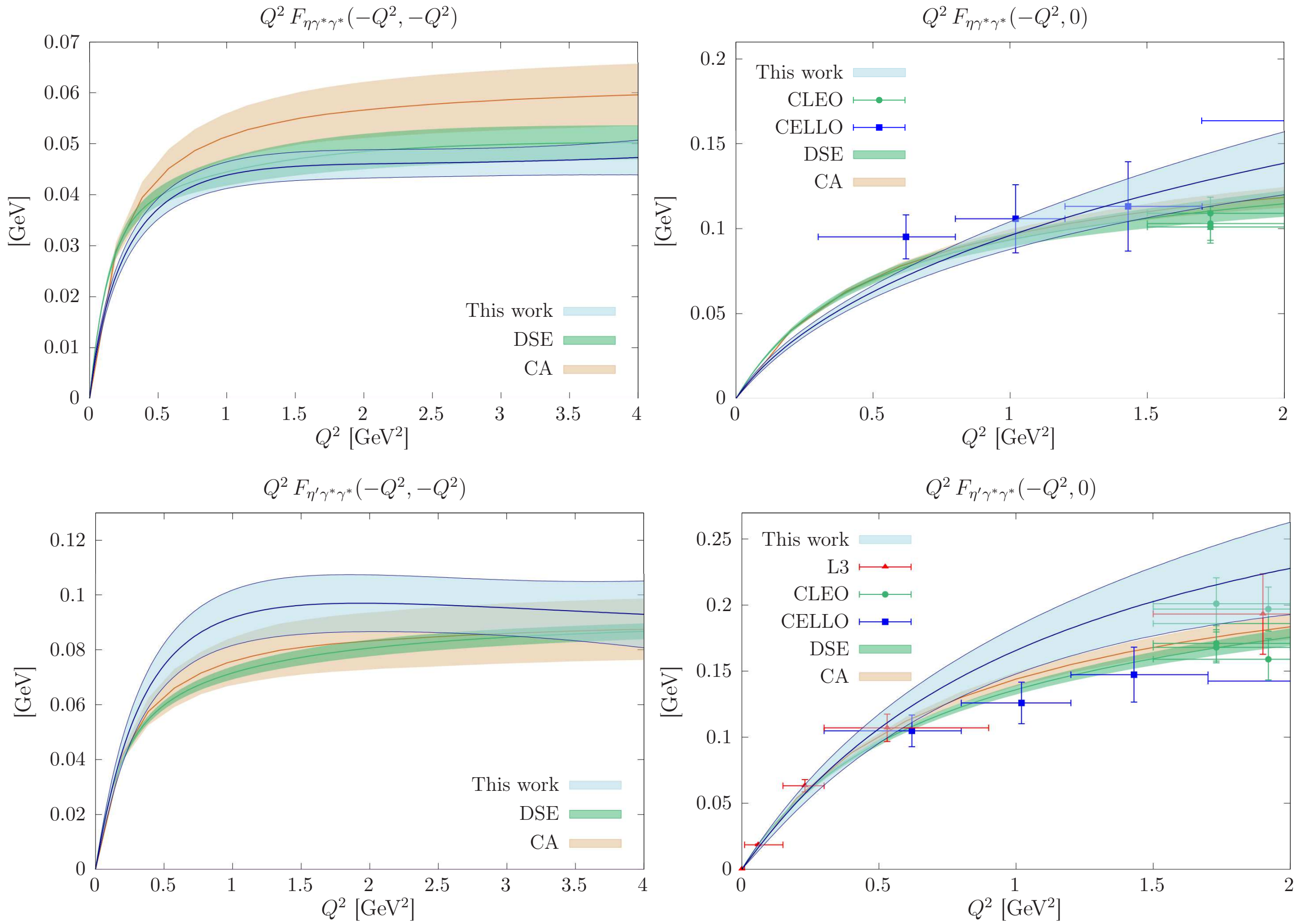
- Double-virtual TFFs



- Next step : continuum extrapolation

  → same strategy as for the pion : z-expansion parametrization

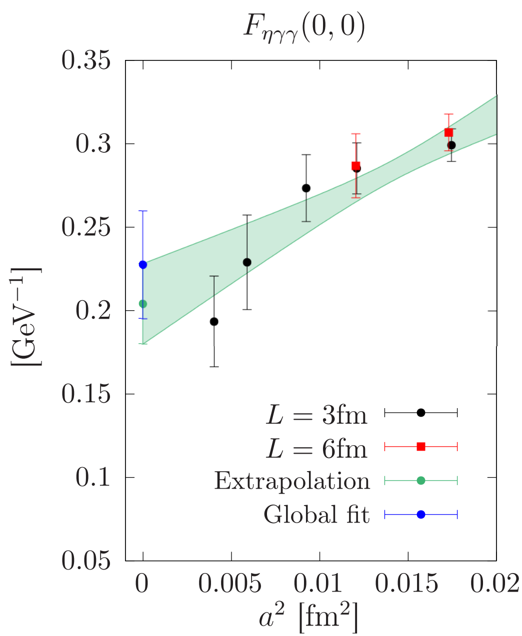
  → $\tilde{c}_{mn}(a) = c_{mn} + \gamma_{mn}^{(1)}\, a^2$

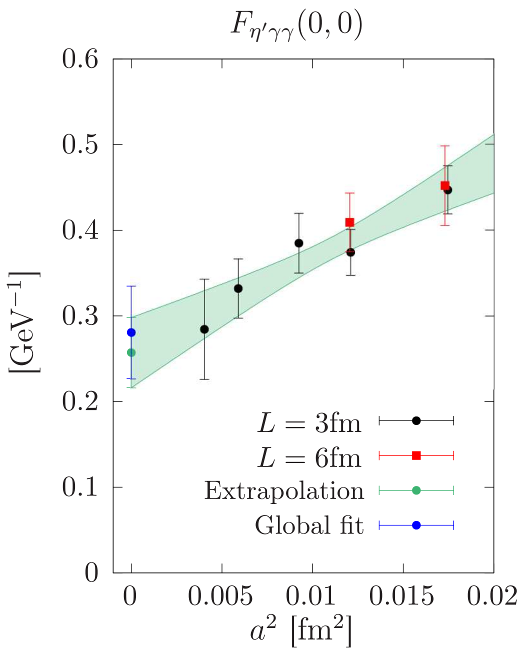


$Q^2 F_{\eta\gamma^*\gamma^*}(-Q^2,-Q^2)$
$Q^2 F_{\eta\gamma^*\gamma^*}(-Q^2,0)$
$Q^2 F_{\eta'\gamma^*\gamma^*}(-Q^2,-Q^2)$
$Q^2 F_{\eta'\gamma^*\gamma^*}(-Q^2,0)$
This work
DSE
CA
CLEO
CELLO
L3
[GeV]
Q^2 [GeV^2]

- two-photon decay width :

$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{\pi\alpha^2 m_{\pi^0}^3}{4}\mathcal{F}^2_{\pi^0\gamma\gamma}(0,0)$$



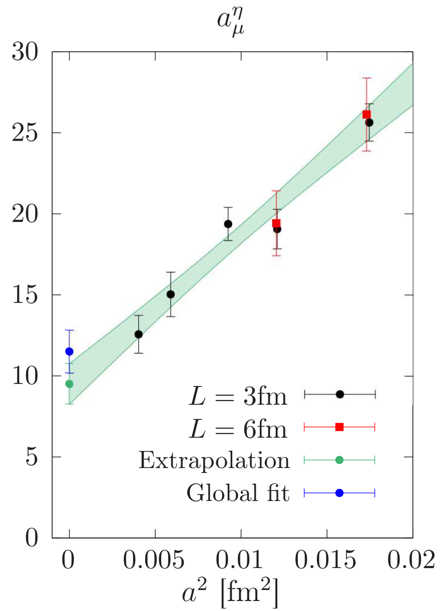


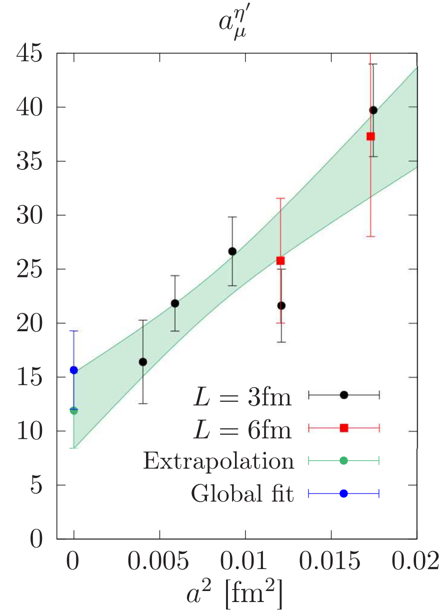
$$\Gamma(\eta \to \gamma\gamma) = 338(94)_{\text{stat}}(35)_{\text{syst}} \text{ eV}$$
$$\Gamma(\eta' \to \gamma\gamma) = 3.4(1.0)_{\text{stat}}(0.4)_{\text{syst}} \text{ keV}$$

$\to$ PDG : $\Gamma(\eta \to \gamma\gamma) = 516(18)$ eV

$\to$ PDG : $\Gamma(\eta' \to \gamma\gamma) = 4.28(19)$ keV

$$a_\mu^{\mathrm{HLbL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_1(Q_1,Q_2,\tau)\, \mathcal{F}_{P\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2)\, \mathcal{F}_{P\gamma^*\gamma^*}(-Q_2^2, 0) + w_2(Q_1,Q_2,\tau)\, \mathcal{F}_{P\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)\, \mathcal{F}_{P\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$



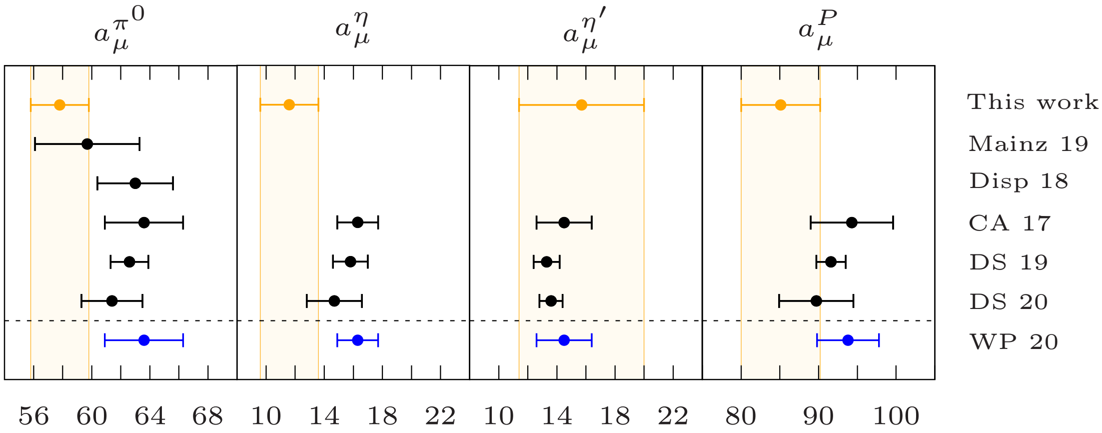


$$a_\mu^{\mathrm{HLbL};\eta} = 11.6(1.6)_{\mathrm{stat}}(0.5)_{\mathrm{syst}}(1.1)_{\mathrm{FSE}} \times 10^{-11}$$
$$a_\mu^{\mathrm{HLbL};\eta'} = 15.7(3.9)_{\mathrm{stat}}(1.1)_{\mathrm{syst}}(1.3)_{\mathrm{FSE}} \times 10^{-11}$$

Canterbury approximants [PRD 95, 054026 (2017)]

$\rightarrow a_\mu^{\mathrm{HLbL};\eta} = 16.3(1.4) \times 10^{-11}$

$\rightarrow a_\mu^{\mathrm{HLbL};\eta'} = 14.5(1.9) \times 10^{-11}$



- Our final estimate

$$a_\mu^{\mathrm{HLbL;ps-poles}} = (85.1 \pm 4.7_{\mathrm{stat}} \pm 2.3) \times 10^{-11} \, .$$

- Pion transition form factor

  $\to$ good agreement with Mainz'19, ETM'23 and with exp. data for $\mathcal{F}_{\pi^0\gamma\gamma}(Q^2, 0)$

  $\to$ calculation of $\mathcal{F}_{\pi^0\gamma\gamma}(0, 0)$ might help to reduce the error (+ comparison with PrimEx)

- $\eta - \eta'$ transition form factors

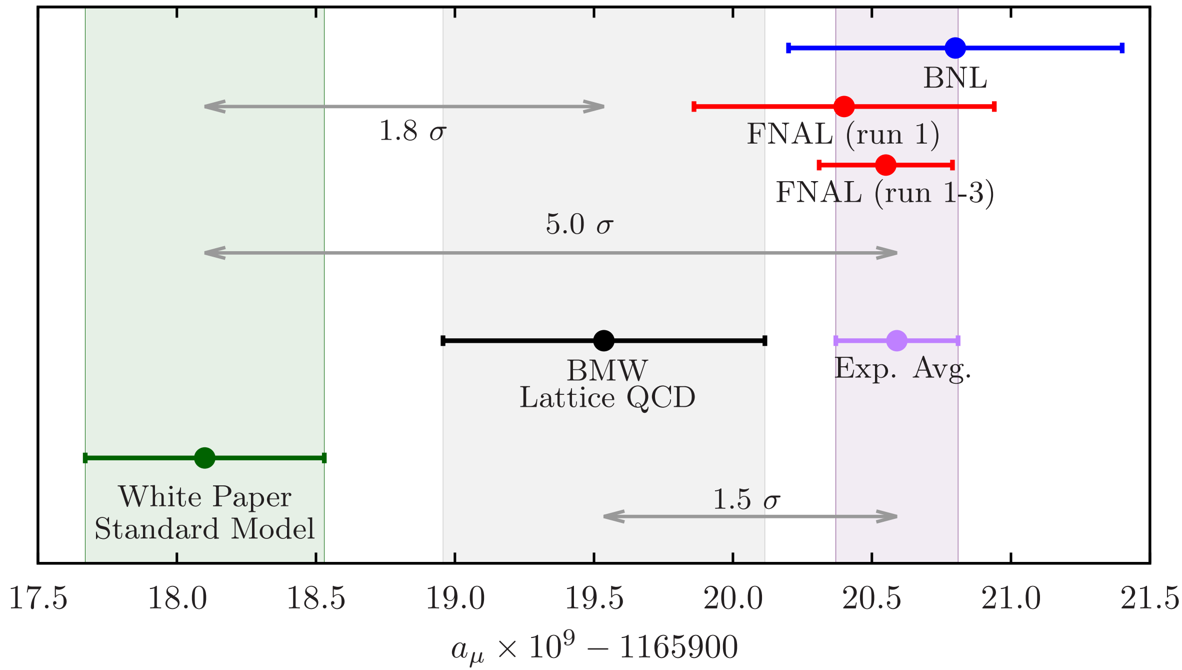
  $\to$ first ab-initio calculation - error dominated by statistics

  $\to$ some tensions for the η TFF at very low virtualities

- Spectroscopy of the η/η' with rooted staggered fermions

Thank you !

Status after runs 1-3 at Fermilab (August 2023)



- Can be measured with very high precision (∼0.2 ppm !)
- Can be computed with similar precision within the Standard Model of particle physics

Good probe to test the SM and to look for possible new physics