Conformal field theory and the hot phase of the 3D $U(1)$ gauge theory

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in collaboration with

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Confinement and center symmetry in pure-gauge theories

In Yang-Mills theories with symmetry gauge group G, the deconfinement transition is related to the breaking of the "center" symmetry

- it is a global symmetry involving transformations z which are part of the center of G
- on the lattice, all the temporal links on a given slice are multiplied by z
- this leaves the action invariant, as for any loop (e.g. plaquette) for every z there is also a z^*

A special case is the Polyakov loop that transforms as

$$\mathcal{P} \to z\mathcal{P}$$

and represents the order parameter of the spontaneous breaking of the center symmetry

$\langle P \rangle = 0$ in the confined phase and $\langle P \rangle \neq 0$ in the deconfined phase

One can think of integrating out all degrees of freedom except the order parameter itself

$\to$ effective field theory with a global symmetry given by the center of G

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The Svetitsky-Yaffe conjecture

The argument of **[Svetitsky,Yaffe; 1982]** is that at $T = T_c$, in the case of a continuous (2nd order) phase transition, there is an equivalence for the critical behaviour of

the theory with gauge symmetry group G in $d+1$ spacetime dimensions	$\Leftrightarrow$	a spin model in d dimensions with the center of G as a global symmetry group

Example: universality class in

$\mathrm{SU}(3)$ in 2+1 dimensions	$\Leftrightarrow$	3-state Potts model in 2 dimensions
$\mathrm{SU}(2)$ in 3+1 dimensions		Ising model in 3 dimensions

Moreover, the phases are "exchanged": the correct mapping is

"hot" phase $\Leftrightarrow$ "ordered" phase

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There is a "special" case:

$\mathrm{U}(1)$ gauge theory in 2+1 dimensions	$\Leftrightarrow$	$\mathrm{U}(1)$ spin model in 2 dimensions XY model

Mermin-Wagner theorem $\Rightarrow$ no spontaneous symmetry breaking $\Rightarrow$ Kosterlitz-Thouless transition

phases are characterized by "topological" order

In addition, the low-temperature phase of the XY model

- is scale-invariant
- has a conformal-field-theory description
- can be considered as a line of fixed points

What happens in the corresponding (hot, $T > T_c$) phase in the $\mathrm{U}(1)$ theory? Can we extend the conjecture?

Test: conformal-field theory predictions for $\mathrm{U}(1)$ observables!

1. $\mathrm{U}(1)$ pure gauge theory in 2+1 dimensions
   - The compact formulation, on the lattice
   - Observables in the dual formulation

2. The XY model
   - The KT transition
   - A conformal description at low-T

3. Numerical results for the hot phase in the $\mathrm{U}(1)$ model

$\mathrm{U}(1)$ pure gauge theory in $2+1$ dimensions

The model

Our theory: $\mathrm{U}(1)$ gauge theory (QED) without matter fields in a three-dimensional spacetime

Action in the continuum:

$$S_{\mathrm{cont}} = -\frac{1}{4} \int \mathrm{d}^2 x \int \mathrm{d}\, t\, F_{\mu\nu} F^{\mu\nu}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Note that in classical electrodynamics in 2 spatial dimensions:

- the Coulombic potential is logarithmic
- the magnetic field is a scalar

Many interesting properties emerge in the compact formulation of this QFT

$\rightarrow$ when the gauge field components A_μ are periodic

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The model on the lattice

The compact formulation is easy to implement when discretizing the spacetime on a lattice Λ

- Wilson action

$$S_W = -\frac{1}{ae^2} \sum_{x \in \Lambda} \sum_{1 \le \mu < \nu \le 3} \mathrm{Re}\, U_{\mu\nu}(x)$$

  with the usual plaquette

$$U_{\mu\nu}(x) = U_\mu(x) U_\nu(x + a\hat{\mu}) U^\star_\mu(x + a\hat{\nu}) U^\star_\nu(x)$$

  and the parallel transporter $U_\mu(x) = \exp\left[ieaA_\mu\left(x + \frac{a}{2}\hat{\mu}\right)\right]$ on the links

- the invariance under

$$A_\mu \to A_\mu + \frac{2\pi k}{ea}$$

  is guaranteed → "compact" $\mathrm{U}(1)$

Field configurations admit topological defects ("magnetic monopoles") **[Polyakov; 1975,1976]**

- the ground state can be thought as a plasma of (anti)monopoles
- this condensation implies confinement of electric charges for all values of $\beta = \frac{1}{ae^2}$ as a dual Meissner effect
- dual superconductor model **[Nambu; 1974],[Mandelstam; 1976],['t Hooft; 1979]**

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How is Maxwell theory recovered?

Analytical proof **[Göpfert,Mack; 1982]** (in the Villain formulation) of the existence of

- a non-zero mass gap

$$m_D a \simeq k_1 \sqrt{\beta} \exp(-k_2 \beta) \quad \text{for } \beta \gg 1$$

- a finite string tension

$$\sigma a^2 \geq \frac{k_3}{\sqrt{\beta}} \exp(-k_2 \beta) \quad \text{for } \beta \gg 1$$

  the model is confining for any value of $\beta = \frac{1}{ae^2}$!

- in the continuum limit ($a \to 0$ means $\beta \to \infty$ at fixed e^2) the Maxwell theory of non-interacting photons is recovered!

Interesting note: the ratio behaves differently with respect to other confining gauge theories (where it is fixed up to a effects)

$$\frac{m_D^2}{\sigma} \propto \sqrt{\beta^3} \exp(-k_2 \beta)$$

$\rightarrow$ one can also take the continuum limit on a "line of constant physics" (i.e. keeping m_D or σ fixed) ...

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A phase transition at finite T

Non-zero temperature setup, temporal direction compactified with

$$T = \frac{1}{N_t a}$$

Similarly to $\mathrm{SU}(N)$ gauge models, a phase transition takes place at a critical temperature T_c

- in the low-temperature phase, the theory is linearly confining
- in the high-temperature region no mass scale (or Debye screening) is present
- " logarithmic confinement" is expected

Interpretation: monopole-antimonopole confining plasma turns into a dipole plasma at T_c **[Chernodub et al.; 2001]**

Recent results confirm the "XY nature" of the transition **[Borisenko et al.; 2015]**

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A different formulation

The usual partition function/path integral

$$\mathcal{Z} = \int \prod_{x,\mu} \mathrm{d}\, U_\mu(x) \exp[-S_W]$$

can be formulated in terms of a spin model with global $\mathbb{Z}$ symmetry

→ integer-valued variables s placed on the sites of the dual lattice

$$\mathcal{Z} = \sum_{\{s\}} \prod_{y,\nu} I_{|s(y)-s(y+a\hat{\nu})|}(\beta)$$

- I_α = modified Bessel function of 1st kind of order α
- $s(y) - s(y + a\hat{\nu})$ difference of $\mathbb{Z}$ variables at the ends of each bond
- $\prod$ taken on bonds of the dual lattice

→ expectation values of interesting quantities can be rewritten in terms of ratios of partition functions!

Observables - Static potential

The **Polyakov-loop correlation function** can be rewritten as

$$\langle \mathcal{P}^\star(r)\mathcal{P}(0)\rangle = \frac{\mathcal{Z}_{N_t \times N_r}}{\mathcal{Z}}$$

with

$$\mathcal{Z}_{N_t \times N_r} = \sum_{\{s\}} \prod_{y,\nu} I_{|s(y)-s(y+a\hat{\nu})+n_\nu(y)|}(\beta)$$

the variables $n_\nu(y)$

- are zero on the oriented bonds of the dual lattice
- but not on the bonds dual to the surface $N_t \times N_r$ bounded by the two Polyakov loops: in that case $n_\nu(y) = 1 \rightarrow$ frustration or "defect"

Note that $\langle \mathcal{P}^\star(r)\mathcal{P}(0)\rangle$ can be rewritten in order to use error-reduction techniques (snake algorithm **[De Forcrand et al.; 2001]**)

In particular we look at

$$\frac{\langle \mathcal{P}^\star(r+a)\mathcal{P}(0)\rangle}{\langle \mathcal{P}^\star(r)\mathcal{P}(0)\rangle} = \prod_{i=0}^{N_t-1} \frac{\mathcal{Z}_{(N_t \times N_r)+i+1}}{\mathcal{Z}_{(N_t \times N_r)+i}} = \prod_{i=0}^{N_t-1} \left\langle \frac{I_{|s(x)-s(x+a\hat{\nu})+1|}(\beta)}{I_{|s(x)-s(x+a\hat{\nu})|}(\beta)} \right\rangle_{(N_t \times N_r)+i}$$

where each factor in the r.h.s. can be computed using hierarchical updates **[Caselle et al.; 2003]**

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Observables - Flux tube profile

Another crucial observable: the **profile of the flux tube** between a pair of static eletric sources

Connected correlator of the field-strength $E(x_t)$ in a background of two Polyakov lines (sources)

$$W(r, x_t) = \frac{\langle \mathcal{P}^\star(r)\mathcal{P}(0)E(x_t)\rangle}{\langle \mathcal{P}^\star(r)\mathcal{P}(0)\rangle} - \langle E(x_t)\rangle$$

where $E(x_t)$ is placed at a transverse distance x_t from the temporal plane of the sources

In the dual formulation it becomes simply

$$W(r, x) = \frac{\langle s(x) - s(x + a\hat{\nu}) + n_\nu(x)\rangle_{N_t \times N_r}}{\sqrt{\beta}}$$

- at $T = 0$ the profile is exponential **[Caselle et al.; 2016]**
- in agreement with the dual superconductor model

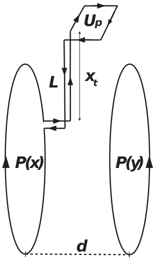

Image from **[EPJ Web Conf. 175 (2018) 12006]**

The XY model

The XY model in two dimensions

The well-known Hamiltonian of the XY model is

$$H = -J \sum_{\langle x,y \rangle} \vec{s}(x) \cdot \vec{s}(y) = -J \sum_{\langle x,y \rangle} \cos[\theta(x) - \theta(y)]$$

where $\vec{s}(x)$ are two-component real vectors of unit length and θ is the angle with a respect to a predefined direction

It has an $O(2)$ internal global symmetry and is ferromagnetic for $J > 0$

- for $T > 0$ the system is in a disordered phase, for which $\langle \vec{s} \rangle = 0$
  $\rightarrow$ Mermin-Wagner theorem
  systems with continuous symmetries are always disordered by thermal fluctuations in 2D
- "conventional" long-range order not allowed at low-T
- however, for $T \rightarrow 0$ a non-conventional "quasi-long-range" order emerges, where the lowest-energy states are spin-waves
- signal of a phase transition at $T = T_{KT}$

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The Kosterlitz-Thouless transition

One can distinguish two different phases by analyzing the behaviour of the spin-spin correlation function

$$G(r) = \langle \vec{s}(x) \cdot \vec{s}(y) \rangle, \qquad \text{with } r = |x - y|$$

- in the high-T phase we have

$$G(r) \sim \exp\left(-\frac{r}{\xi}\right)$$

where ξ is a temperature-dependent correlation length that for $T \to T_{KT}$

$$\frac{\xi}{a} \sim \exp\left(\frac{b}{\sqrt{\tau}}\right), \qquad \tau = \frac{T}{T_{KT}} - 1$$

$\to$ infinite-order phase transition!

- while in the low-T phase

$$G(r) \sim \left(\frac{r}{L}\right)^{-\eta}$$

where η depends on T as well

$$\eta(T) = \frac{T}{2\pi J} = \frac{1}{2\pi K} \qquad \eta(T_{KT}) = 1/4$$

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The Kosterlitz-Thouless transition

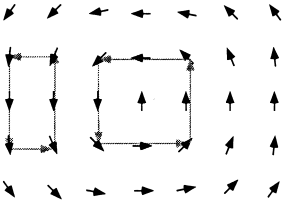
Where does this behaviour come from?

→ "vortex" solutions: when the θ field winds around a point (vortex center) n times

$$\oint \nabla\theta \cdot \mathrm{d}\, l = 2\pi n$$

and to each vortex we associate

$$\text{energy} \sim \pi n^2 J \ln\left(\frac{L}{a}\right)$$



If the entropy goes just like $S \sim \ln\left((L/a)^2\right)$ then the free energy for an isolated vortex is

$$F_v \sim (\pi J - 2T) \ln\left(\frac{L}{a}\right)$$

and so the creation of a free vortex becomes possible when

$$\frac{T_{KT}}{J} = \frac{\pi}{2}$$

more complicated with more than one vortex: the actual value is

$$\frac{T_{KT}}{J} = 0.89294(8)$$

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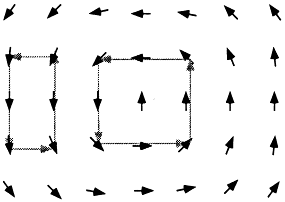

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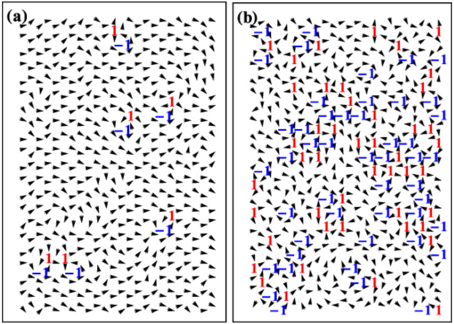

Image from **[J Phys Condens Matter 25, 065501 (2013)]**

at low temperatures: only vortex-antivortex pairs are energetically possible in the thermodynamic limit

at high temperatures: the entropy increase compensates the energy for the creation of an isolated vortex $\rightarrow$ free vortices proliferate and interact with each other with a logarithmic Coulomb potential

A conformal description for the low-T phase of the XY model

We know that for $T < T_{KT}$

$$G(r) \sim \left(\frac{r}{L}\right)^{-\eta}$$

$\rightarrow$ the correlation function is scale-invariant!

for any length x, the system is invariant under global dilatation transformations

$$x \rightarrow \lambda x$$

For classical systems with local interactions: invariance under global dilatations + translations + rotations $\Rightarrow$ conformal invariance

$$x \rightarrow \lambda(x) x$$

a simple way of understanding conformal transformations: they leave invariant the relative angle between vectors

- a conformal field theory (CFT) description of the XY model in the low-T phase is possible
- it is the theory of a free, massless compact bosonic field with central charge $c = 1$
- the field is identified with the θ phase

Note that conformal invariance implies the lack of a mass gap or of Debye screening: consistent with hot phase of U(1)

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A conformal description for the U(1) hot phase - $\mathcal{P}$ correlator

Crucial feature of conformal invariance → constraints on correlation functions

We consider *quasi-primary* fields Q_i that transform under general conformal mapping as

$$Q_i(x) = \left| \frac{\partial x'}{\partial x} \right|^{\Delta_i/D} Q_i(x')$$

Δ_i = "scaling dimension"

For the two-point function conformal invariance tells us

$$\langle Q_a(x_1) Q_b(x_2) \rangle = \frac{\delta_{ab}}{|x_1 - x_2|^{2\Delta_a}}$$

For the spin field $\vec{s}$

$$\langle \vec{s}(r) \cdot \vec{s}(0) \rangle \sim \frac{1}{r^\eta}$$

We identify the spins $\vec{s}$ with the Polyakov loop $\mathcal{P}$ and we investigate the ratio

$$H(r) = \frac{\langle \mathcal{P}^\star(r+a)\mathcal{P}(0)\rangle}{\langle \mathcal{P}^\star(r)\mathcal{P}(0)\rangle} = \left(1 + \frac{1}{r/a}\right)^{-\eta}$$

(where a is the lattice spacing) so that the normalization does not enter the analysis

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$$\langle \vec{s}(r) \cdot \vec{s}(0)\rangle \sim \frac{1}{r^\eta}$$

We identify the spins $\vec{s}$ with the Polyakov loop $\mathcal{P}$ and we investigate the ratio

$$H(r) = \frac{\langle \mathcal{P}^\star(r+a)\mathcal{P}(0)\rangle}{\langle \mathcal{P}^\star(r)\mathcal{P}(0)\rangle} = \left(1 + \frac{1}{r/a}\right)^{-\eta}$$

(where a is the lattice spacing) so that the normalization does not enter the analysis

A conformal description for the $\mathrm{U}(1)$ hot phase - $\mathcal{P}$ correlator

Crucial feature of conformal invariance → constraints on correlation functions

We consider *quasi-primary* fields Q_i that transform under general conformal mapping as

$$Q_i(x) = \left| \frac{\partial x'}{\partial x} \right|^{\Delta_i / D} Q_i(x')$$

$\Delta_i =$ "scaling dimension"

For the two-point function conformal invariance tells us

$$\langle Q_a(x_1) Q_b(x_2) \rangle = \frac{\delta_{ab}}{|x_1 - x_2|^{2\Delta_a}}$$

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A conformal description for the hot phase - flux tube

For the three point-function conformal invariance tells us that

$$\langle Q_a(x_1) Q_b(x_2) Q_c(x_3) \rangle = \frac{c_{abc}}{x_{12}^{\Delta_a - \Delta_b - \Delta_c} x_{13}^{\Delta_b - \Delta_a - \Delta_c} x_{23}^{\Delta_c - \Delta_a - \Delta_b}}$$

c_{abc} = structure constant of the algebra of the fields

For the flux-tube profile we have that

$$W(r, x) = \frac{\langle \vec{s}(r) \vec{s}(0) \phi(x) \rangle}{\langle \vec{s}(r) \vec{s}(0) \rangle} - \langle \phi \rangle = \frac{c_{ss\phi}}{(r/4)^{\Delta_\phi}} \left(1 + \frac{4x^2}{r^2}\right)^{-\Delta_\phi} = C(r) \left(1 + \frac{4x^2}{r^2}\right)^{-\Delta_\phi}$$

where

- Δ_ϕ is the scaling dimension of the operator ϕ that corresponds to the field-strength
- $\phi(x)$ is placed in the perpendicular line that goes through the midpoint between the two charges
- $\langle \phi \rangle = 0$ due to scale invariance
- any dependence on $2\Delta_s = \eta$ has disappeared

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Numerical results for the hot phase in the $\mathrm{U}(1)$ model

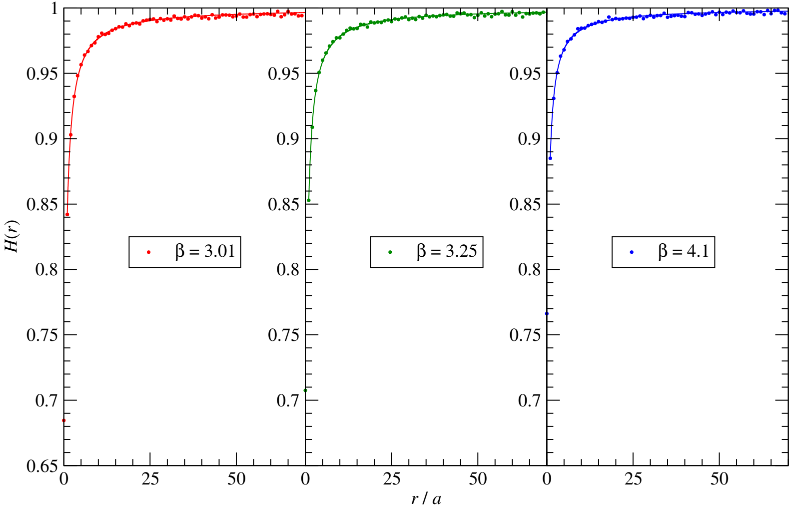
GOAL: describe quantities in the high-T region of the $\mathrm{U}(1)$ pure gauge theory using functional forms dictated by conformal invariance

- $N_s \gg N_t$ to avoid finite-size effects
- values of β for each N_t chosen considering the β_c from **[Borisenko et al.; 2015]**
- e.g. $\beta_c = 3.005$ for $N_t = 4 \to \beta \in [3.01, 4]$

We start with the Polyakov loop correlator ratio

$$H(r) = \frac{\langle \mathcal{P}^\star(r+a)\mathcal{P}(0)\rangle}{\langle \mathcal{P}^\star(r)\mathcal{P}(0)\rangle} = \left(1 + \frac{1}{r/a}\right)^{-\eta}$$

Fits of $H(r)$



$\eta = 0.2469(12)$ $\eta = 0.2299(10)$ $\eta = 0.1750(9)$

Behaviour of η

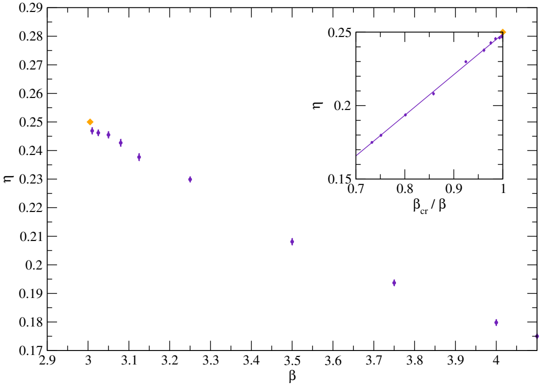
We recall that

$$\eta(T_{KT}) = 1/4$$

A (very) crude ansatz for $\eta(T)$

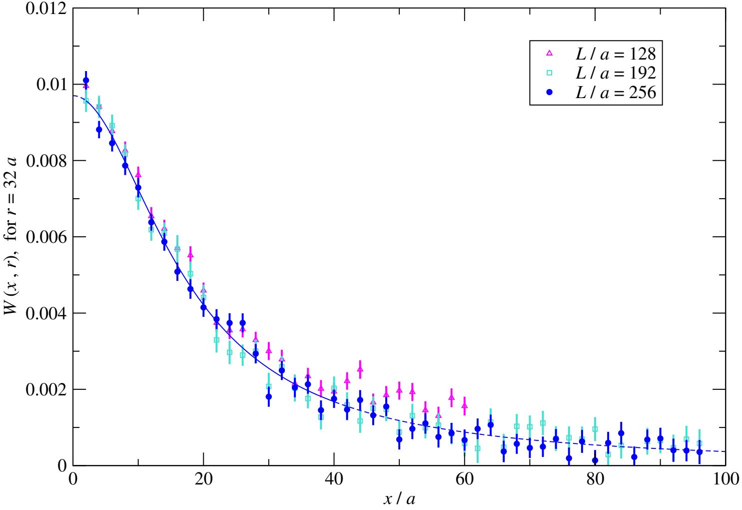
$$\eta = a_1 \frac{\beta_c}{\beta} + a_0$$

works surprisingly well



Now we look at the profile of the flux-tube ($\beta = 4.0$, $N_t = 4$)

$$W(r, x) = C(r)\left(1 + \frac{4x^2}{r^2}\right)^{-\Delta_\phi} \text{ with } C(r) \text{ and } \Delta_\phi \text{ as free parameters}$$



fit for fixed $r/a = 32$ in the $2 < x/a < 40$ range

$\Delta_\phi = 0.933(7)$ coming from weighted average at several r

Field-strength correlator

2-parameter fits of $W(x, r)$ not very precise at large r

There is a direct way of getting Δ_ϕ: looking at the field-strength correlator

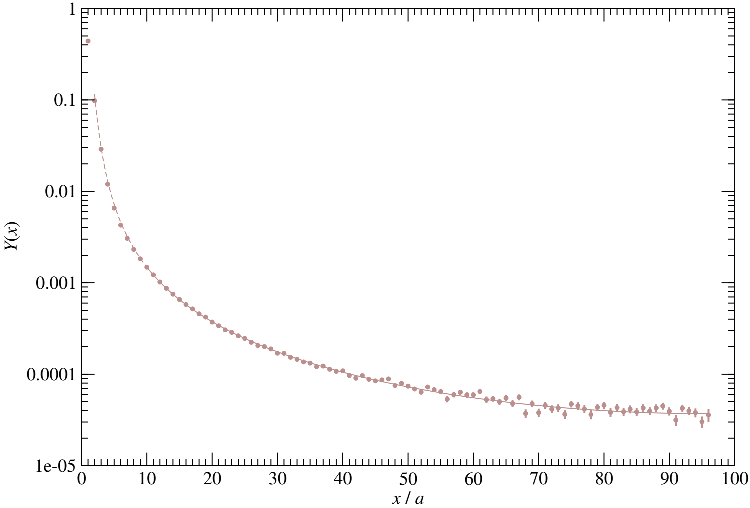
$$Y(x) = \langle E(x)E(0) \rangle$$

which we fit with

$$Y(x) = b_0 \cdot \left[\frac{1}{x^{2\Delta_\phi}} + \frac{1}{(L-x)^{2\Delta_\phi}} \right] + b_1 \cdot \left[\frac{1}{x^4} + \frac{1}{(L-x)^4} \right]$$

- we include contributions from
    - the operator ϕ (scaling dimension Δ_ϕ)
    - the marginal operator associated to action density (exact scaling dimension $\Delta_m = 2$)
- finite spatial size L contribution included

$\beta = 4.0,\ N_s = 192,\ N_t = 4$



the fit for $Y(x)$ yields $\Delta_\phi = 0.946(5)$

Fitting $W(x)$ again

- fix $\Delta_\phi = 0.946(5)$ from the field-strength correlator
- study again the flux-tube profile $W(x, r)$ with a one-parameter fit

$$W(r, x) = C(r) \left(1 + \frac{4x^2}{r^2}\right)^{-\Delta_\phi}$$

- extract $C(r)$ for several values of r
- fit it again in Δ_ϕ and $c_{ss\phi}$ knowing that

$$C(r) = \frac{c_{ss\phi}}{(r/4)^{\Delta_\phi}}$$

- the fit yields $\Delta_\phi = 0.948(6)$: self-consistent result!

Conclusions

- "expanding" the Svetitsky-Yaffe conjecture: the whole hot phase of the $\mathrm{U}(1)$ pure gauge theory in 3D can be mapped to the low-T phase of the XY model in 2D
- numerical study of the behaviour of
  - the Polyakov loop (static charges) correlator
  - the flux-tube profile (field strength in a background of static charges)
- they are effectively described by the same functional forms used for the XY model
- directly descend from constraints obtained in conformal field theory!

Moreover

- the critical index η behaves roughly as expected when the moving away from the critical point at $T > T_c$
- the scaling dimension Δ_ϕ can be extracted independently from the flux-tube profile and from the field-strength correlator: good agreement
- not only: they can be used together to draw a consistent picture over different observables

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Thank you for the attention!