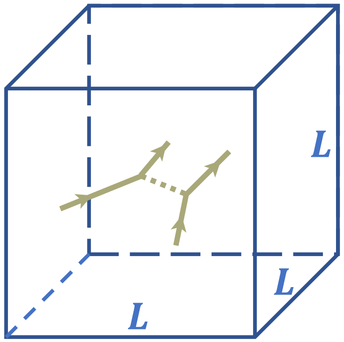
RUB

Evgeny Epelbaum, Ruhr University Bochum

Lattice seminar, Berlin, 5. February 2024

Two-particle scattering from finite-volume energies

Based on work done in collaboration with Lu Meng, Vadim Baru, Arseniy Filin, Ashot Gasparyan



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The Lüscher method

Lüscher's quantization condition for moving systems: $\det\left[\underbrace{M^{(\Gamma,\boldsymbol{P})}_{ln,l'n'}} - \delta_{ll'}\delta_{nn'}\cot\delta_l\right] = 0$

Lüscher '86; Rummukainen, Gottlieb '95; Kim et al. '05; Luu, Savage'11; Göckeler et al. '12; Fu '12; Lescovec, Prelovsek '12; …

infinite-dimensional matrix written in terms of the Lüscher zeta functions $\mathscr{Z}^{\mathbf{P}}_{lm}(1,q^2)$

Limitations and issues:

- Requires sufficiently large volumes $(M_\pi L \gtrsim 3)$
- Partial-wave mixing effects
  - — Extreme truncation (1 PW): one-to-one correspondence between FV energies and $\delta_l(q)$
  - — Several PWs: Need a **parametrization** of $T_l(q)$ like e.g. the ERE Morningstar et al.'17
    Pole extraction from $\delta_l(q_i)$ is also usually carried out using the ERE.
    But the ERE is only valid for $|q| < M_\pi/2$… Is addressed for NN in 1504.07852, 1604.02551 and for the T_{cc} state in 2023.09441
- The t-channel cut problem
  - — For FV energy levels below the LHC, the above Lüscher's QC is invalid

Actually, all above limitations are ultimately related to the LHC-physics (as for PW mixing effects, they start becoming relevant at $q \sim M_\pi$, since $T_l(q) \xrightarrow{q\to 0} \sim (q/M_\pi)^{2l}$)

Low-energy theorems in the infinite volume

Idea: Benefit from the known long-range interaction due to the one-pion exchange to:

— Construct ***model-independent*** parametrization of $T_l(E)$ valid well beyond the ERE (useful for analyzing lattice data at physical & unphysical M_π)

— Provide an efficient method tor reconstructing the 2-body scattering amplitude from finite-volume energies, which is ***applicable also below the t-channel cut***

ERE and MERE

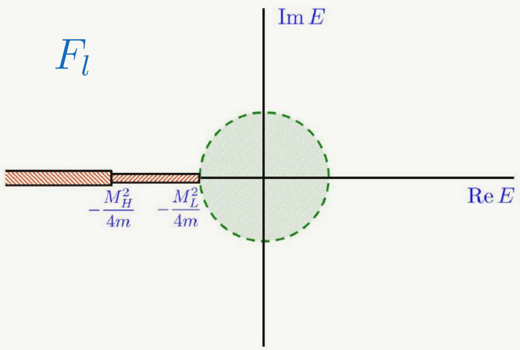
Consider two-range potential $V(r) = V_L(r) + V_S(r)$ with $M_L^{-1} \gg M_H^{-1}$

- Effective range expansion Landau, Smorodinsky '44, Bethe '49

$$T_l(k) = -\frac{16\pi^2}{m} \frac{k^{2l}}{F_l(k) - ik^{2l+1}}$$

$\equiv k^{2l+1} \cot \delta_l$ — *real meromorphic function of* k^2 *in* $|k| \le M_L/2$

$$\Rightarrow \text{ ERE: } F_l = -\frac{1}{a} + \frac{1}{2} r k^2 + v_2 k^4 + v_3 k^6 + \ldots$$



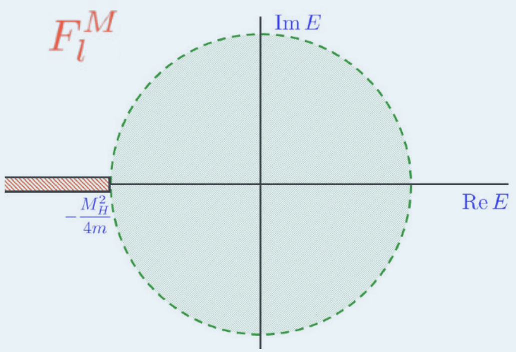
- Modified effective range expansion van Haeringen, Kok '82

$$F_l^M(k^2) \equiv M_l^L(k) + \frac{k^{2l+1}}{|f_l^L(k)|^2} \cot\left[\delta_l(k) - \delta_l^L(k)\right]$$

$$f_l^L(k) = \lim_{r \to 0} \left(\frac{l!}{(2l)!} (-2ikr)^l f_l^L(k, r) \right)$$

Jost function for $V_L(r)$ *Jost solution for* $V_L(r)$

$$M_l^L(k) = Re\left[\frac{(-ik/2)^l}{l!} \lim_{r \to 0} \left(\frac{d^{2l+1}}{dr^{2l+1}} \frac{r^l f_l^L(k, r)}{f_l^L(k)} \right) \right]$$



Per construction, F_l^M reduces to F_l for $V_L = 0$ and is meromorphic in $|k| < M_H/2$

MERE and low-energy theorems

Example: proton-proton scattering

$$F_C(k^2) = C_0^2(\eta)\, k\, \cot[\delta(k) - \delta^C(k)] + 2k\,\eta\, h(\eta) = -\frac{1}{a^M} + \frac{1}{2} r^M k^2 + v_2^M k^4 + \ldots$$

where $\underbrace{\delta^C \equiv \arg\Gamma(1+i\eta)}_{\text{Coulomb phase shift}}$, $\eta = \frac{m}{2k}\alpha$, $\underbrace{C_0^2(\eta) = \frac{2\pi\eta}{e^{2\pi\eta}-1}}_{\text{Sommerfeld factor}}$, $h(\eta) = \mathrm{Re}\Big[\underbrace{\Psi(i\eta)}_{\text{Digamma function}}\Big] - \ln(\eta)$

Digamma function $\Psi(z) \equiv \Gamma'(z)/\Gamma(z)$

MERE and low-energy theorems

Long-range forces impose correlations between the ER coefficients ($\equiv$ low-energy theorems)

Cohen, Hansen '99; Steele, Furnstahl '00; Baru, EE, Filin, Gegelia '15

$$F_l^M(k^2) \equiv M_l^L(k) + \frac{k^{2l+1}}{|f_l^L(k)|^2} \cot\left[\delta_l(k) - \delta_l^L(k)\right]$$

$F_l^M(k^2)$: meromorphic for $k^2 < (M_H/2)^2$

$M_l^L(k)$, $f_l^L(k)$, $\delta_l^L(k)$: can be computed if the long-range force is known

- approximate $F_l^M(k^2)$ by first 1,2,3,… terms of the Taylor expansion in k^2
- calculate all "soft" quantities
- reconstruct $\delta_l^L(k)$ and predict all coefficients in the ERE

Toy model: Low-energy theorems

$$V(r) = \underbrace{v_L e^{-M_L r} f(r)}_{V_L} +$$

where $f(r) = \dfrac{(M_H r)^2}{1 + (M_H r)^2}$

and $M_L = 1.0\,,\ v_L = -0.875\,,$

ERE and MERE

	a	r	v_2	v_3	v_4
F_0 [fm^n]	5.458	2.432	0.113	0.515	−0.993
F_0^M [M_S^{-n}]	1.710	−1.063	−0.434	−0.680	2.624

Low-Energy Theorems

	LO	NLO	NNLO	"Exp"
r				2.432197161
v_2				0.112815751
v_3				0.51529
v_4				−0.9928

The simplest implementation: use E(F)T with explicit OPEP and let QM do the job…

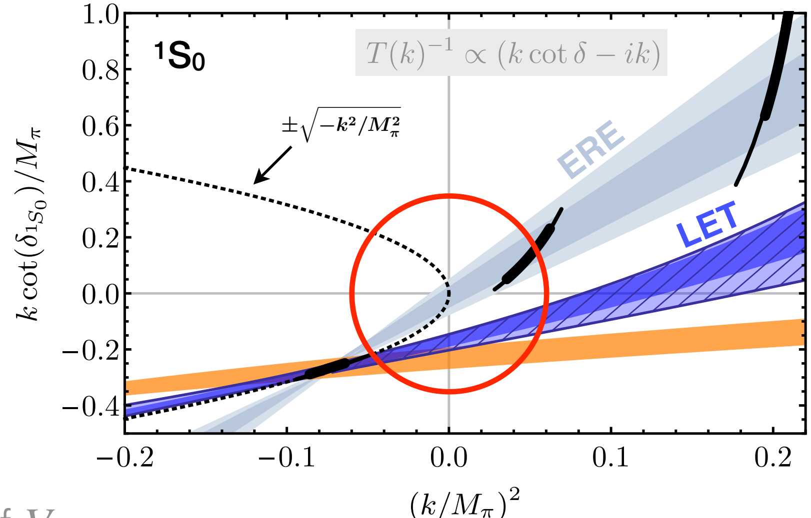
Application 1: LETs as a sanity check of LQCD results

Baru, EE, Filin, Phys. Rev. C94 (2016) 014001

2015 NPLQCD study of NN at $M_\pi = 450$ MeV:

Orginos et al., Phys. Rev. D92 (2015) 114512

$$B_{nn} = 12.5\left(^{+3.2}_{-4.9}\right)\ \text{MeV}$$



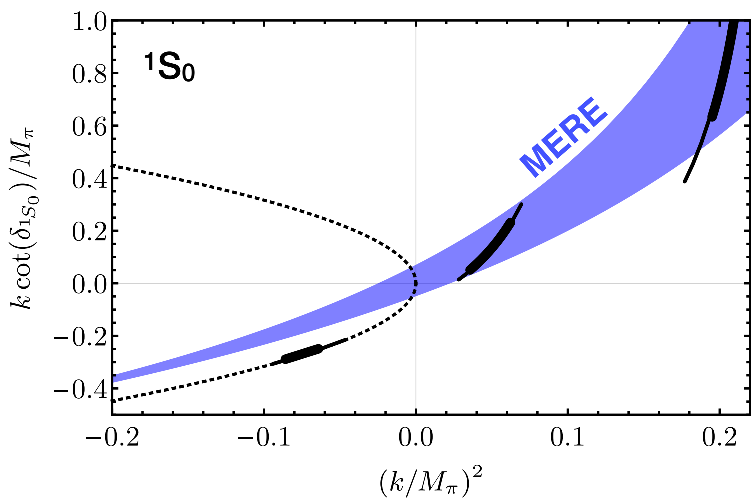
The ERE-fit yielded the large effective range:

$$r_{1S0} \simeq 6.7\, M_\pi^{-1}$$

If true, this would suggest:

— the interaction range (much) longer than that of $V_{1\pi}$

— or the appearance of a pole in $k\cot\delta$ near the threshold



⇒ In any case, no reason to expect the ERE to be valid for $|k| > 2/r \sim M_\pi/4$.

Use MERE (valid in a large kinematical range):

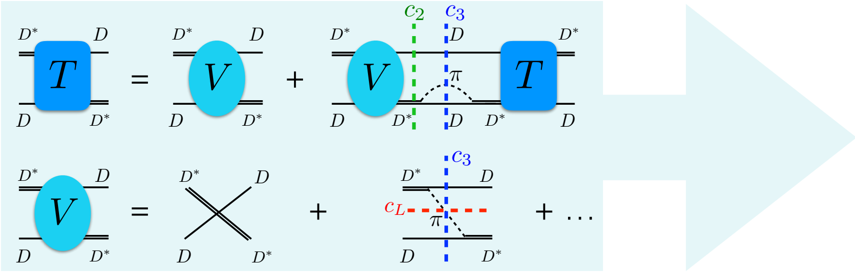
- LET prediction based on the BE inconsistent with NPLQCD scattering phase shifts
- MERE for to the lowest-energy $\delta(k)$ cannot accommodate for BE ~ 12 MeV

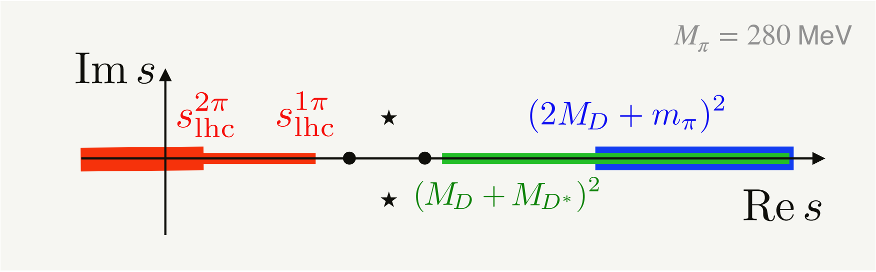
2021 re-analysis by NPLQCD (Illa et al.) is consistent with the MERE and LETs

Application 2: DD* scattering and the $T_{cc}(3875)^+$ state

Du, Filin, Baru, Dong, EE, Guo, Hanhart, Nefediev, Nieves, Wang, PRL 131 (2023) 131903

Re-analyzed LQCD data for DD* scattering at $M_\pi = 280$ MeV of Padmanath, Prelovsek PRL 129 (2022) 032002 by taking into account effects from the LHC



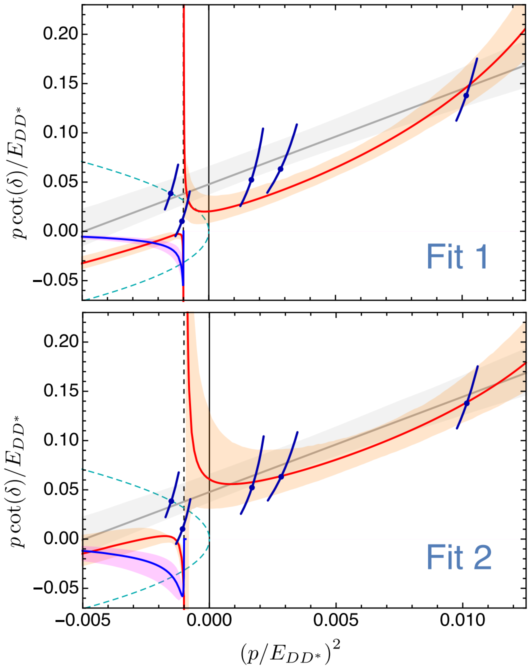


For this LQCD setup, the LHC starts at $p_{\rm lhc}^2 \simeq \frac{1}{4}\left[(M_{D^\star} - M_D)^2 - M_\pi^2\right] \sim -(126\ \text{MeV})^2$

Notice:

— Below the LHC, $p \cot\delta$ acquires an imaginary part

— The ER function possesses a near-threshold pole (accidentally) located very close to the LHC



Fit 1 (all points): a pair of virtual states

Fit 2 (3 points above threshold): narrow resonance

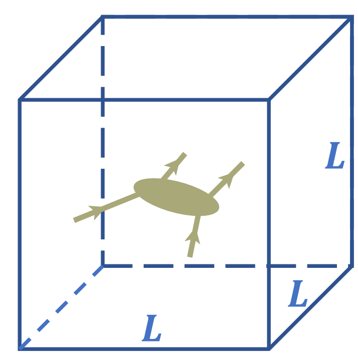
But how reliable are the phase shifts extracted using the Lüscher method given the nearby LHC?

LETs in a finite volume: 2-particle scattering from FV spectra without Lüscher

Lu Meng, EE, JHEP 10 (2021) 051; Lu Meng et al., e-Print: 2312.01930

- EFT-based Hamiltonian: long-range (1π) + contact interactions

- LECs depend on UV (cutoff) but not on IR (volume) physics $\Rightarrow$ same EFT can be used in the infinite and in finite volume
- Solve the theory in a box using the plain wave basis and fix the LECs from FV energies
- Extract real-world scattering observables by solving the theory in the infinite volume



Formalism

Infinite volume: Lippmann-Schwinger (or 3-dim-reduced Bethe-Salpeter equation) in the plane-wave basis (CMS)

$$T(\boldsymbol{p},\boldsymbol{p}';E) = V(\boldsymbol{p},\boldsymbol{p}';E) + \int_{q<q_{\max}} \frac{d^3\boldsymbol{q}}{(2\pi)^3} V(\boldsymbol{p},\boldsymbol{q};E)G(q;E)T(\boldsymbol{q},\boldsymbol{p}';E)$$

Finite volume: Momenta $\boldsymbol{p}_1,\ \boldsymbol{p}_2 = \boldsymbol{P} - \boldsymbol{p}_1$ are quantized $\boldsymbol{P} = \frac{2\pi}{L}\boldsymbol{d}, \quad \boldsymbol{d} \in Z^3$

$$\int \frac{d^3\boldsymbol{q}^*}{(2\pi)^3} f(\boldsymbol{q}^*) = \int \mathcal{J} \frac{d^3\boldsymbol{q}}{(2\pi)^3} f[\boldsymbol{q}^*(\boldsymbol{q})] \xrightarrow{FV} \frac{1}{L^3} \sum_{\boldsymbol{n}\in Z^3} \mathcal{J} f[\boldsymbol{q}^*(\boldsymbol{q_n})]$$

Döring et al., EPJA 48 (12) 114;
Li et al., PRD 103 (21) 094518

LS equation reduces to the matrix equation $\underbrace{\mathbb{T}}_{\text{block-diagonal according to } \Gamma_a} = \mathbb{V} + \mathbb{V}\mathbb{G}\mathbb{T}$; T-matrix poles from

$$\det\left(\mathbb{G}^{-1} - \mathbb{V}\right) = 0 \quad \text{or} \quad \det(\mathbb{H} - E\mathbb{I}) = 0 \quad \text{or just} \quad \underbrace{\mathbb{H}\,|\Psi\rangle = E\,|\Psi\rangle}$$

only real, positive momenta needed ⇒ no LHC problem!

Lüscher's Quantization Condition: $\underbrace{\det\left\{\mathbb{M}^{(\Gamma,P)}(E) - \text{diag}[\cot\delta_l(E)]\right\} = 0}$

relationship between FV energy levels E (T-matrix poles) and the ***on-shell amplitude*** $\cot\delta_l(E)$*, which possesses the LHC*

Alternative solution: Raposo, Hansen, 2311.18793

Application 1: Two-nucleon scattering (spin-0)

Lu Meng, EE, JHEP 10 (2021) 051

Motivation:

- Quantify partial-wave mixing effects from the 1π-exchange at physical pion masses
- Test the method to extract phase shifts from FV energies

Instead of lattice-QCD data, use FV energy spectra from realistic NN potential models.

Benchmarking (no partial-wave mixing)

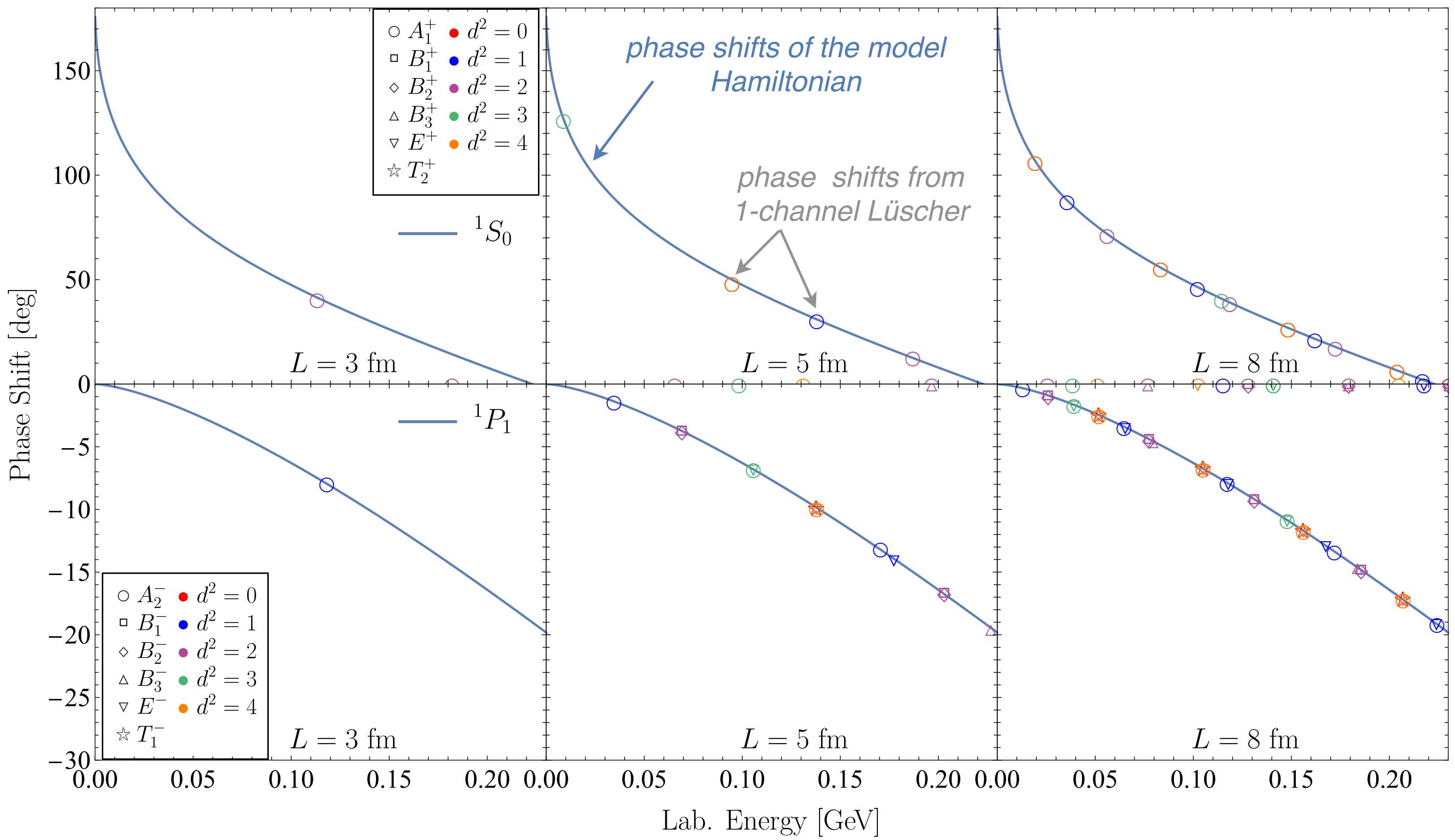
To test the approach, use π-less EFT at NLO. The Hamiltonian in the infinite volume:

$$V_{1S0}(p',p) \equiv \langle l=0,p'|\hat{V}|l=0,p\rangle = \left[\tilde{C}_{1S0} + C_{1S0}(p'^2+p^2)\right] e^{-\frac{p'^2+p^2}{\Lambda^2}}$$

$$V_{1P1}(p',p) \equiv \langle l=1,p'|\hat{V}|l=1,p\rangle = C_{1P1}\, p' p\, e^{-\frac{p'^2+p^2}{\Lambda^2}}$$

The LECs are chosen to mimic (qualitatively) the behavior of NN phase shifts

Per construction, no partial wave mixing…

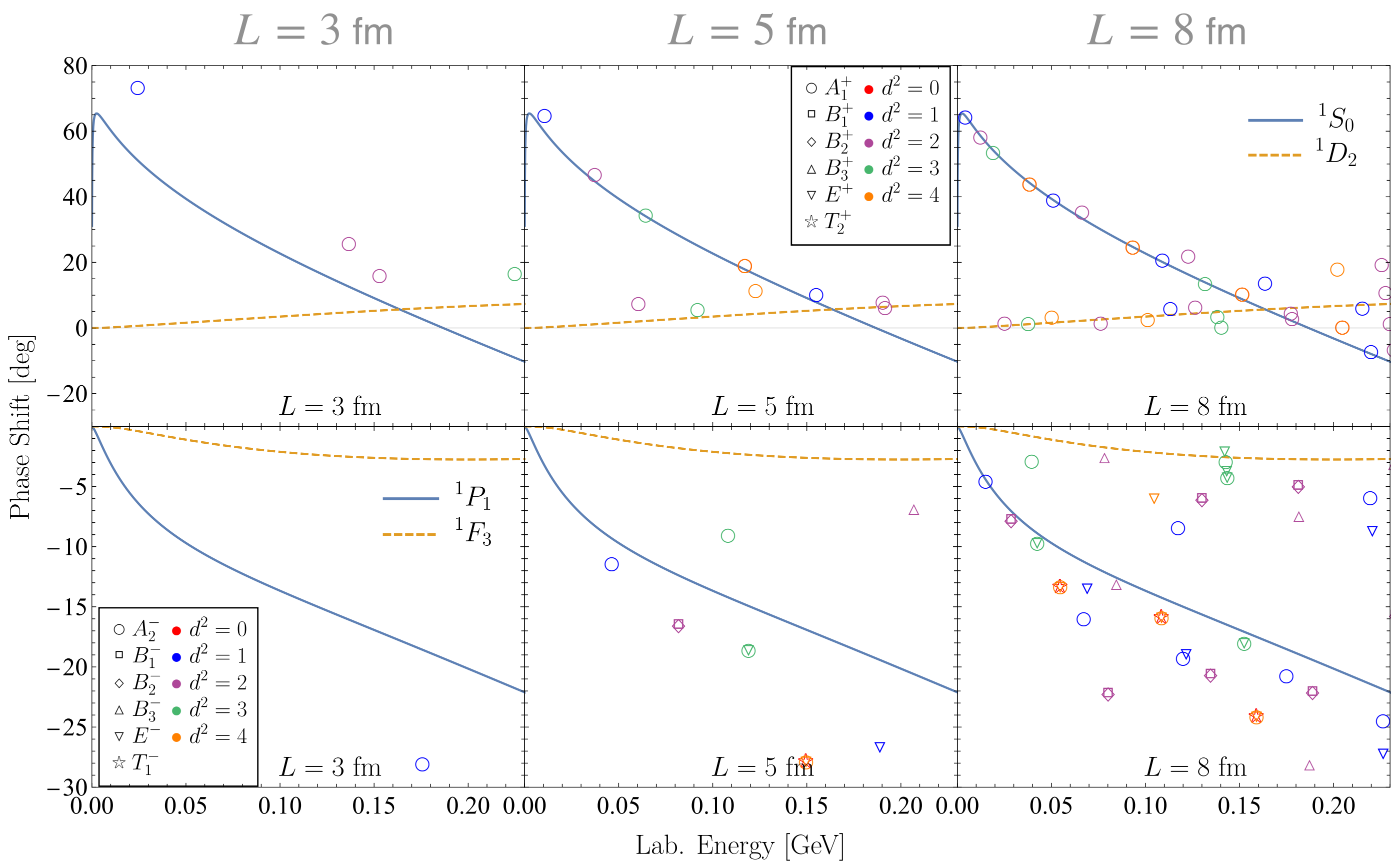


positive parity (i.e., S-wave)

negative parity (i.e., P-wave)

Partial-wave mixing effects at the physical M_π

Chiral EFT@N²LO as an example of realistic NN interaction: $V = V_{1\pi} + V_{2\pi} +$ contact terms



positive parity (S-, D-, …, waves)

negative parity (P-, F-, …, waves)

— deviations for the smallest box $L = 3$ fm to be expected ($M_\pi L \sim 2$)

— fairly small PW mixing effects for positive-parity states

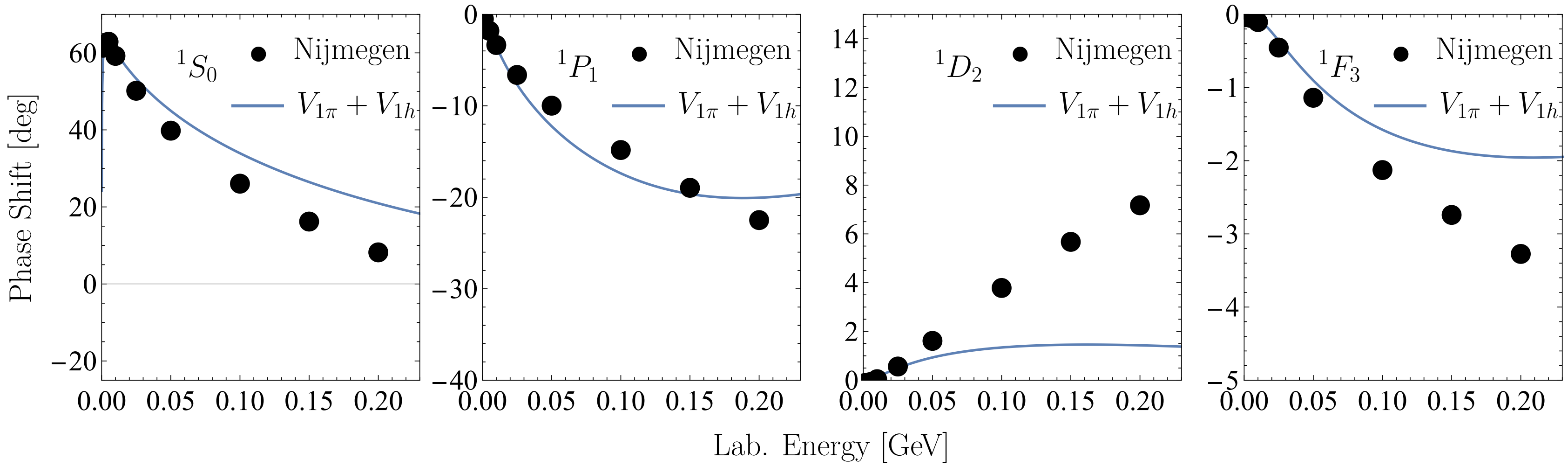
— surprisingly large PW mixing effects for negative-parity states, no improvement for large L

Phase shifts from FV-energies using EFT

- Toy-model for spin-0 NN interaction:

$$V_{\text{toy}} = V_{1\pi} + V_{1h} = \left[\underbrace{-\left(\frac{g_A}{2F_\pi}\right)^2 \frac{M_\pi^2}{\boldsymbol{q}^2 + M_\pi^2} \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2}_{\text{long-range}} + \underbrace{(c_{h1} + c_{h2}\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) \frac{1}{\boldsymbol{q}^2 + m_h^2}}_{\text{short-range}} \right] e^{-\frac{p^2 + p'^2}{\Lambda^2}}$$

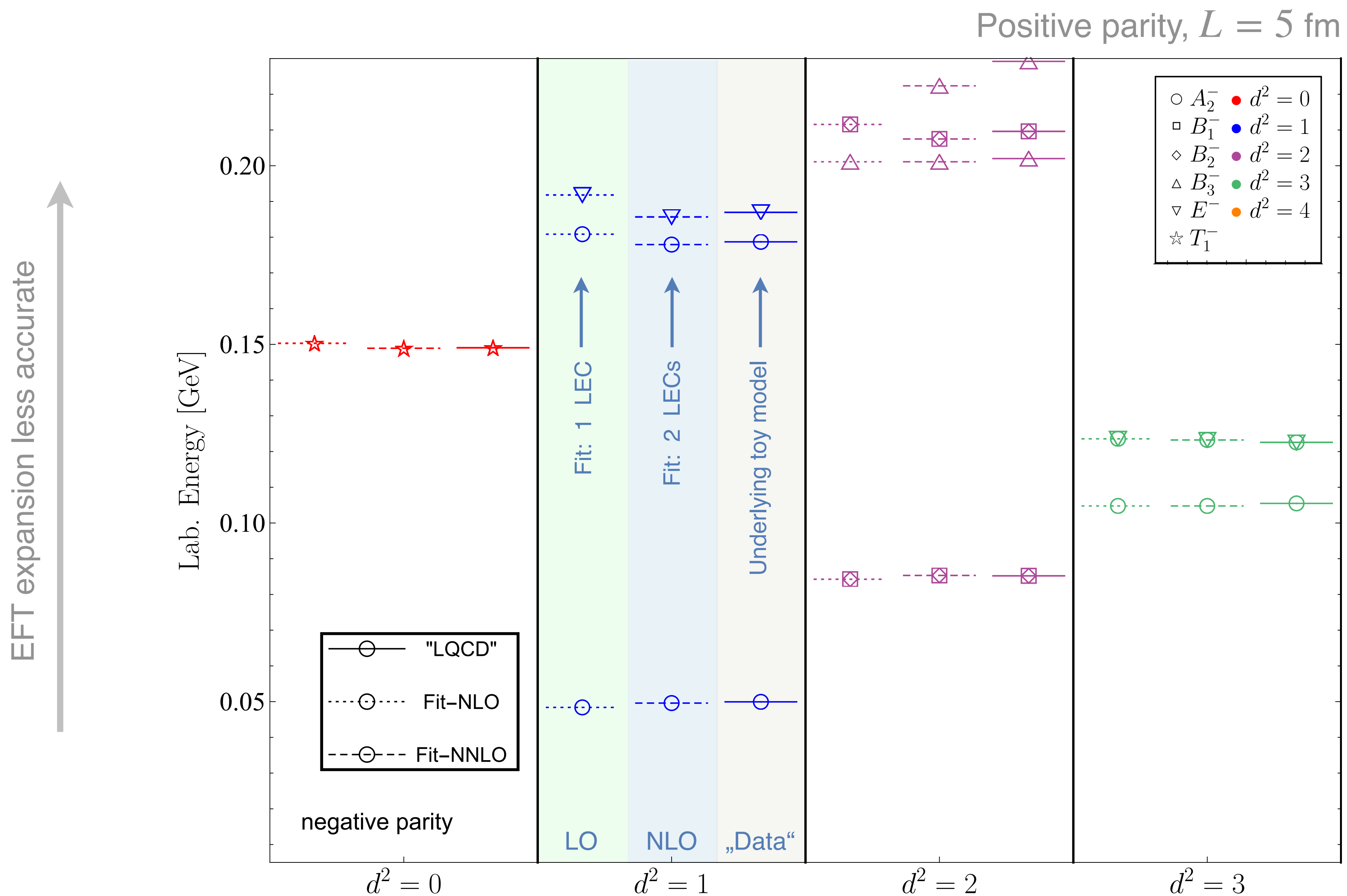
Reproduces (qualitatively) the empirical phase shifts:



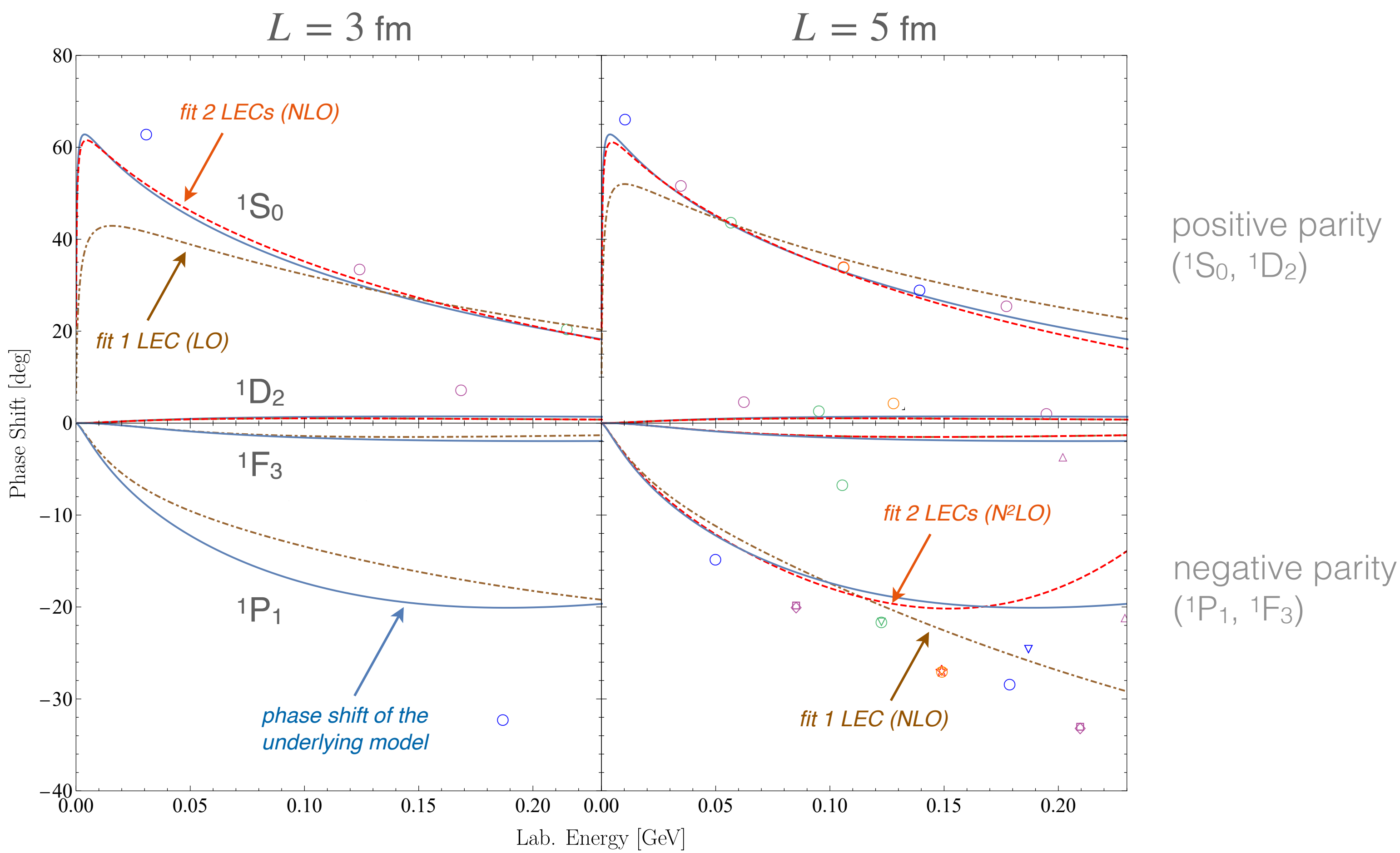
- Effective theory: $$V_{\text{EFT}} = V_{1\pi} + \underbrace{V_{\text{cont}}^{(0)}}_{\text{1 LEC (S)}} + \underbrace{V_{\text{cont}}^{(2)}}_{\text{+2 LECs (S,P)}} + \underbrace{V_{\text{cont}}^{(4)}}_{\text{+3 LECs (S,P,D)}} + \ldots$$

⟵ *tuned to reproduce the FV energy spectrum*

Phase shifts from FV-energies using EFT



Phase shifts from FV-energies using EFT



— Using an EFT, tuned to FV energies, allows one to avoid complications from PW mixing

— Systematically improvable, good convergence for $p \sim M_\pi$ (as we know from χEFT for NN)

— Works even for $L = 3$ fm (lattice size actually limited by the number of low-lying states…)

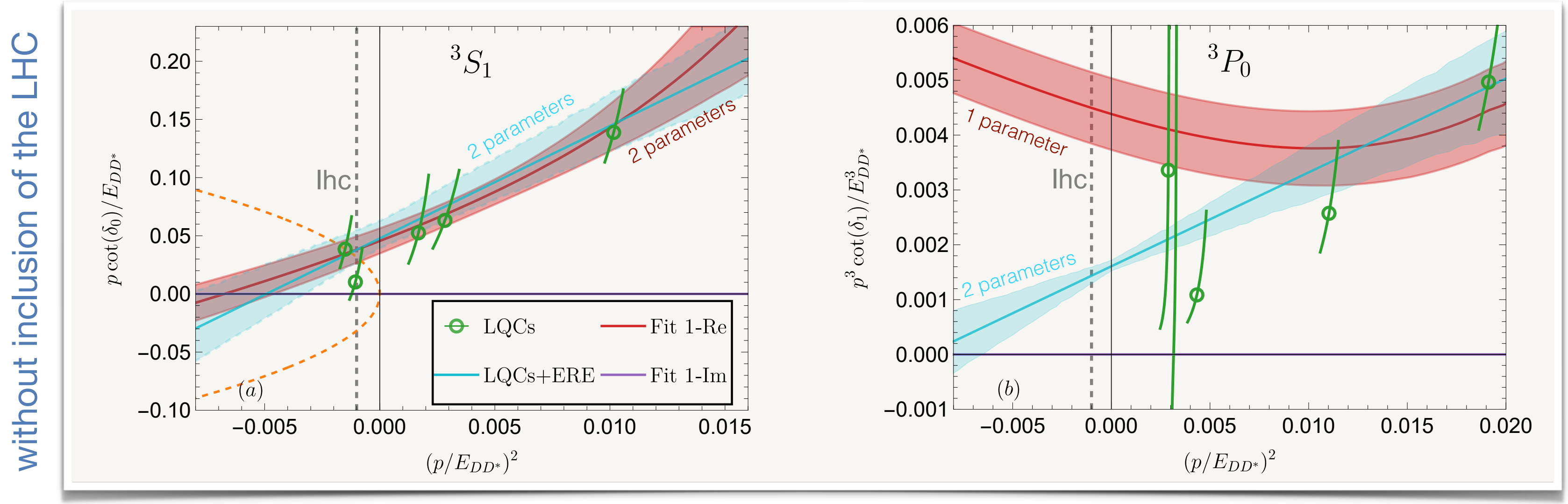
Application 2: $T_{cc}(3875)^+$ from FV energy levels

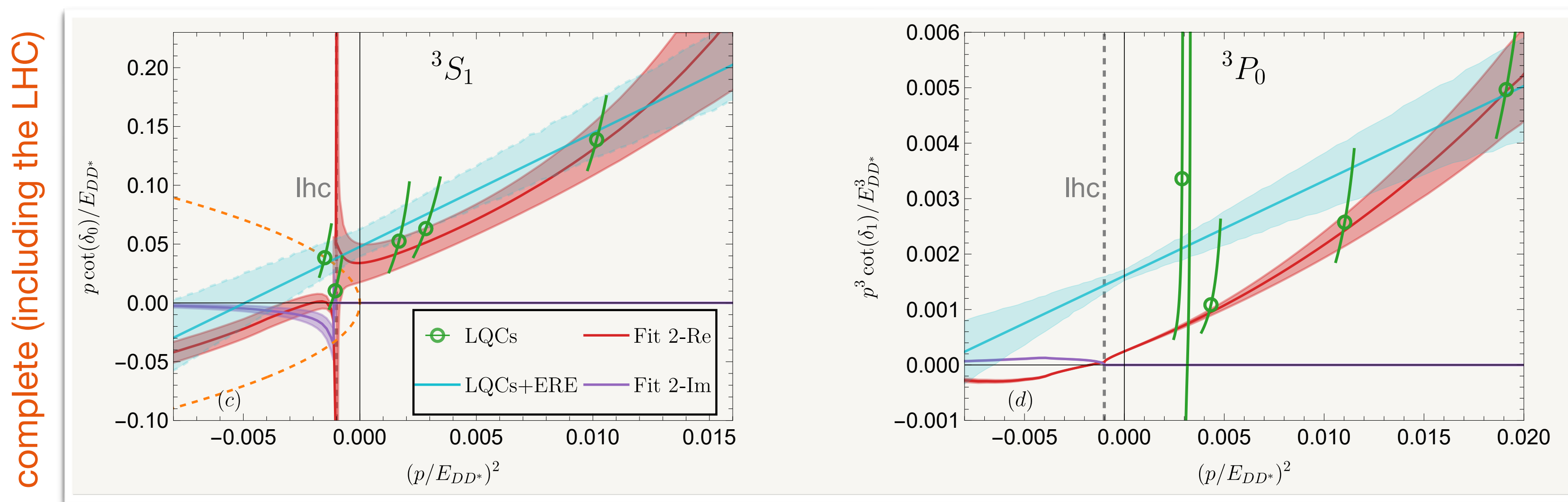
Lu Meng, Vadim Baru, EE, Arseniy Filin, Ashot Gasparyan, e-Print: 2312.01930

Goal: Analyze the FV L-QCD data from Padmanath, Prelovsek PRL 129 (22) 032002
with no reliance on the Lüscher method

$T_{cc}(3875)^+$ from FV energy levels

Lu Meng, Vadim Baru, EE, Arseniy Filin, Ashot Gasparyan, e-Print: 2312.01930





Summary and conclusions

The knowledge of long-range physics (1π-exchange) allows one to go beyond the ERE when parametrizing $T_l(E)$ and beyond the „exponentially suppressed corrections" when relating FV energies and 2-body scattering amplitude.

EFT-based Hamiltonian with 1π-exchange + PW basis + matching to FV energy spectra:

— no issues with partial-wave mixing

— no left-hand cut problem

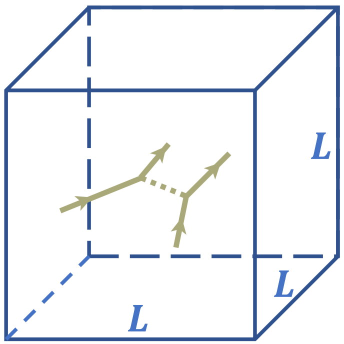
— smaller boxes

First applications have revealed:

— strong partial-wave mixing effects in NN scattering @physical M_π due to $V_{1\pi}$, the 1-channel Lüscher formula is a no-go

— 1π-exchange plays an important role for analyzing LQCD data for DD* scattering and will be even more important for smaller M_π:

$$M_{\rm eff}^2 = (M_{D^\star} - M_D)^2 - M_\pi^2$$



Future: Other systems, 3-body scattering…