

Perturbative quantities on the lattice QCD and the tensor renormalization group

[K.N, and S.Hashimoto arXiv:1606.01002]

[K.N, H.Fukaya, and S.Hashimoto arXiv:1804.06695]

[D.Kadoh, K.N. arXiv:1803.07960]

[D.Kadoh, K.N. arXiv:1812.10642]

[D. Kadoh, K.N. arXiv:1912.xxxxx]

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● Contents

(1): Introduction

(2): Perturbative quantities on the lattice

(3): Tensor renormalization group for higher dimension

(4): Summary

● Why Lattice QFT?

What is the role of Lattice QFT?

- ◆ Non-perturbative calculation:
 - Precise determination of physical quantities.
(e.g. hadron mass, quark mass, form factor, $g-2$...)
- ◆ Theoretical formulation of QFT without perturbation:
 - Fundamental interest for the definition.
(e.g. $SU(N)$ chiral gauge fermion still unknown)
- ◆ Interdisciplinary study:
 - cond-mat, quant-ph, and any discrete systems.
(e.g. Domain wall fermion $\longleftrightarrow$ Topological insulator)

● Contents

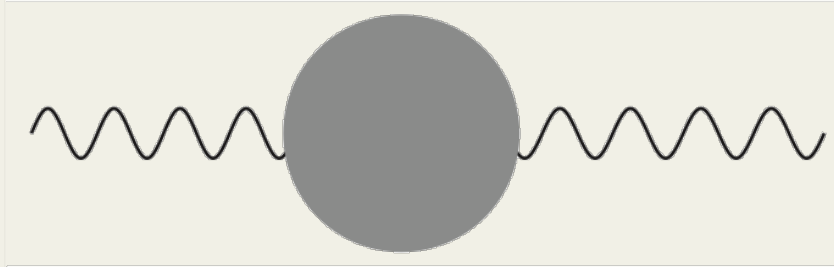
(1): Introduction

(2): Perturbative quantities on the lattice

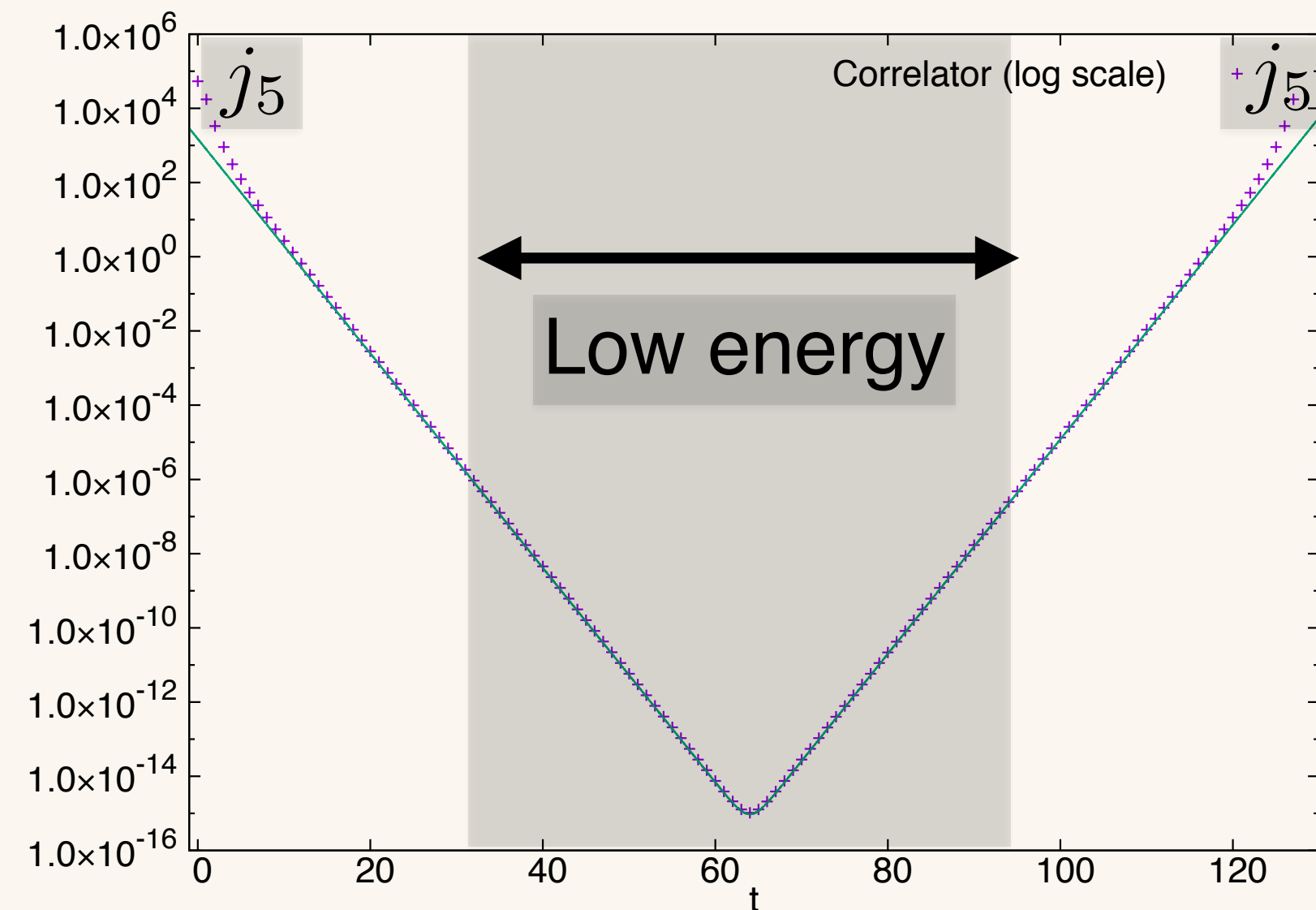
(3): Tensor renormalization group for higher dimension

(4): Summary

Current correlator



$$q^2 \Pi(q^2) = i \int dx e^{iqx} \langle j_5(x) j_5(0) \rangle$$



◇ Time development
(→ Euclidean)

$$e^{iHt} \rightarrow e^{-m_{\text{eff}} t}$$



$$m_{\text{eff}}^{\text{lattice}} = 2.9928(5) \text{ GeV}$$

$$m_{\eta_c}^{\text{experiment}} = 2.9834(5) \text{ GeV}$$

● Perturbative information in Lattice calculations

◇ Any other information?

Non-perturbative value (meson mass) from Low energy

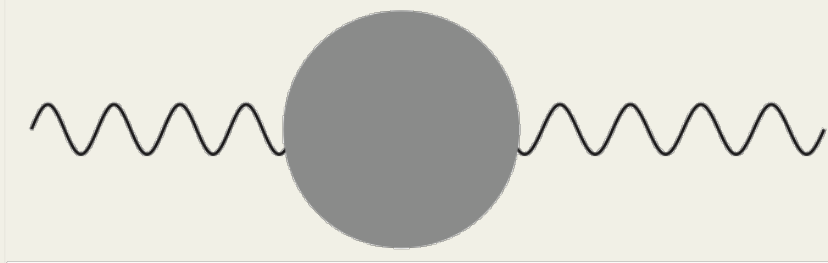


Perturbative value (quark mass) from High energy

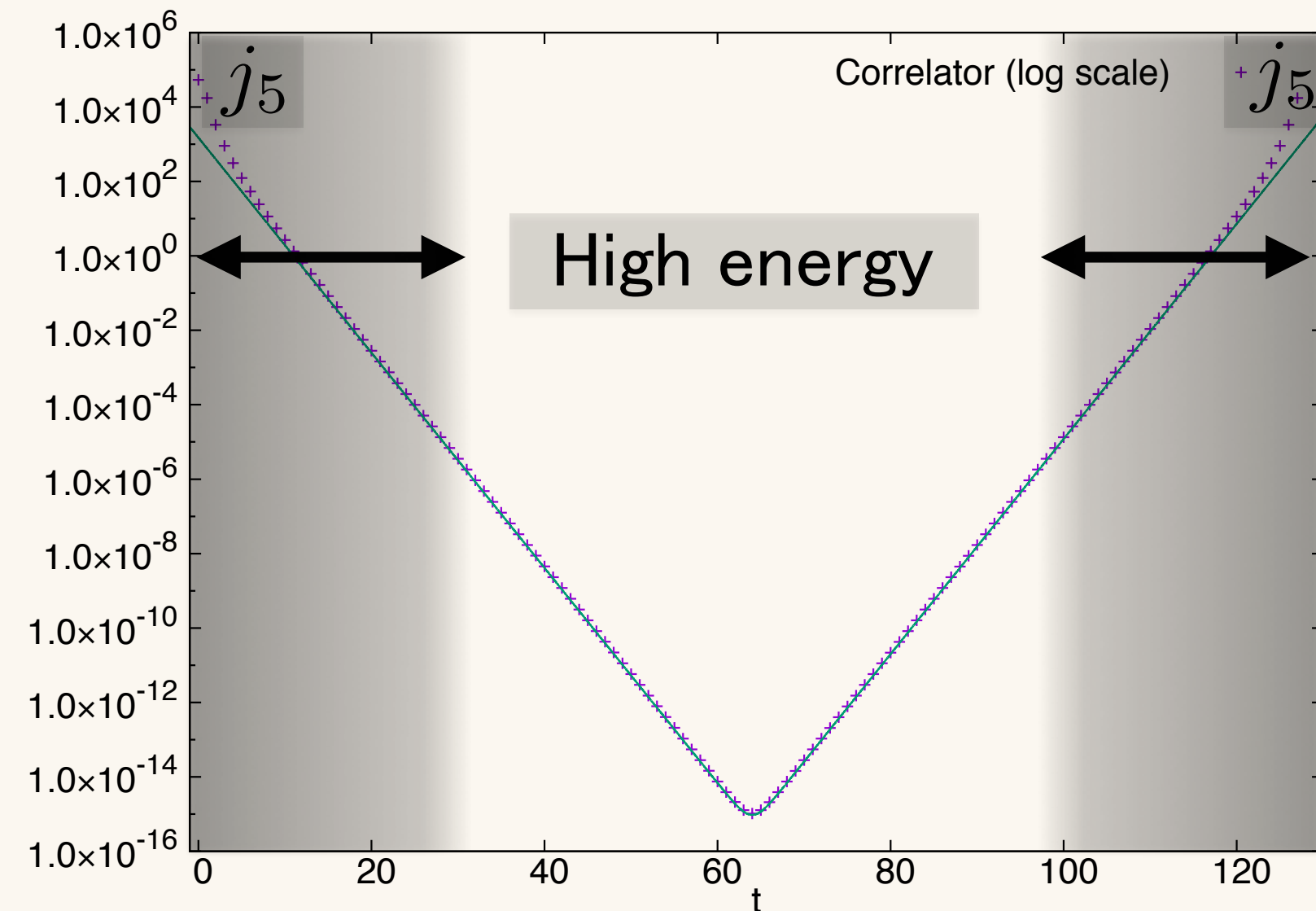
→ Moments (Derivative of correlator, or Adler function)

$$g_{2n} = \frac{1}{n!} \left(\frac{\partial}{\partial q^2} \right)^n \left(q^2 \Pi(q^2) \right)_{q^2=0}$$

Current correlator



$$G(t) = a^6 \sum_{\mathbf{x}} (am_{0c})^2 \langle j_5(x) j_5(0) \rangle$$



◇ Moment
(→ Differential)

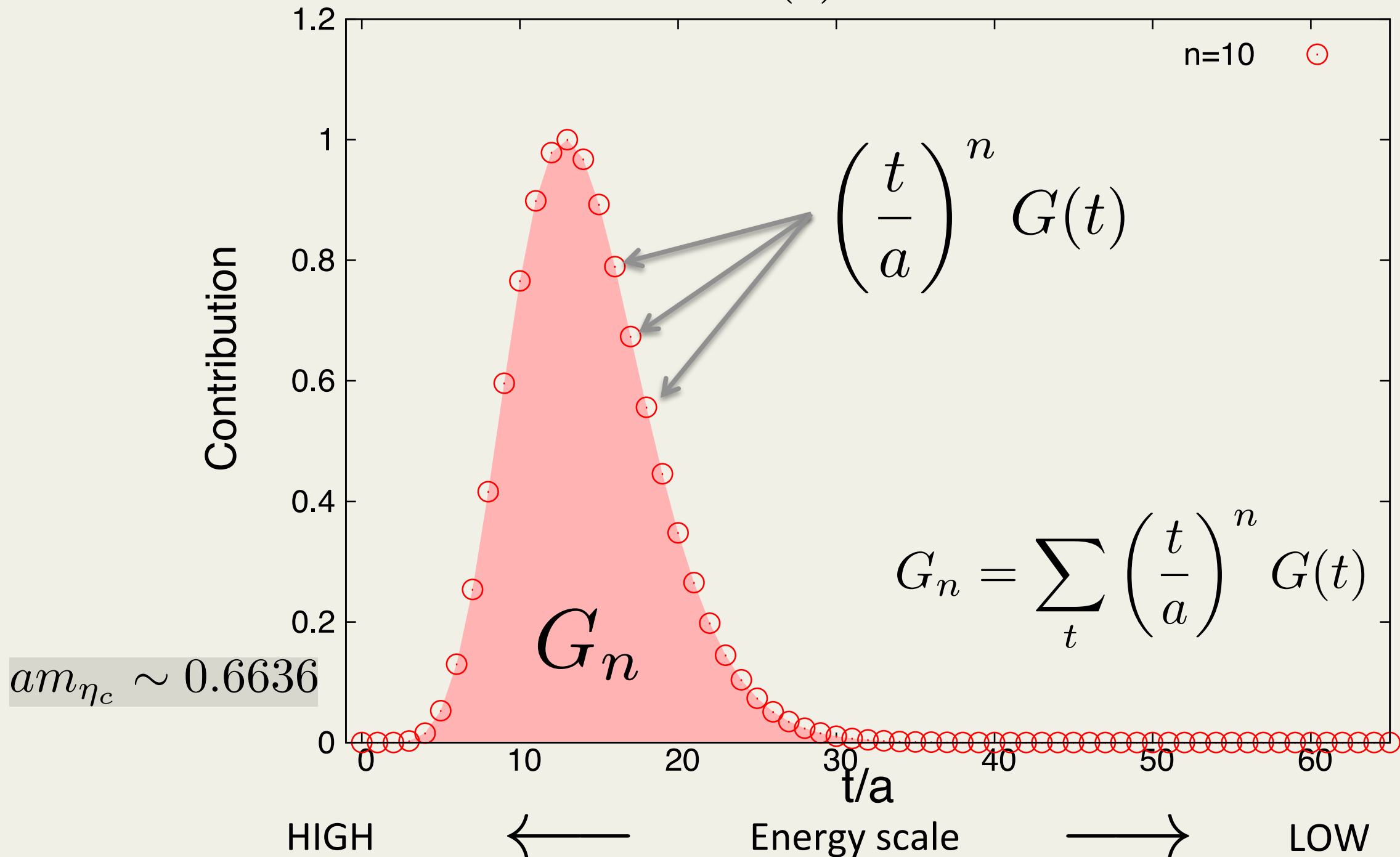
$$g_{2n} = \frac{1}{n!} \left(\frac{\partial}{\partial q^2} \right)^n (q^2 \Pi(q^2))_{q^2=0}$$



$$G_n = \sum_t \left(\frac{t}{a} \right)^n G(t)$$

What's the Moment?

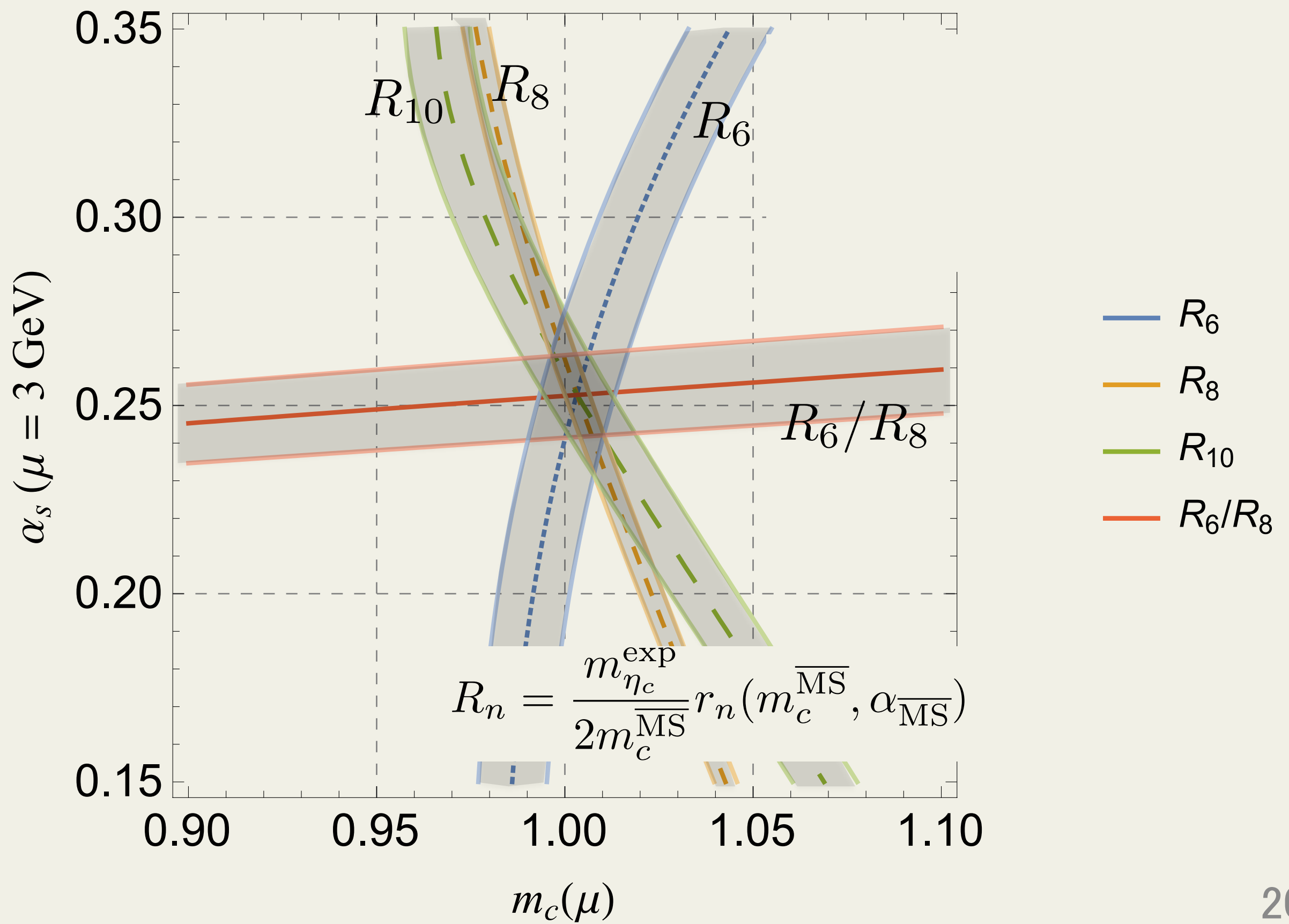
- Moment is defined by $G(t)$ at typical energy scale



$$g_{2n}^{\text{lattice}} = G_{2n}^{\text{perturbative}}(\alpha_s^{\overline{\text{MS}}}, m_c^{\overline{\text{MS}}}; \mu)$$

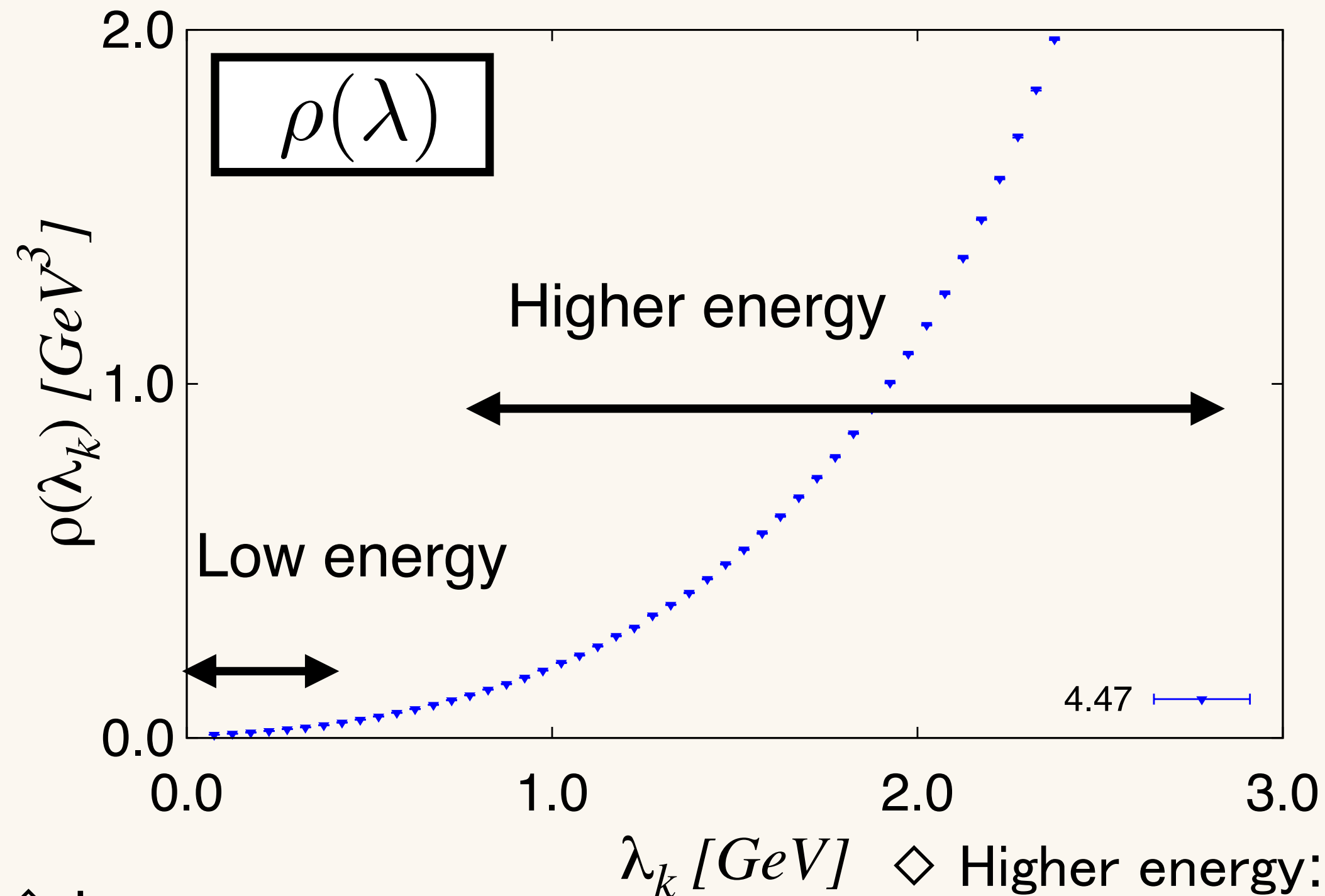
$$G_{2n}^{\text{perturbative}}(\alpha_s^{\overline{\text{MS}}}, m_c^{\overline{\text{MS}}}; \mu)$$

$$\begin{aligned} &= 1 \\ &+ \left(3.9 + 2.0 \log \frac{m_c(\mu)^2}{\mu^2} \right) \frac{\alpha_s}{\pi} \\ &+ \left(13.6 + 3.0 \log \frac{m_c(\mu)^2}{\mu^2} - 0.08 \left(\log \frac{m_c(\mu)^2}{\mu^2} \right)^2 \right) \left(\frac{\alpha_s}{\pi} \right)^2 \\ &+ \left(13.2 + 14.2 \log \frac{m_c(\mu)^2}{\mu^2} + 1.03 \left(\log \frac{m_c(\mu)^2}{\mu^2} \right)^2 + 0.06 \left(\log \frac{m_c(\mu)^2}{\mu^2} \right)^3 \right) \left(\frac{\alpha_s}{\pi} \right)^3 \end{aligned}$$



		pert	$t_0^{1/2}$	stat	$O(a^4)$	vol	$m_{\eta_c}^{\text{exp}}$	disc	EM
$m_c(3\text{GeV})$ [GeV]	1.0033(96)	(77)	(49)	(4)	(30)	(4)	(3)	(4)	(6)
$\alpha_s(3\text{GeV})$	0.2528(127)	(120)	(32)	(2)	(26)	(1)	(0)	(0)	(1)

● Another example: Dirac eigenvalue density



◇ Low energy:
→ Chiral condensate

◇ Higher energy:
→ Strong coupling

● Perturbative calculation up to $O(\alpha_s^3)$

◇ Eigenvalue density from chiral condensate

$$\rho(\lambda) = -\frac{1}{2\pi} \langle \bar{q}q(m = i\lambda + \epsilon) \rangle - \langle \bar{q}q(m = i\lambda - \epsilon) \rangle$$

◇ Known up to $O(\alpha_s^3)$ ($n_f = 3$)

[J.Kneur, A.Neveu arXiv:1506.07506]

$$\rho(\mu = \lambda) = \frac{3\lambda^3}{4\pi^2} \left(1 + 1.06\alpha_s - 2.14\alpha_s^2 - 5.98\alpha_s^3 + O(\alpha_s^4) \right)$$

◇ Exponent $\frac{d \log \rho(\lambda)}{d \log \lambda}$ can be predicted up to $O(\alpha_s^4)$.

$$\begin{aligned}
& -\frac{\rho_5}{\pi^5} = c_5 \\
& -\frac{4}{15}L_\lambda^5\gamma_0 [3\beta_0^4 - 25\beta_0^3\gamma_0 + 70\beta_0^2\gamma_0^2 - 80\beta_0\gamma_0^3 + 32\gamma_0^4] \\
& +\frac{1}{3}L_\lambda^4 [3\beta_0^4c_1 + 4\beta_0\gamma_0^2(13\beta_1 - 20c_1\gamma_0 + 60\gamma_0^2 - 48\gamma_1)] \\
& +\frac{1}{3}L_\lambda^4 [\beta_0^3(-25c_1\gamma_0 + 25\gamma_0^2 - 12\gamma_1) - 16\gamma_0^3(3\beta_1 - 2c_1\gamma_0 + 8\gamma_0^2 - 8\gamma_1)] \\
& +\frac{1}{3}L_\lambda^4 [\beta_0^2\gamma_0(-13\beta_1 + 70c_1\gamma_0 - 140\gamma_0^2 + 88\gamma_1)] \\
& +\frac{1}{3}L_\lambda^3 [\beta_0^3(12c_2 - 13c_1\gamma_0) + \beta_0^2(13\beta_1c_1 - 52c_2\gamma_0 + 88c_1\gamma_0^2 - 70\gamma_0^3 - 36c_1\gamma_1 + 88\gamma_0\gamma_1 - 24\gamma_2)] \\
& +\frac{1}{3}L_\lambda^3 [-4\beta_0(3\beta_2\gamma_0 + \beta_1(13c_1\gamma_0 - 13\gamma_0^2 + 7\gamma_1))] \\
& +\frac{1}{3}L_\lambda^3 [-4\beta_0(-6(3c_2\gamma_0^2 - 7c_1\gamma_0^3 + 10\gamma_0^4 + 5c_1\gamma_0\gamma_1 - 16\gamma_0^2\gamma_1 + 2\gamma_1^2 + 4\gamma_0\gamma_2))] \\
& +\frac{1}{3}L_\lambda^3 [-2\gamma_0(3\beta_1^2 - 12\beta_1(2c_1\gamma_0 - 4\gamma_0^2 + 3\gamma_1))] \\
& +\frac{1}{3}L_\lambda^3 [-2\gamma_0(4(-3\beta_2\gamma_0 + 4(c_2\gamma_0^2 + 3(-c_1\gamma_0^3 + 2\gamma_0^4 + c_1\gamma_0\gamma_1 - 4\gamma_0^2\gamma_1 + \gamma_1^2 + \gamma_0\gamma_2))))] \\
& +L_\lambda^2 \left[\frac{3\beta_1^2c_1}{2} + \beta_0^2(6c_3 - 7c_2\gamma_0 + \frac{9c_1\gamma_0^2}{2} - 3c_1\gamma_1) \right] \\
& +L_\lambda^2 [\beta_1(\beta_0(7c_2 - 6c_1\gamma_0) - 2(5c_2\gamma_0 - 7c_1\gamma_0^2 + 6\gamma_0^3 + 4c_1\gamma_1 - 9\gamma_0\gamma_1 + 3\gamma_2))] \\
& +L_\lambda^2 [\beta_0(3\beta_2c_1 - 2(7c_3\gamma_0 - 11c_2\gamma_0^2))] \\
& +L_\lambda^2 [-2\beta_0(+12c_1\gamma_0^3 - 10\gamma_0^4 + 6c_2\gamma_1 - 17c_1\gamma_0\gamma_1 + 24\gamma_0^2\gamma_1 - 6\gamma_1^2 + 5c_1\gamma_2 - 12\gamma_0\gamma_2 + 4\gamma_3)] \\
& +L_\lambda^2 [-2(\beta_3\gamma_0 + \beta_2(3c_1\gamma_0 - 3\gamma_0^2 + 2\gamma_1) - 4(c_3\gamma_0^2 - 2c_2\gamma_0^3 + 3c_1\gamma_0^4 - 4\gamma_0^5 + 2c_2\gamma_0\gamma_1))] \\
& +L_\lambda^2 [-2(-4(-6c_1\gamma_0^2\gamma_1 + 12\gamma_0^3\gamma_1 + c_1\gamma_1^2 - 6\gamma_0\gamma_1^2 + 2c_1\gamma_0\gamma_2 - 6\gamma_0^2\gamma_2 + 2\gamma_1\gamma_2 + 2\gamma_0\gamma_3))] \\
& +L_\lambda [\beta_3c_1 + 2\beta_2c_2 + 3\beta_1c_3 + 4\beta_0c_4 - \beta_2c_1\gamma_0 - 2\beta_1c_2\gamma_0 - 3\beta_0c_3\gamma_0 - 4c_4\gamma_0 + \beta_1c_1\gamma_0^2 + 2\beta_0c_2\gamma_0^2] \\
& +L_\lambda [4c_3\gamma_0^2 - \beta_0c_1\gamma_0^3 - 4c_2\gamma_0^3 + 4c_1\gamma_0^4 - 4\gamma_0^5 - \beta_1c_1\gamma_1 - 2\beta_0c_2\gamma_1 - 4c_3\gamma_1 + 2\beta_0c_1\gamma_0\gamma_1] \\
& +L_\lambda [8c_2\gamma_0\gamma_1 - 12c_1\gamma_0^2\gamma_1 + 16\gamma_0^3\gamma_1 + 4c_1\gamma_1^2 - 12\gamma_0\gamma_1^2 - \beta_0c_1\gamma_2 - 4c_2\gamma_2 + 8c_1\gamma_0\gamma_2 - 12\gamma_0^2\gamma_2] \\
& +L_\lambda [8\gamma_1\gamma_2 - 4c_1\gamma_3 + 8\gamma_0\gamma_3 - 4\gamma_4]
\end{aligned}$$

● Pauli-Villars mass generalization

◇ Overlap operator construction

[D.B.Kaplan (1992)]

$$aD_{\text{OV}}(m_f = 0) \equiv [\mathcal{P} D_{\text{DW}}^{-1}(1) D_{\text{DW}}(m_f = 0) \mathcal{P}]$$

$$= \frac{1 + \gamma_5 \text{sign}_{L_5} [\gamma_5 a D_{\text{Mobius}}]}{2}$$

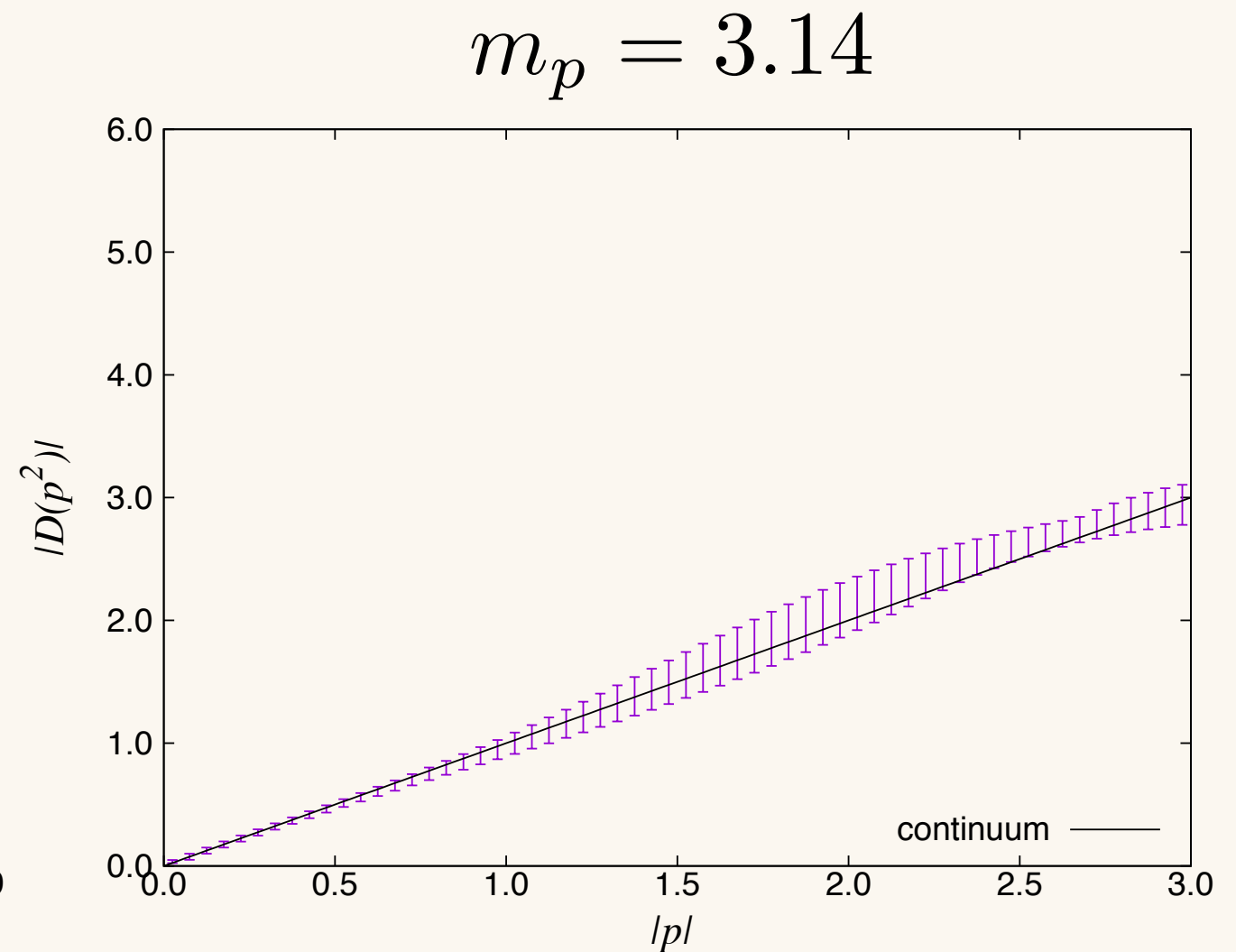
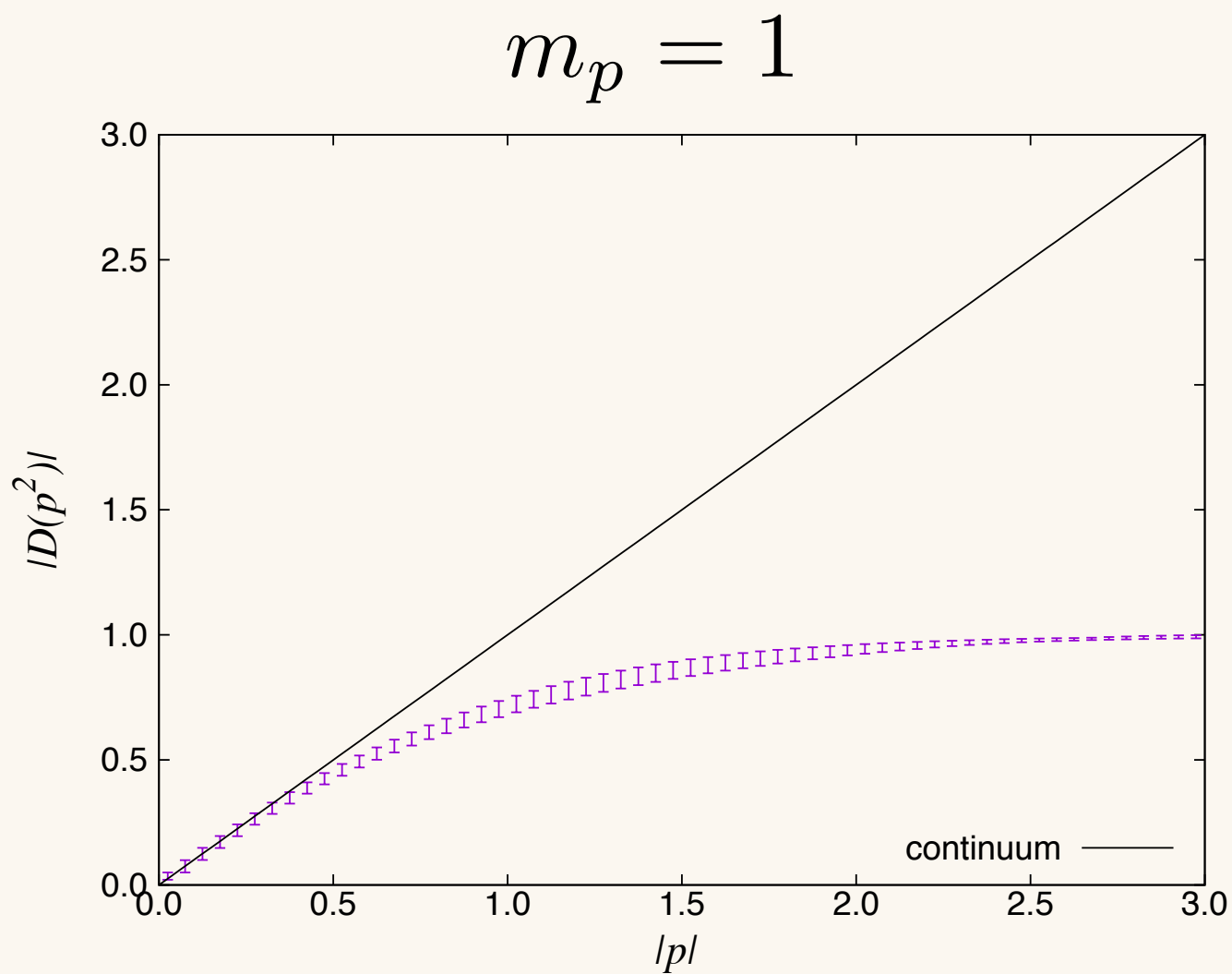


General Pauli-Villars mass

$$aD_{\text{OV}}(m_f = 0) \equiv [\mathcal{P} D_{\text{DW}}^{-1}(m_p) D_{\text{DW}}(m_f = 0) \mathcal{P}]$$

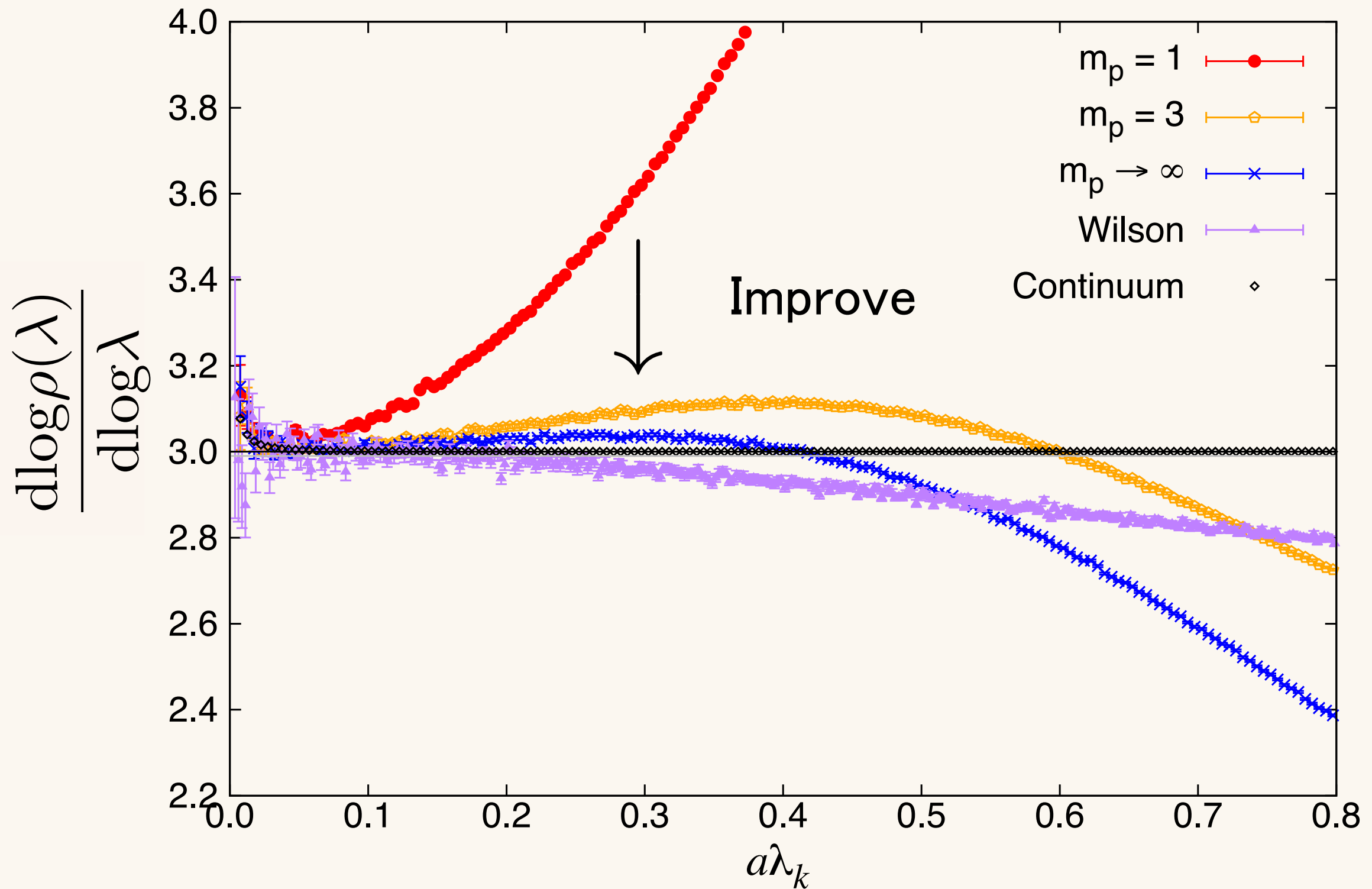
$$= m_p \frac{1 + \gamma_5 \text{sign}_{L_5} [\gamma_5 a D_{\text{Mobius}}]}{(1 + m_p) + (1 - m_p) \gamma_5 \text{sign}_{L_5} [\gamma_5 a D_{\text{Mobius}}]}$$

● Pauli-Villars mass as an eigenvalue cutoff (Tree)



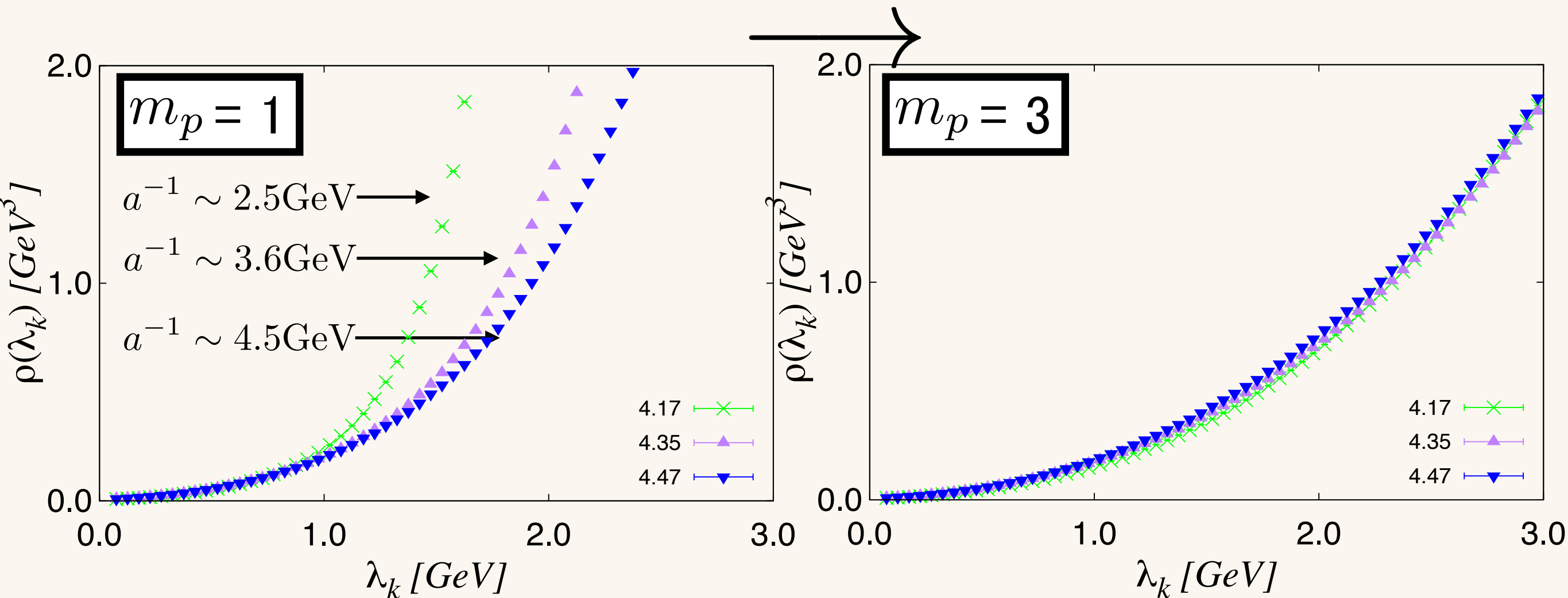
$$|aD_{\text{OV}}| \leq m_p$$

● Lattice power of eigenvalue density (Tree level)



● Reducing discretization effect

◇ a dependence is reduced by $m_p = 1 \rightarrow 3$



$$a^2 \lambda^2(m_p) = \frac{a^2 \lambda^2(m_p = 1)}{1 - (1 - 1/m_p^2) a^2 \lambda^2(m_p = 1)}$$

◇ Eigenvalue

$$a^2 \lambda^2(m_p) = \frac{a^2 \lambda^2(m_p = 1)}{1 - (1 - 1/m_p^2) a^2 \lambda^2(m_p = 1)}$$

◇ Chirality (Ginsparg-Wilson)

$$\{D_{\text{OV}}^{-1}, \gamma_5\} = 2a\gamma_5 \quad [\text{P.H.Ginsparg, K.G.Wilson (1982)}]$$



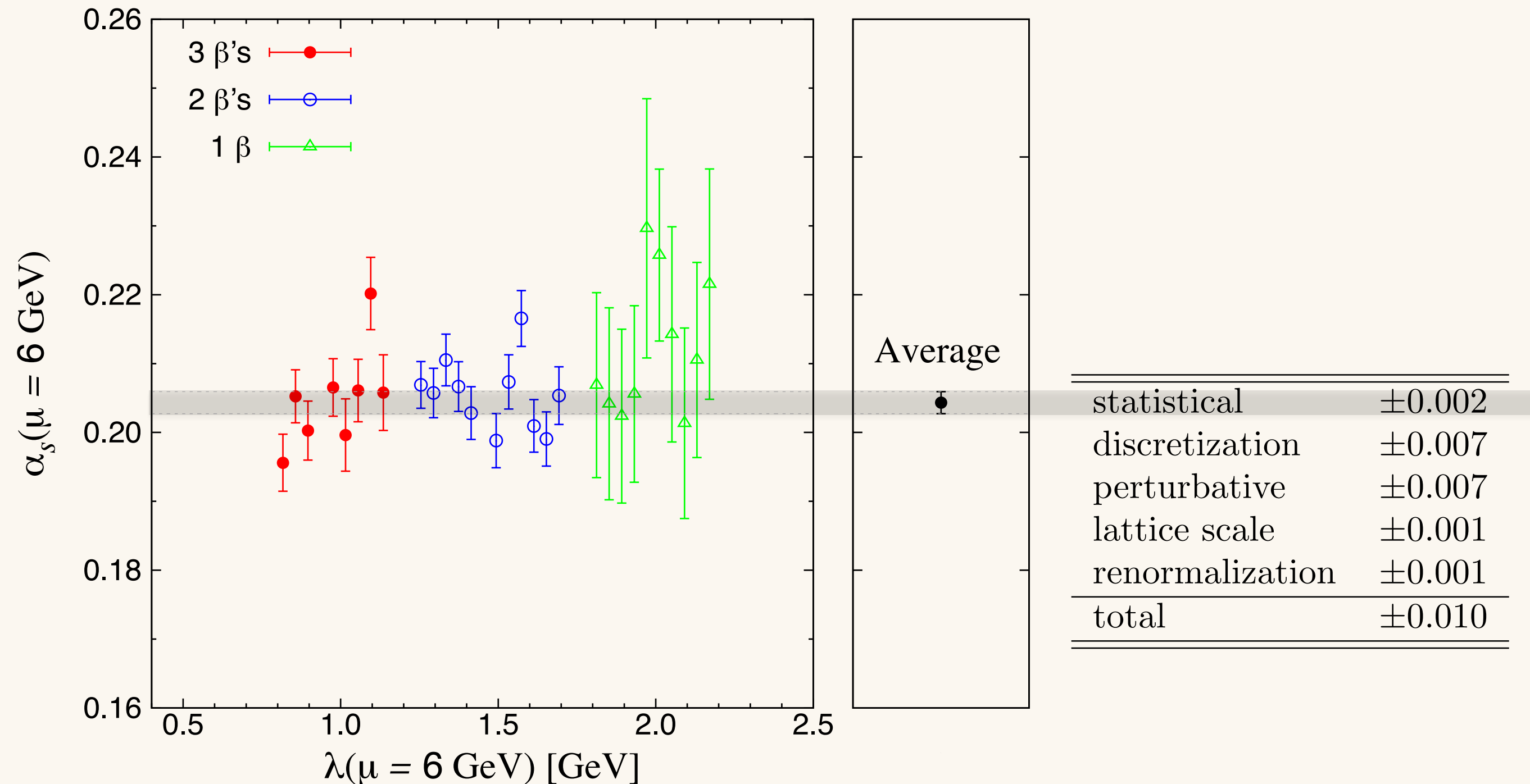
General Pauli–Villars mass

$$\{D_{\text{OV}}^{-1}, \gamma_5\} = \frac{2a\gamma_5}{m_p}$$

◇ Locality → Exponential locality is maintained

(Exponential locality of sign function) [P.Hernandez et.al.(1999)]

● Strong coupling constant determination



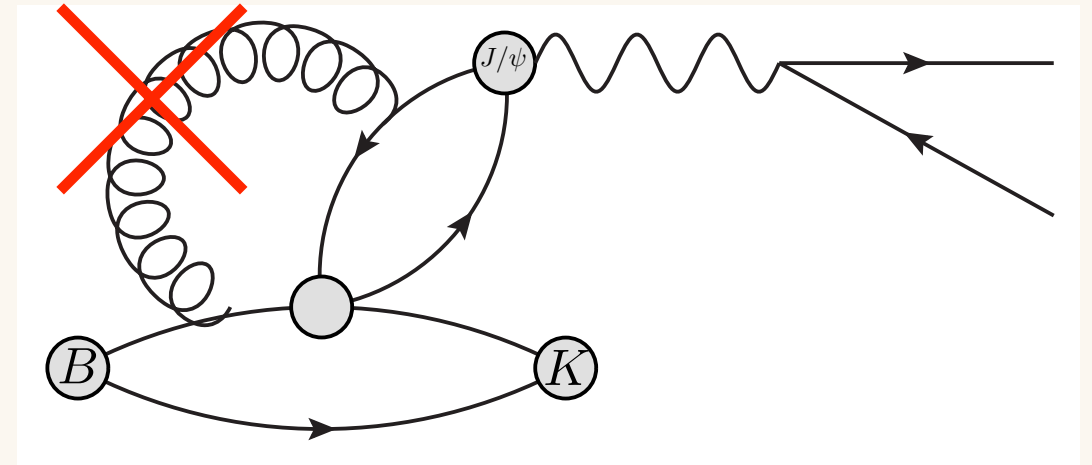
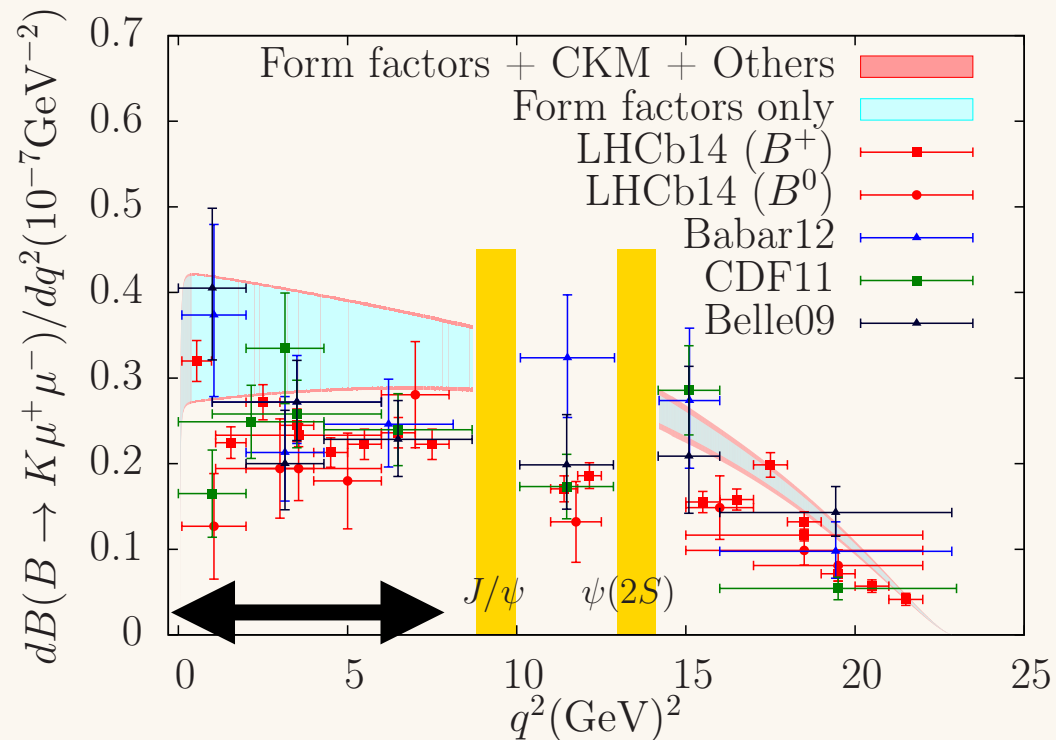
$\alpha_s(\mu=6 \text{ GeV})$
0.204(10)

(stat.) (pert.) ($O(a^4)$) (δt_0) (δZ_m)
(2) (7) (7) (2) (2)

PDG
0.191(3)

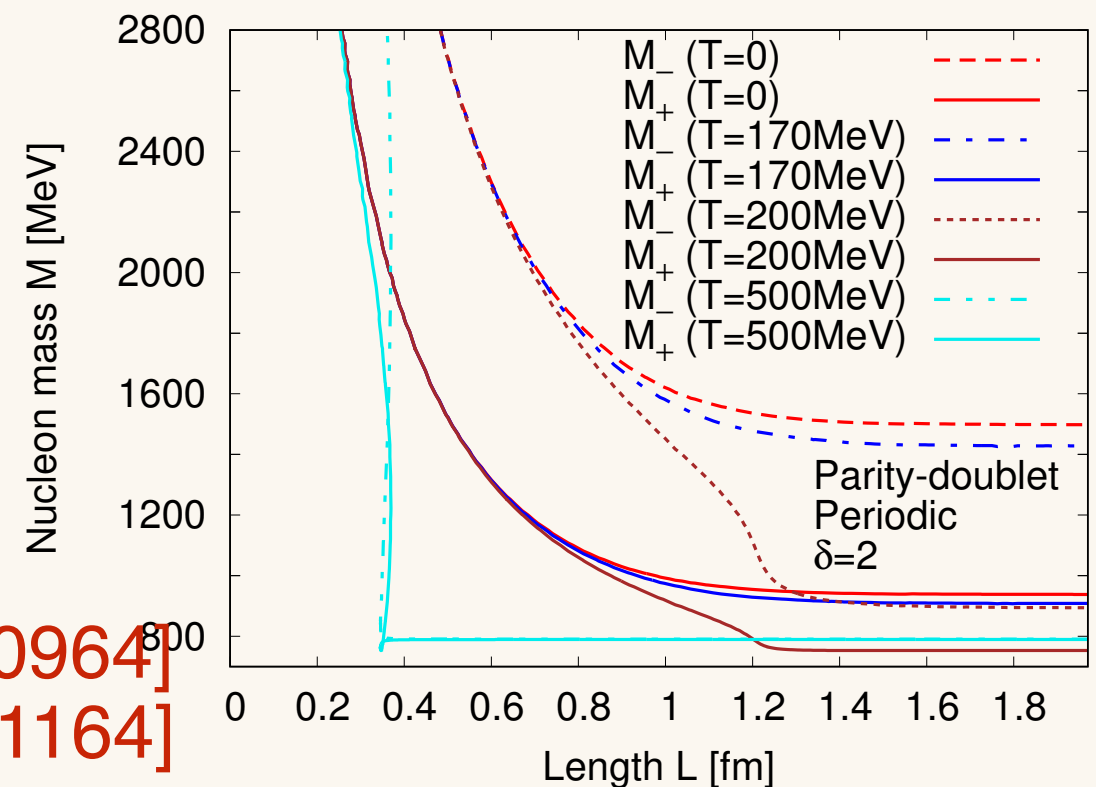
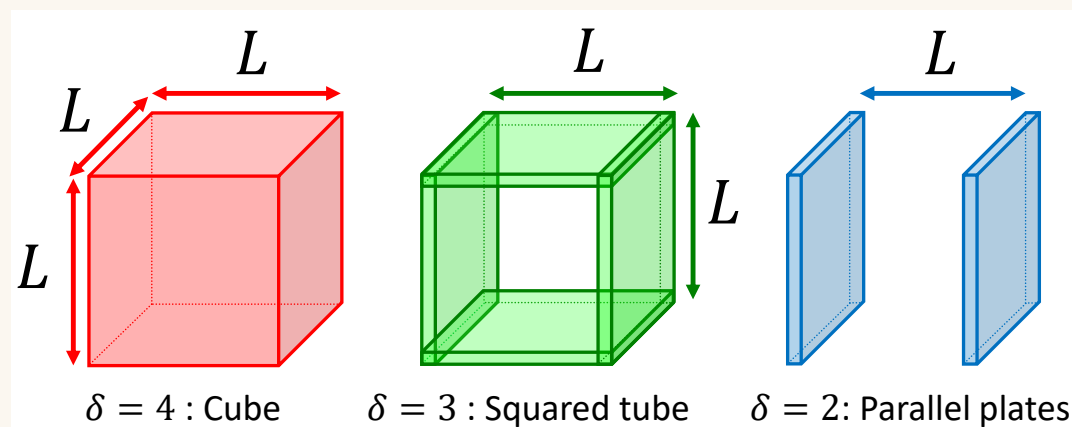
● Others

◇ Non-factorizable charmonium contribution in $B \rightarrow K$ II



[K.N., S. Hashimoto arXiv:1901.08784]
[D. Du et al. (Fermilab, MILC) 1510.02349]

◇ Finite Volume (Casimir) effects in the model of nucleon



[T. Ishikawa, K.N., K. Suzuki arXiv:1812.10964]
[T. Ishikawa, et al. arXiv:1905.11164]

● Summary for QCD part

- ◆ Since lattice calculations are a (first principle) numerical calculation, it provides physical quantities without perturbative method.
- ◆ There are many source of uncertainty, but under control.
- ◆ The error from the truncation of perturbative expansion is often dominant.
- ◆ The systematic error of the fermion formulation is also important.

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● Theoretical aspects of Lattice QFT

◇ What is the Lattice QFT (again)?

Classical field theory on the discrete spacetime

+

Path integral quantization

$$Z = \int \mathcal{D}\phi e^{-S_{\text{lattice}}}$$



$$\int dx \rightarrow \sum_x$$

Lattice quantum field theory

● Differential ∂ and symmetric difference ∇

$$a\partial_\mu\psi(x) \rightarrow \frac{\psi(x+a) - \psi(x-a)}{2} = (\sinh a\partial_\mu)\psi(x) \\ = \nabla_\mu\psi(x)$$

◇ Leibniz rule violation

$$\partial(fg) = \partial f g + f \partial g \rightarrow \nabla(fg) = \nabla f \underline{M[g]} + \underline{M[f]} \nabla g$$

◇ Propagator

$$\frac{1}{i\gamma_\mu a p_\mu + M} \rightarrow \frac{1}{i\gamma_\mu \sin a p_\mu + M}$$

● Classical correspondence

Lattice QFT



Condensed matter

◆ Lagrangian L_{lattice}

◆ Mass m_{eff}

◆ Gamma matrix γ_μ

◆ Hamiltonian H_{cond}

◆ Magnetic field M

◆ Pauli operator σ_i

→ No need to consider path integral (explicitly)

● Classical correspondence

Lattice QFT



Condensed matter

◆ Staggard

◆ Minimal doubling

◆ Domain-wall

◆ Quantum simulation

◆ Honeycomb (graphen)

◆ Topological Insulator

→ No need to consider path integral (explicitly)

● Quantum correspondence

◇ Is there Quantum correspondence?

→ Tensor Renormalization group (for Fermion, Boson),
World sheet representation (for Gauge)

$$\int D\bar{\psi} D\psi \exp \left[- \sum_x \mathcal{L}(\bar{\psi}_x, \psi_x) \right] \longleftrightarrow \sum_{\sigma_i} \exp \left[-\beta \left(J \sum_i \sigma_i \sigma_{i+1} + h \sum_i \sigma_i \right) \right]$$

$$\{\bar{\psi}, \psi\} \longleftrightarrow \{\sigma_i\}$$

→ Need to consider path integral measure

● Fermionic field/Spin variables

◇ Field variable integration (Fermion part)

$$\begin{aligned}\langle \psi_N \bar{\psi}_k \rangle &= \int D\psi D\bar{\psi} \psi_N \bar{\psi}_k e^{\sum_{t=1}^N \bar{\psi}_t A_t \psi_t + \bar{\psi}_t B_t \psi_{t-1}} \\ &= \int D\psi D\bar{\psi} \psi_N \bar{\psi}_k e^{\sum_{t=1}^N \bar{\psi}_t A_t \psi_t} \prod_{t=1}^N \sum_{b_t=0}^1 (\bar{\psi}_t B_t \psi_{t-1})^{b_t}\end{aligned}$$

◇ To spin variable by field integration

$$\{\bar{\psi}_t, \psi_t\} \rightarrow \{b_t\}$$

→ Efficient numerical integration using ‘locality’ of action, and Grassman property

● Scalar field/Spin variables

◇ Loop formulation

$$S_{\Lambda}^B = \sum_x \{ -w \cdot \phi_x \phi_{x-1} + V(\phi_x) \}$$

(occupation number n)

$$e^{-S_{\Lambda}^B} = \prod_x \left(\sum_{n^b(x)=0}^{\infty} \frac{(w \cdot \phi_{x-1} \phi_x)^{n^b(x)}}{n^b(x)!} \right) \prod_x e^{-V(\phi_x)}$$

$\{\phi_x\} \rightarrow \{n^b(x)\}$

→ Straightforward extension to bosonic field with cutoff

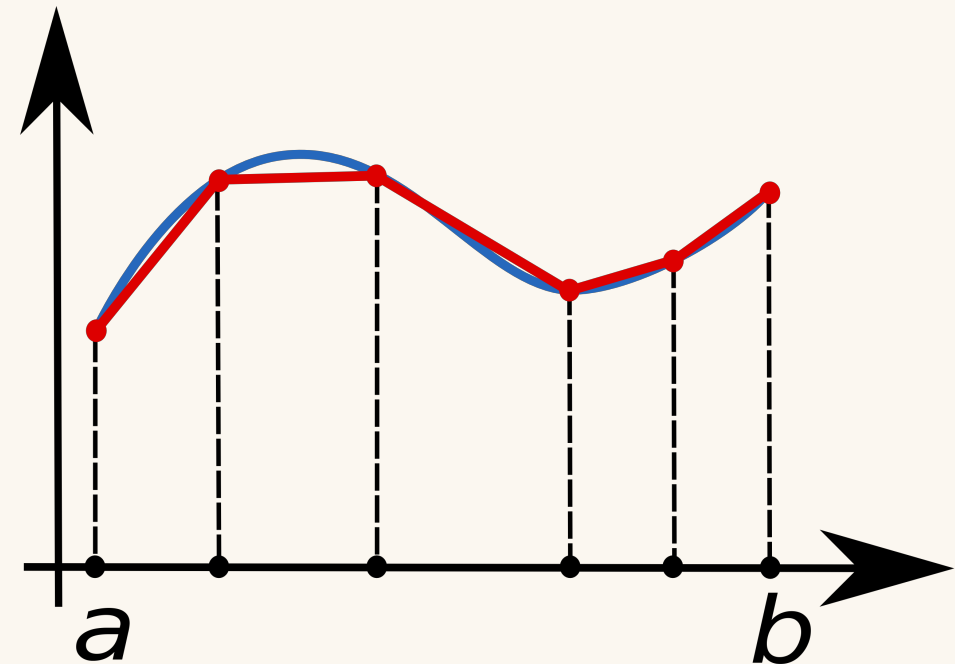
[D.Baumgartner, U.Wenger arXiv:1503.05232]

● Scalar field/Spin variables

◇ Multiple integration

→ Cost grows exponentially

(e.g.) Simpson rule $O(n^{\frac{d}{8}})$



Discretization $\int d\phi F(\phi) \simeq \sum_{\phi \in S_K} g(\phi) F(\phi)$

◇ tensor (matrix) rep.

$$\int d\phi_1 \dots d\phi_N \prod_{t=1}^N e^{-\frac{1}{2}(\phi_t - \phi_{t-1})^2} \rightarrow \text{tr} \left[\hat{F}_{\phi_t \phi_{t-1}}^N \right]$$

$$O(n^{\frac{N}{8}}) \rightarrow O(\log N)$$

→ Dramatic cost reduction [D.Kadoh, K.N. arXiv:1803.07960]

● Gauge field/flux tubes

◇ World sheet representation

$$S_G[U] = -\frac{\beta}{3} \sum_{x, \mu < \nu} \text{Re Tr } U_{x, \mu} U_{x+\hat{\mu}, \nu} U_{x+\hat{\nu}, \mu}^\dagger U_{x, \nu}^\dagger$$

$$\int D[U] \prod_{x, \mu < \nu} \prod_{a, b, c, d} \sum_{n_{x, \mu \nu}^{abcd}=0}^{\infty} \sum_{\bar{n}_{x, \mu \nu}^{abcd}=0}^{\infty} \frac{(\beta/6)^{n_{x, \mu \nu}^{abcd} + \bar{n}_{x, \mu \nu}^{abcd}}}{n_{x, \mu \nu}^{abcd}! \bar{n}_{x, \mu \nu}^{abcd}!} (U_{x, \mu}^{ab} U_{x+\hat{\mu}, \nu}^{bc} U_{x+\hat{\nu}, \mu}^{dc*} U_{x, \nu}^{ad*})^{n_{x, \mu \nu}^{abcd}!} (c.c.)^{\bar{n}_{x, \mu \nu}^{abcd}!}$$

$$\{U_{x, \mu \nu}^{abcd}, U_{x, \mu \nu}^{abcd*}\} \rightarrow \{n_{x, \mu \nu}^{abcd}, \bar{n}_{x, \mu \nu}^{abcd}\}$$

→ Integrate out can be done at least SU(3) case.

[C.Marchis, C.Gattringer. arXiv:1712.07546]

● Interest

◇ What are the points?

◇ Advantages and disadvantages

○ Sign problem

○ Another representation

○ Low cost (low dim)

○ Partition function

× High cost (high dim)

△ Systematic error

→ We study SUSYQM as a good example.

[D.Kadoh, K.N. arXiv:1803.07960]

[D.Kadoh, K.N. arXiv:1812.10642]

● 1-dim SUSYQM

$$S = \sum_{t=1}^N \left[\frac{1}{2} (\nabla \phi_t + W(\phi_t))^2 + \bar{\psi}_t (\nabla + W'(\phi_t)) \psi_t \right]$$

$$Q_1 \phi = \psi$$

$$Q_1 \psi = -\partial_t - W$$

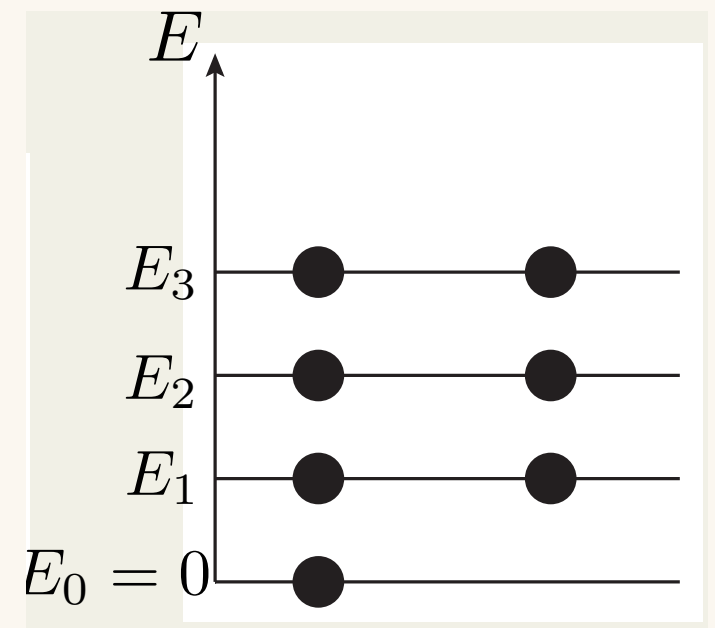
$$Q_1 \bar{\psi} = 0$$

◇ Q-exact form

$$Q_1 S = 0 \quad \text{Latticize} \quad Q_1 S = 0$$

$$Q_2 S = 0 \quad \longrightarrow \quad Q_2 S \neq 0$$

$$\{Q_1, Q_2\} = 2\partial_t \quad \{Q_1, Q_2\} = 2\nabla_t$$



● 1-dim SUSYQM

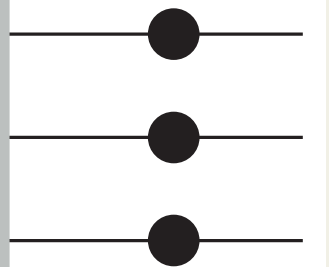
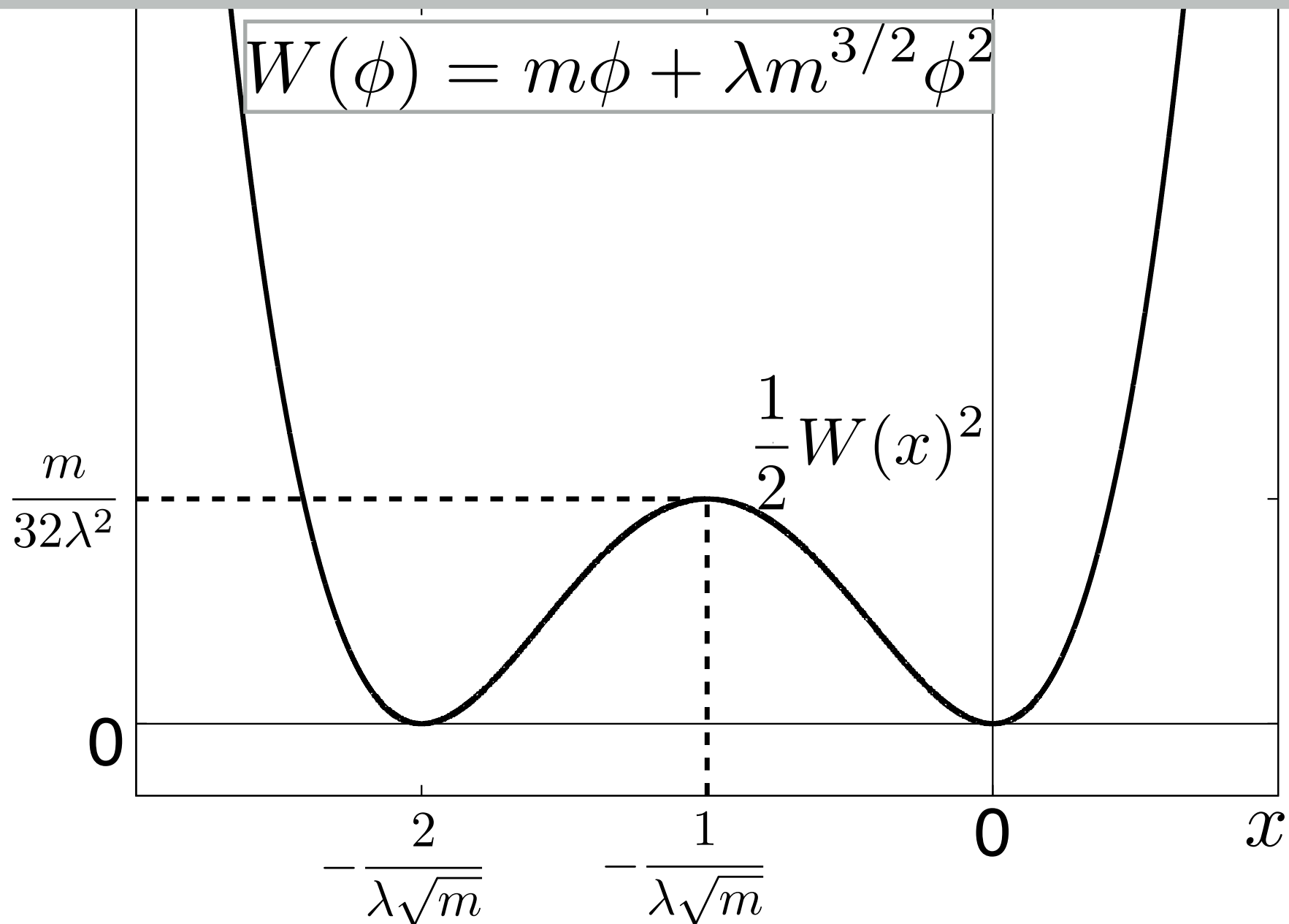
$$S = \sum_{t=1}^N \left[\frac{1}{2} (\nabla \phi_t + W(\phi_t))^2 + \bar{\psi}_t (\nabla + W'(\phi_t)) \psi_t \right]$$

◇ Q-e

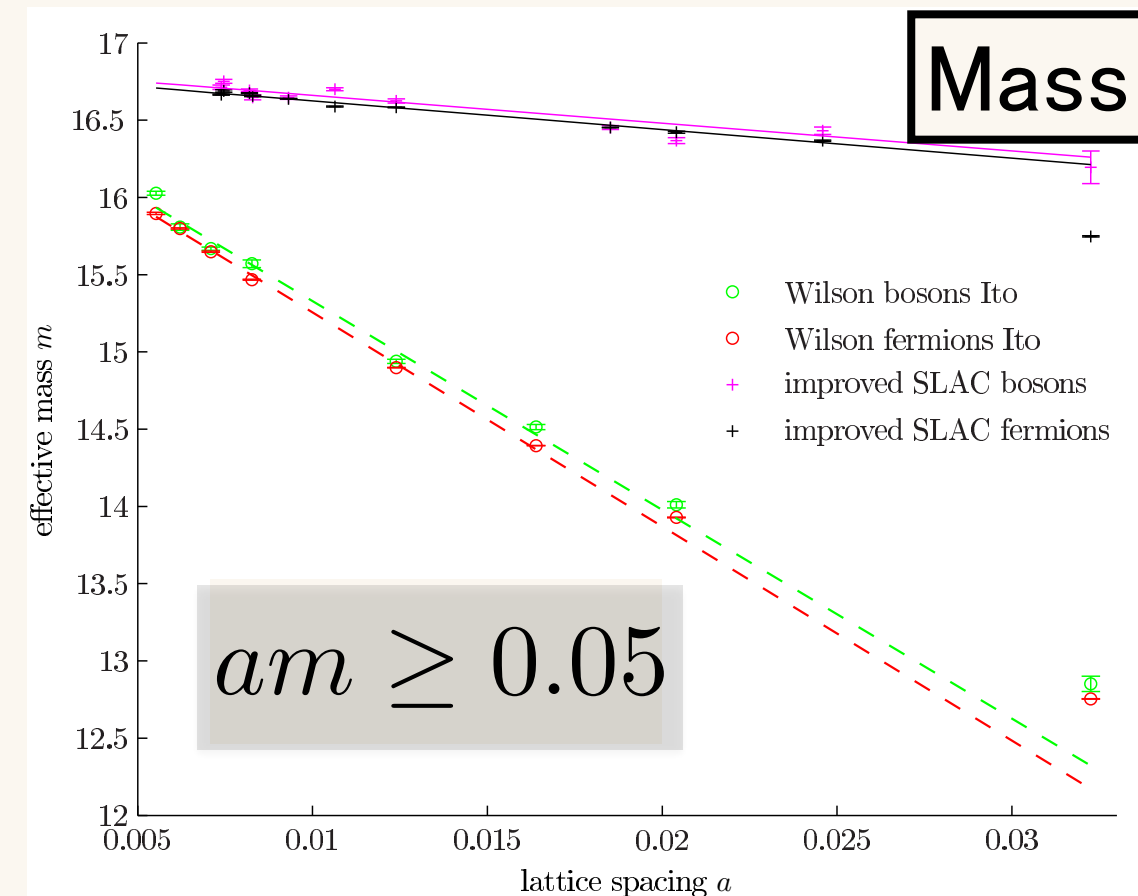
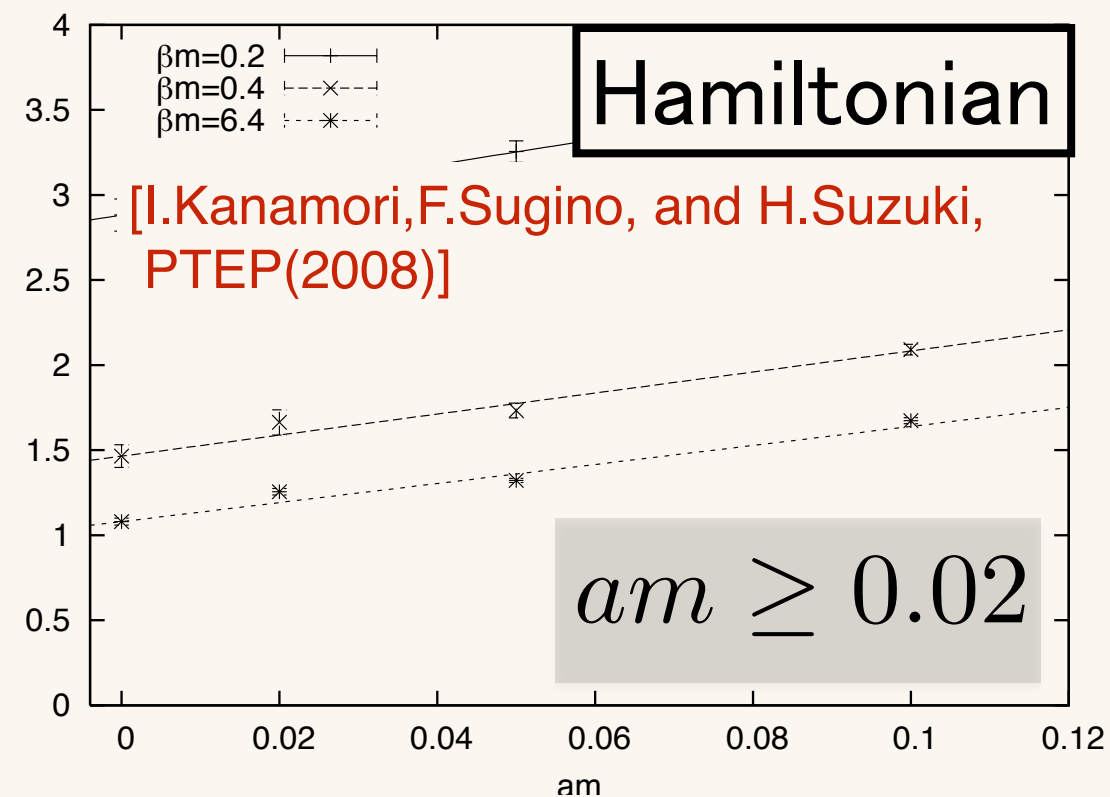
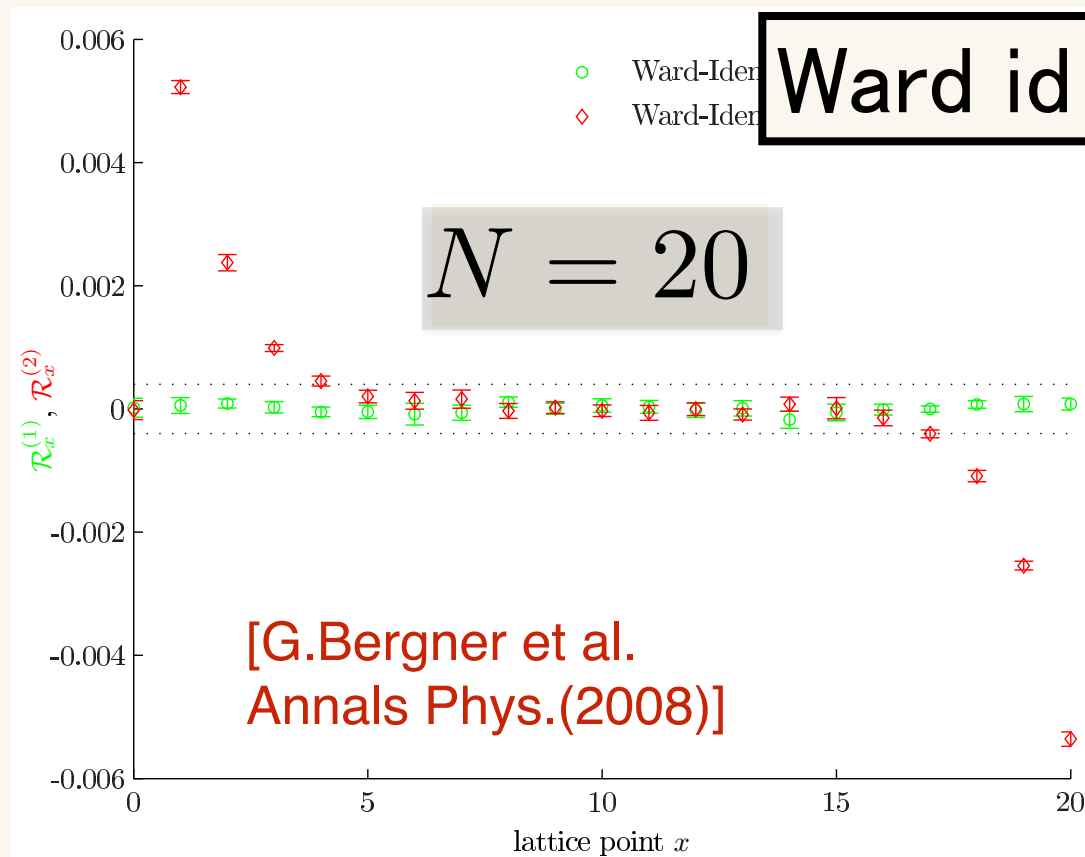
Q

Q

$\{Q_1, Q_2\}$



● Comparing to Monte-Carlo



Our result:

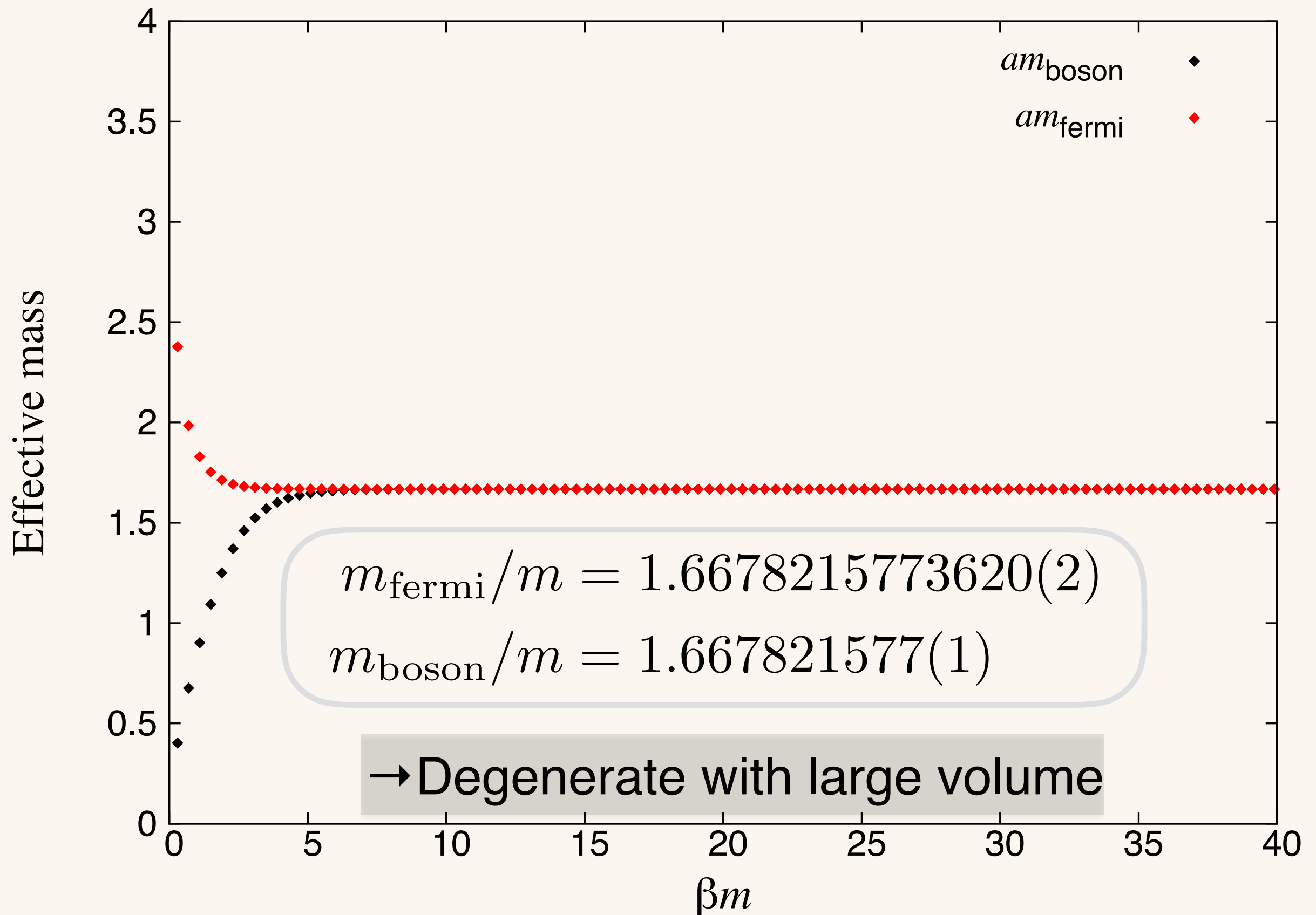
$$N = 1500 \sim 30000$$

$$am \geq 0.001$$

→ Great improvement

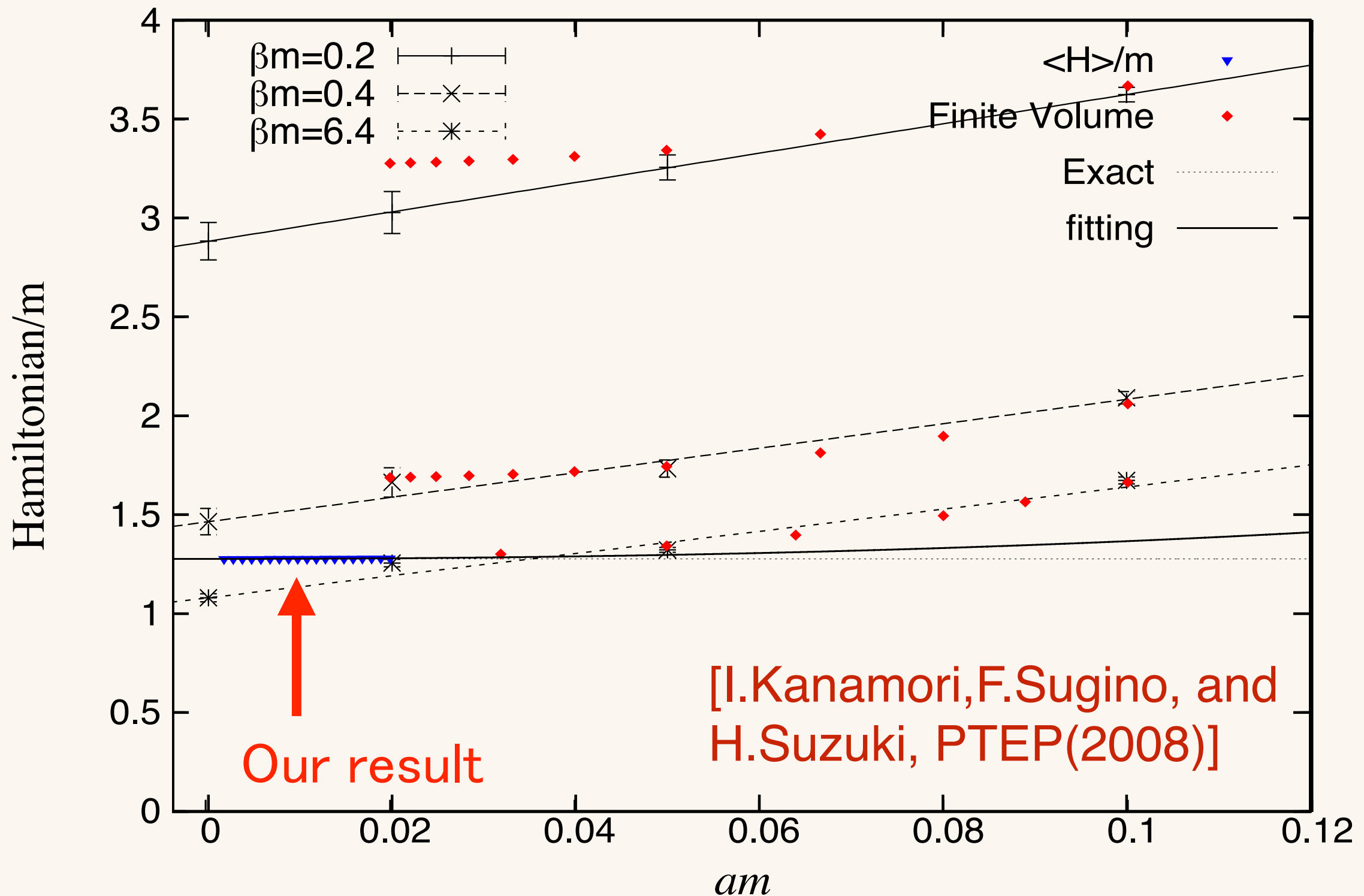
● Spectral degeneration

$$W(\phi) = m\phi + \lambda m^2 \phi^3$$



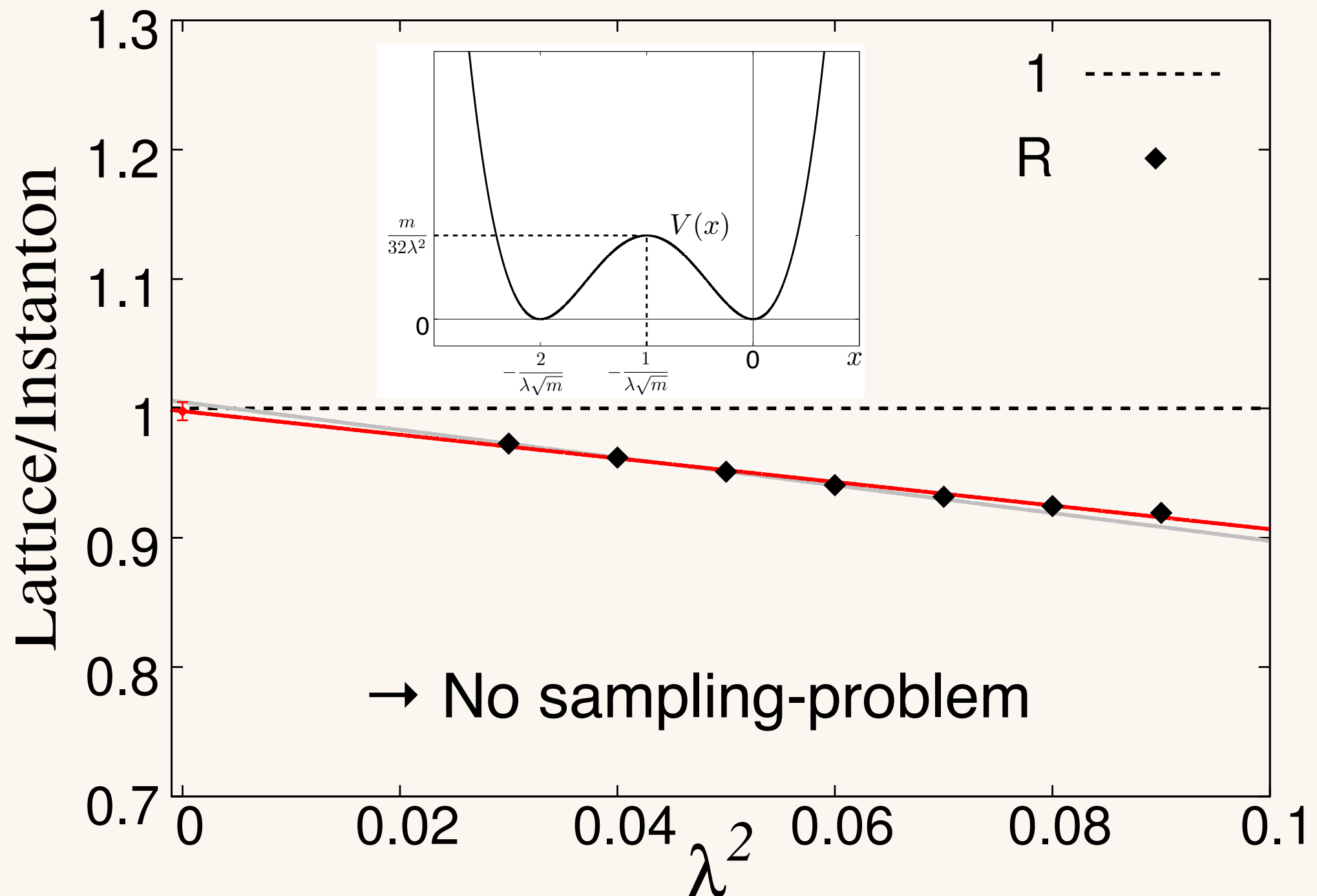
● Hamiltonian (extrapolation)

$$W(\phi) = m\phi + \lambda m^{3/2} \phi^2$$



● Hamiltonian (Weak coupling regime)

$$W(\phi) = m\phi + \lambda m^{3/2} \phi^2$$



● Tensor representation for Lattice

MPS

Deterministic
usual Vol.
w/o sign-prob.

~2 or 3 dim

No rigorous error

State base

TRG

Deterministic
exponentially large
w/o sign-prob.

~2 or 3 dim

No rigorous error

Measurement base

Warm algorithm

Sampling
usual Vol
It depends.

any-dim

statistical error

Measurement base

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● Examples: Variational TRG

(Simple) Tensor renormalization group (TRG)

[M. Levin and C. P. Nave arXiv:cond-mat/0611687]

[S. Morita, R. Igarashi et al. arXiv:1712.01458]

Higher-Order TRG (HOTRG)

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Triad RG (Tri-RG)

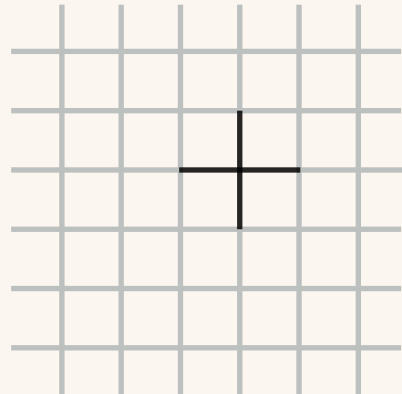
[D. Kadoh and K.N. arXiv:1911.xxxxx]

● (Simple) Tensor renormalization group (TRG)

[M. Levin and C. P. Nave arXiv:cond-mat/0611687]

◇ Starting point: network representation

$$Z = \sum_{a,b} \prod_{x,y} T_{a_{x,y}, a_{x+1,y}, b_{x,y}, b_{x,y+1}}$$

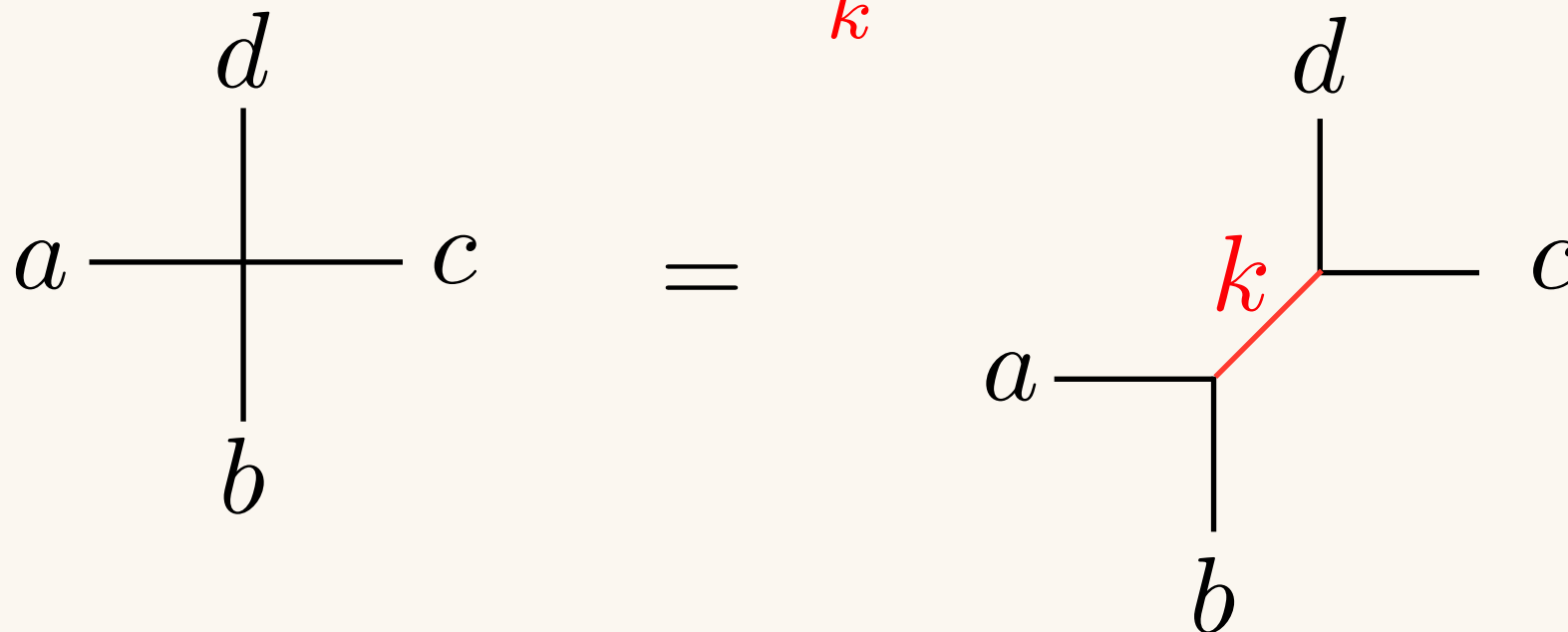
$$T_{a_{x,y}, a_{x+1,y}, b_{x,y}, b_{x,y+1}} = \begin{array}{c} b_{x,y+1} \\ | \\ a_{x,y} \text{ --- } | \text{ --- } a_{x+1,y} \\ | \\ b_{x,y} \end{array}$$


Naive contraction costs $\propto \{\dim(a)\dim(b)\}^{\text{Volume}}$

→ How can we calculate (approximate) these contractions?

● Singular Value Decomposition (SVD)

$$T_{abcd} = \sum_k^D A_{ab}^k \lambda^k B_{cd}^k$$

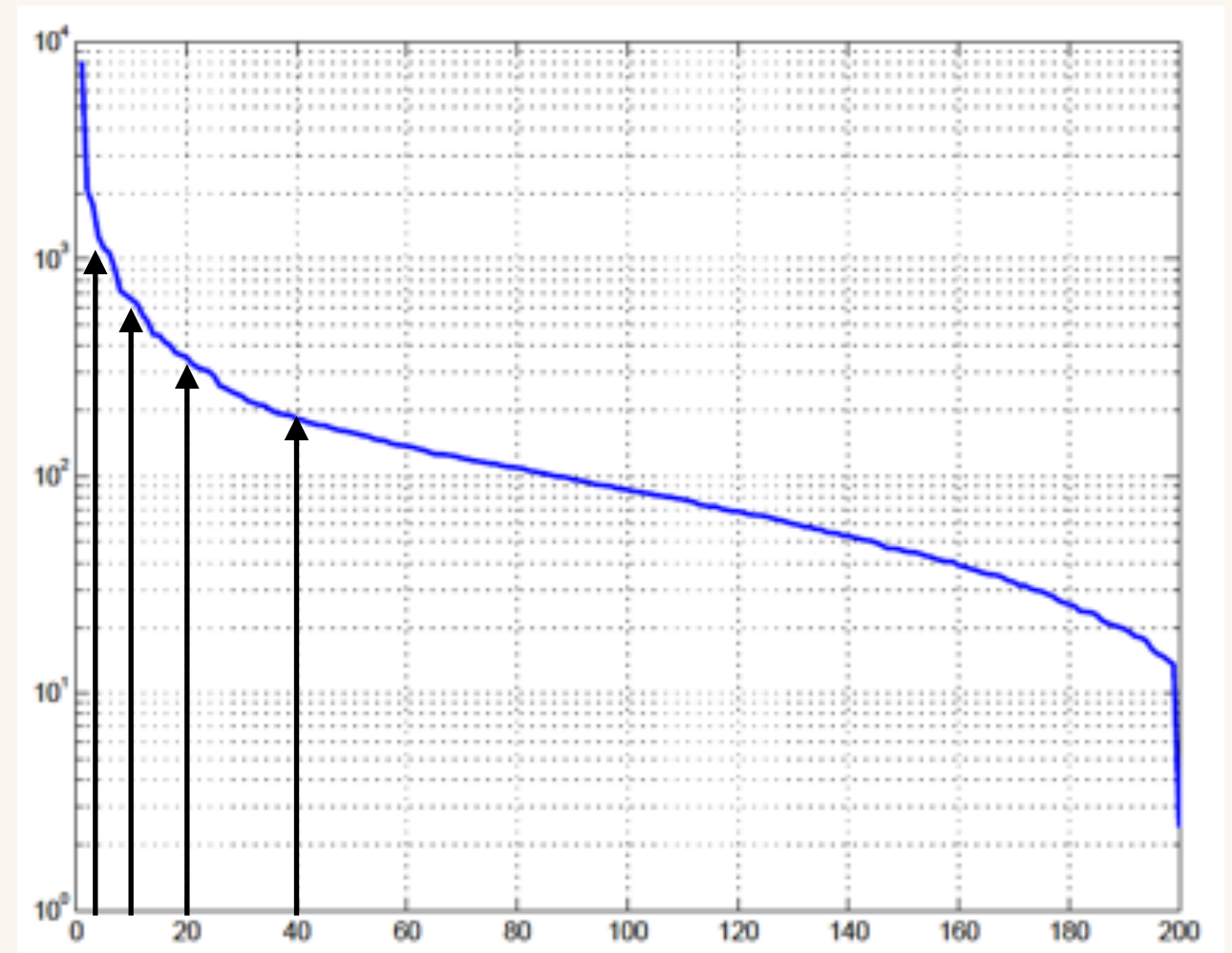
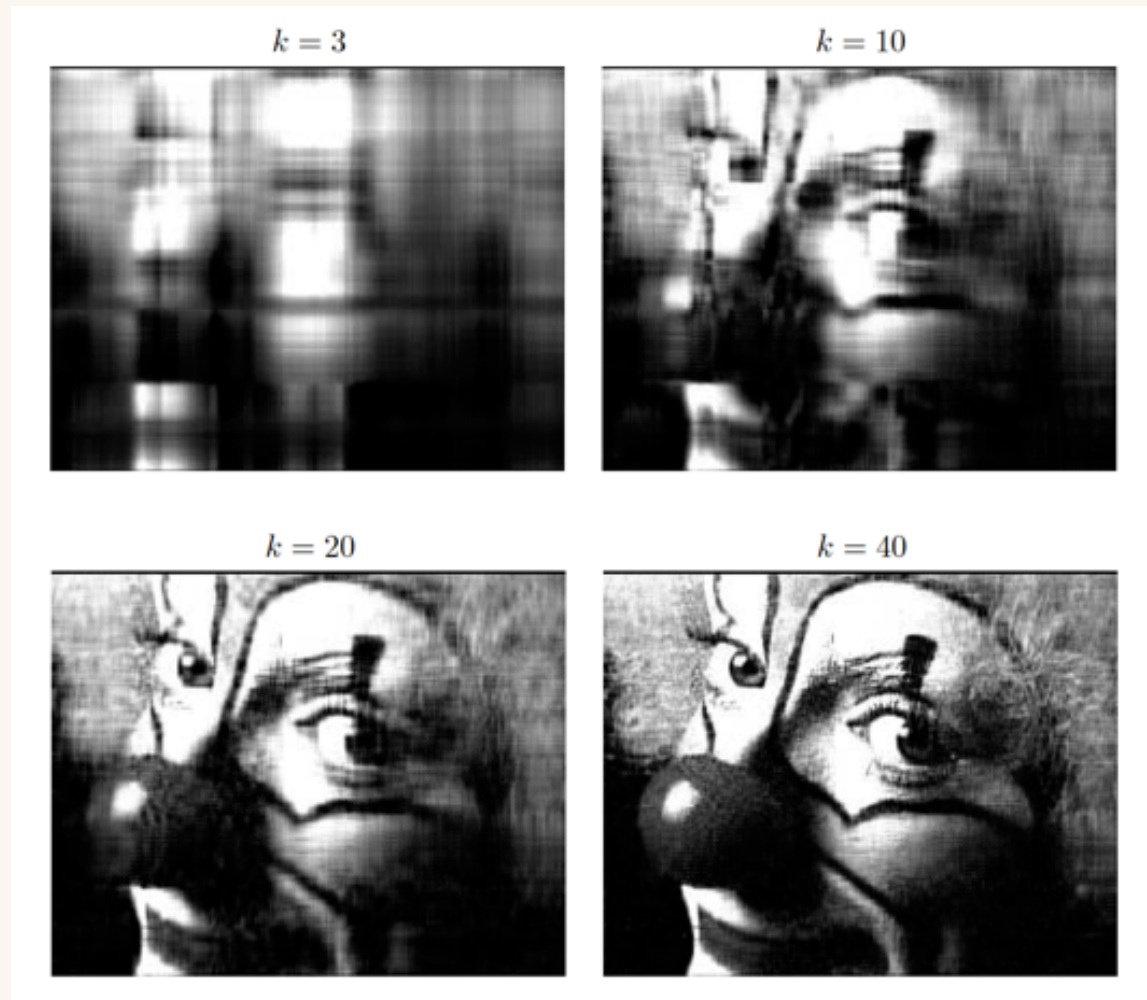


◇ Larger singular values λ have much “information” of T
→ (Frobenius norm)

→ We can approximate the matrix by the cutoff of index k

$$\dim(k) = \dim(a)\dim(b) \rightarrow D$$

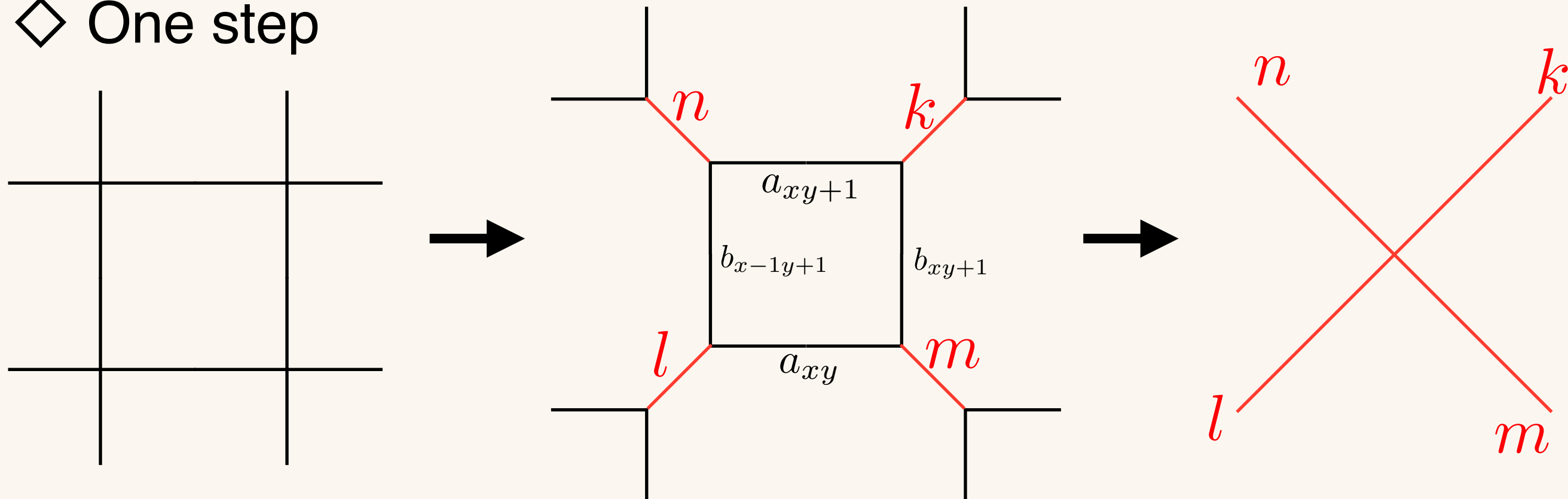
● SVD for a coarse graining (e.g. Image)



[<http://www.na.scitec.kobe-u.ac.jp/~yamamoto/lectures/cse-introduction2009/cse-introduction090512.PPT>]

● SVD for a coarse graining (e.g. Ising)

◇ One step



$$T_{abcd} = \sum_k A_{ac}^k B_{bd}^k$$

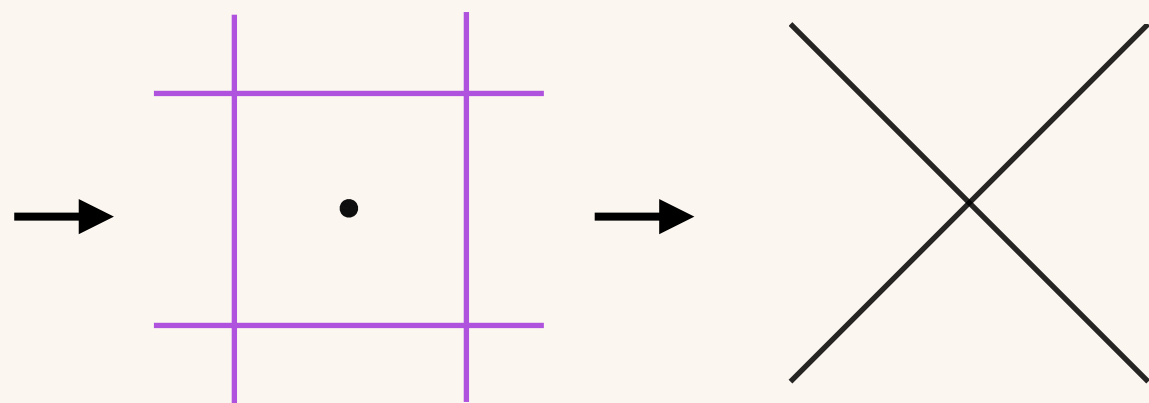
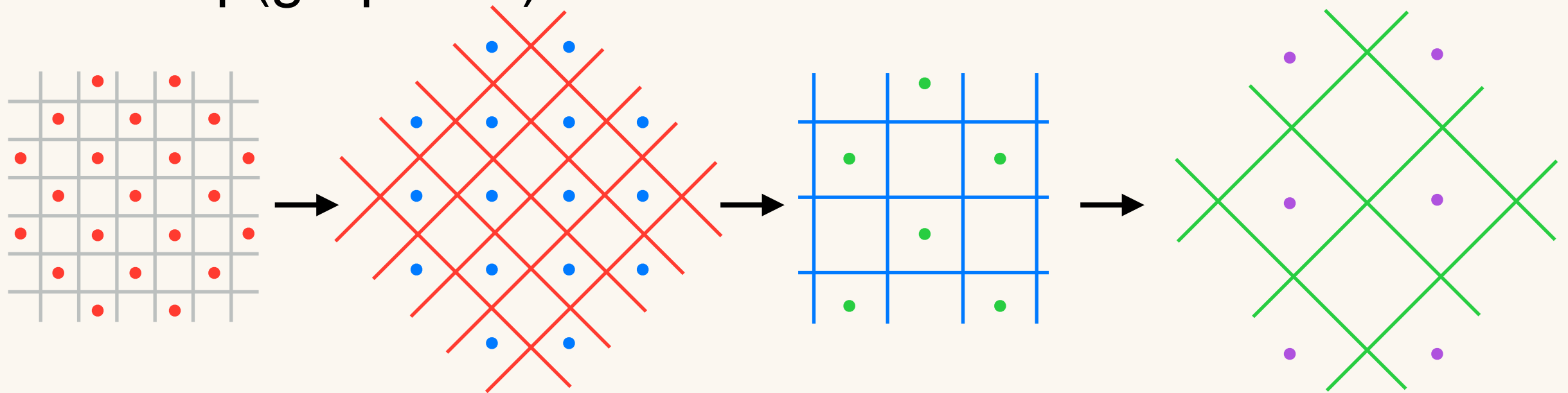
$$T_{abcd} = \sum_k C_{ad}^k D_{bc}^k$$

$$T_{klmn}^{(n+1)} = \sum_{a,b} A_{a_{xy+1}b_{xy+1}}^k B_{a_{xy}b_{x-1y+1}}^l C_{a_{xy}b_{xy+1}}^m D_{a_{xy+1}b_{x-1y+1}}^n$$

→ Exponential Volume reduction $V \rightarrow \frac{V}{2}$

● SVD for a coarse graining (e.g. Ising)

◇ ALL step(graphical)



$$V \rightarrow \frac{V}{2^{\# \text{step}}}$$

$$Z = \sum_{a,b} \prod_{x,y} T_{a_{x,y}, a_{x+1,y}, b_{x,y}, b_{x,y+1}} \rightarrow \sum_{a,b} T'_{aabb}$$

● Numerical costs for simple TRG

◇ How fast?

SVD:

$$T_{abcd} = \sum_k A_{ac}^k B_{bd}^k$$

$$O(D^6)$$

Contraction:

$$T_{klmn}^{(n+1)} = \sum_{a,b} A_{a_{xy+1}b_{xy+1}}^k B_{a_{xy}b_{x-1y+1}}^l C_{a_{xy}b_{xy+1}}^m D_{a_{xy+1}b_{x-1y+1}}^n$$

$$O(D^6)$$

$$2^{2V} \rightarrow O((\log V) \times D^6)$$

◇ Large D is still difficult.

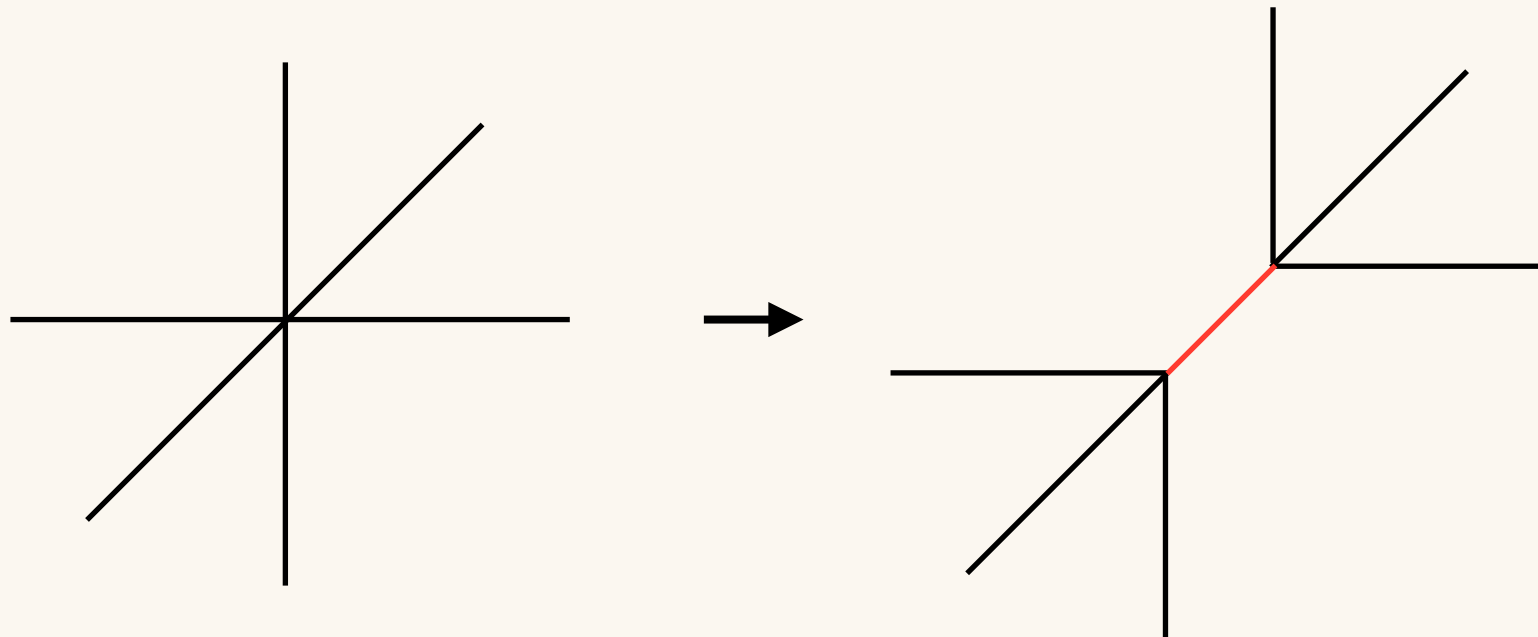
(Higher dimension, complicated system...)

→ We need more sophisticated algorithm.

● Cost for Higher-dimensional system

◇ Can we generalize (simple) TRG for higher dimension?

→ Formally we can, but bad approximation (and Cost)



Cutoff:	$D^3 \rightarrow D$
Cost:	$O(D^{12})$

We need more faster method (for more interesting system)

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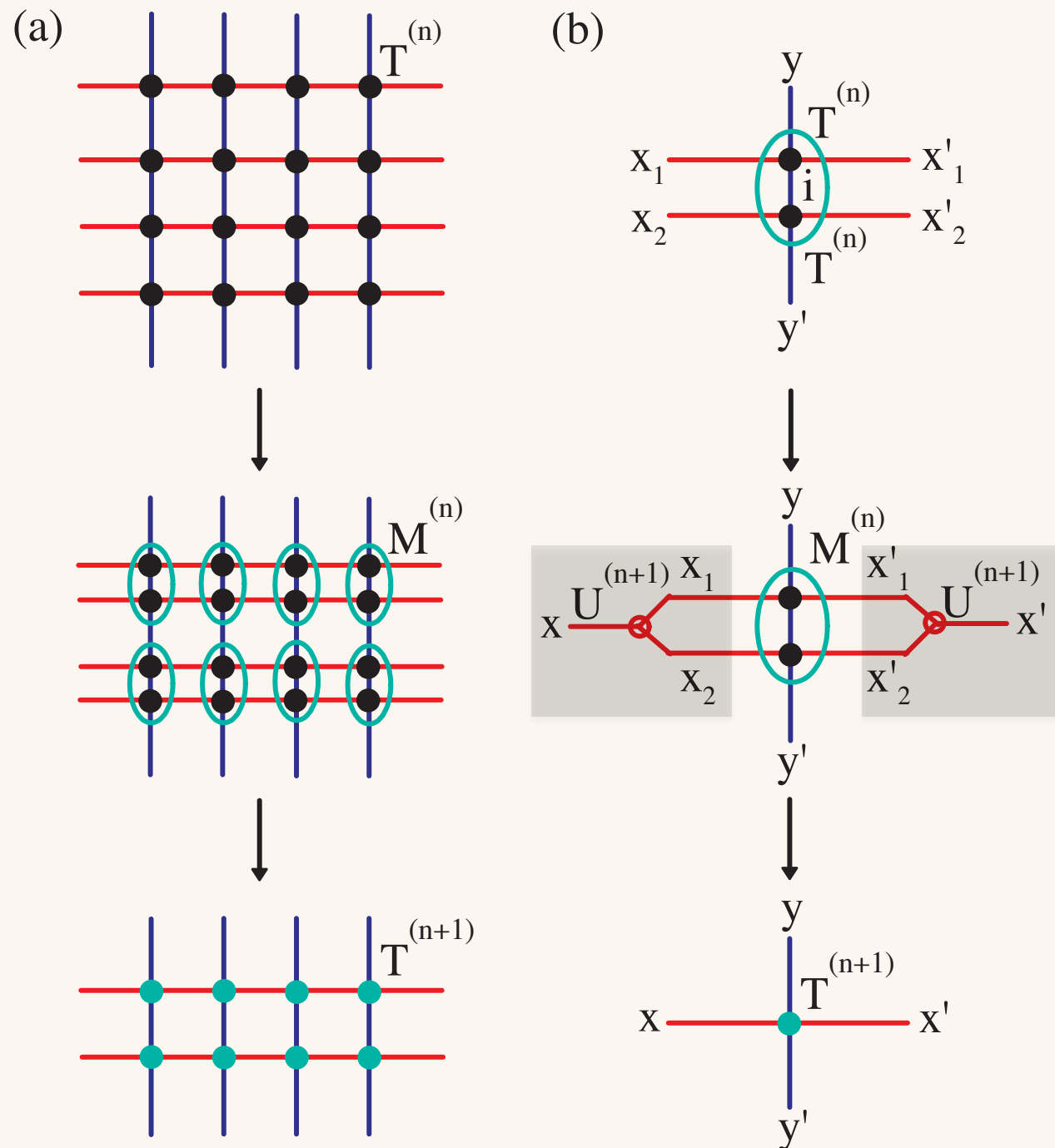
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◇ Contraction by projection operator U



→ U is made by SVD

$$M_{x_1 x_2 x'_1 x'_2 y y'} \equiv \sum_k T_{x_1 x'_1 y k} T_{x_2 x'_2 k y'}$$

$$[M^t M]_{[x_1 x_2][\chi_2 \chi_2]} \equiv \sum_{abcd} M^t_{[x_1 x_2][abcd]} M_{[abcd][\chi_1 \chi_2]}$$

SVD Cutoff: $D^2 \rightarrow D$
Cost: $O(D^{11})$

$$[M^t M]_{[x_1 x_2][\chi_2 \chi_2]} = \sum_k U^k_{[x_1 x_2]} \lambda^k U^k_{[\chi_1 \chi_2]}$$

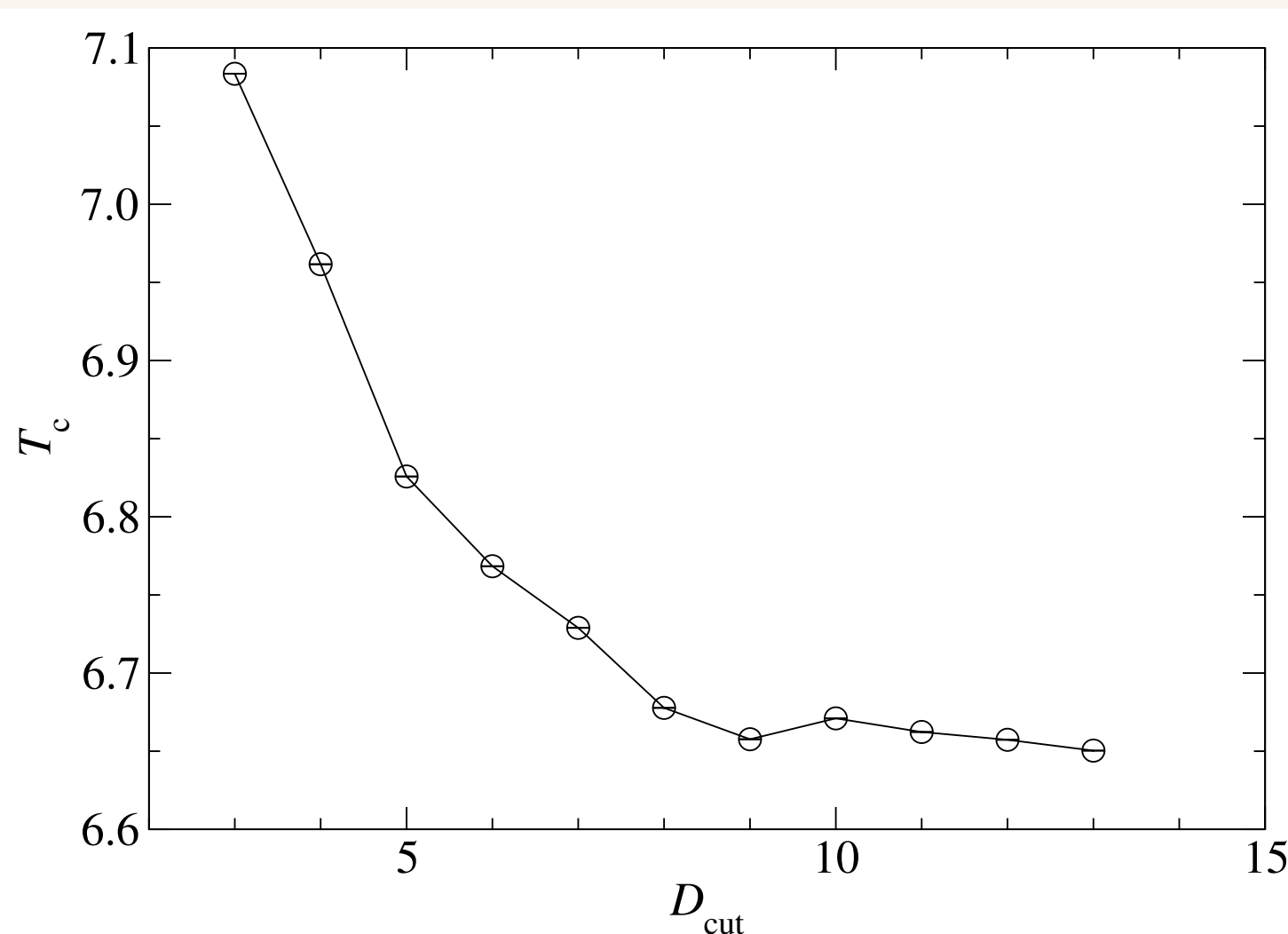
Contraction Cost: $O(D^{4\text{dim}-1})$ 50

● Numerical costs for HOTRG

◇ More higher dimension? → still difficult.

Cost: $O(D^{4\text{dim}-1})$

◇ 4-dim Ising $O(D^{15})$ [S.Akiyama, Y.Kuramashi et al. arXiv:1906.06060]



$D = 13$ @Oakforest-PACS

$$L^4 \leq 1024^4$$

$$T_c(D = 13) = 6.650365(5)$$

Monte-Carlo

$$L^4 \leq 80^4 \quad (200)$$

$$T_c = 6.68026(2)$$

[P.H. Lundow and K. Markstrom
arXiv:1202.3031]

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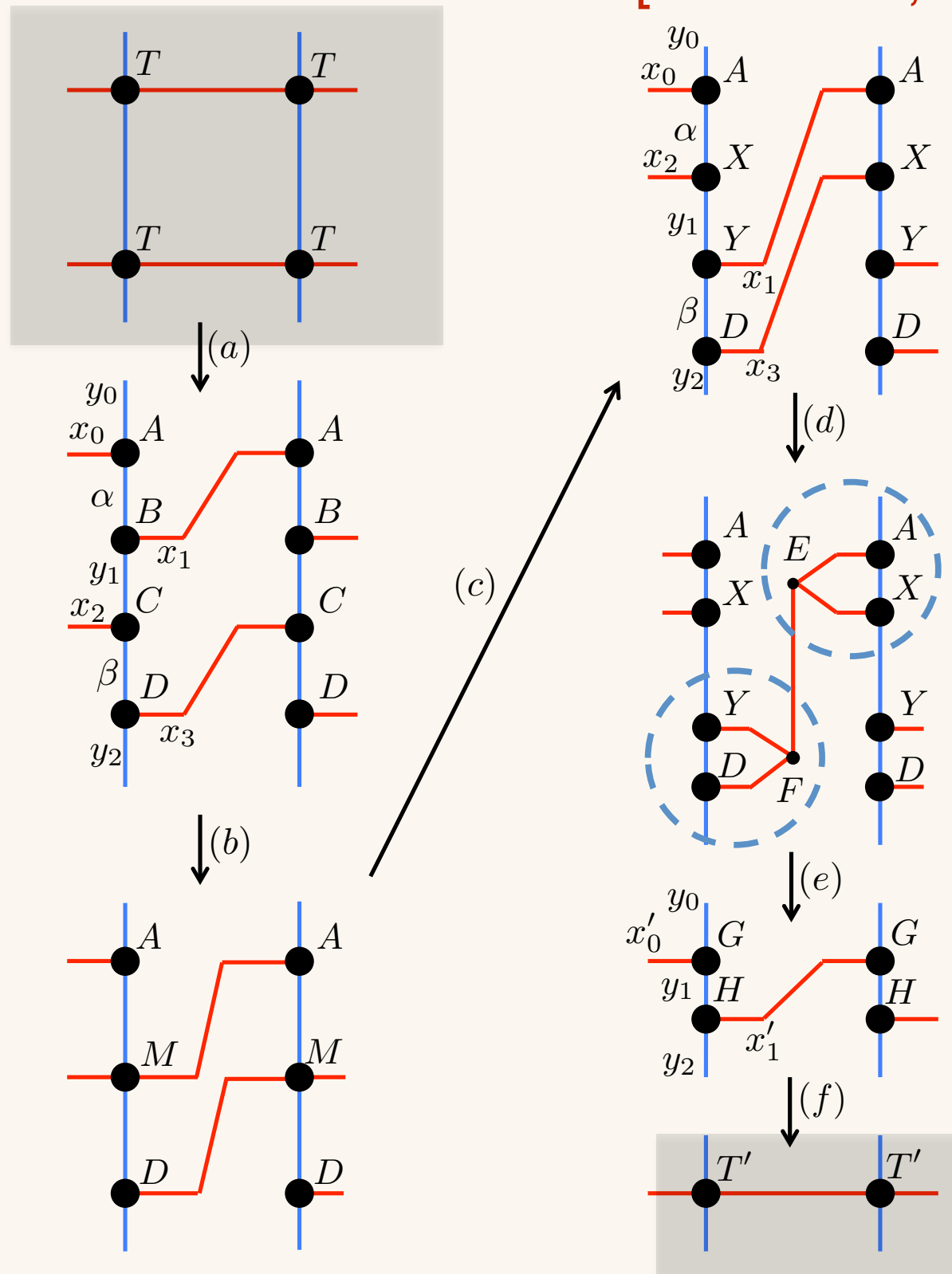
$$O(D^{4\text{dim}-1}) \rightarrow O(D^{2\text{dim}+1})$$

Triad RG (Tri-RG)

[D. Kadoh and K.N. arXiv:1911.xxxxx]

● Anisotropic TRG (ATRG)

[D. Adachi, T. Okubo, and S. Todo. arXiv:1906.02007]



◇ Auxiliary SVD before SVD
(In order to reduce indeces)

$$O(D^{4\text{dim}-1}) \rightarrow O(D^{2\text{dim}+1})$$

$$\begin{array}{c} d \\ | \\ a - \text{---} c \\ | \\ b \end{array} = \begin{array}{c} d \\ | \\ a - \text{---} c \\ | \\ b \end{array}$$

The diagram shows a tensor contraction where a central point is connected to four points a, b, c, d . The left side shows a standard cross-like contraction. The right side shows a contraction where the connection between a and b is highlighted in red and labeled k , representing a low-rank representation.

◇ Key-point: Low rank rep.
reduces the leading cost.

● Examples: Variational TRG

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$$O(D^{4\text{dim}-1}) \rightarrow O(D^{2\text{dim}+1})$$

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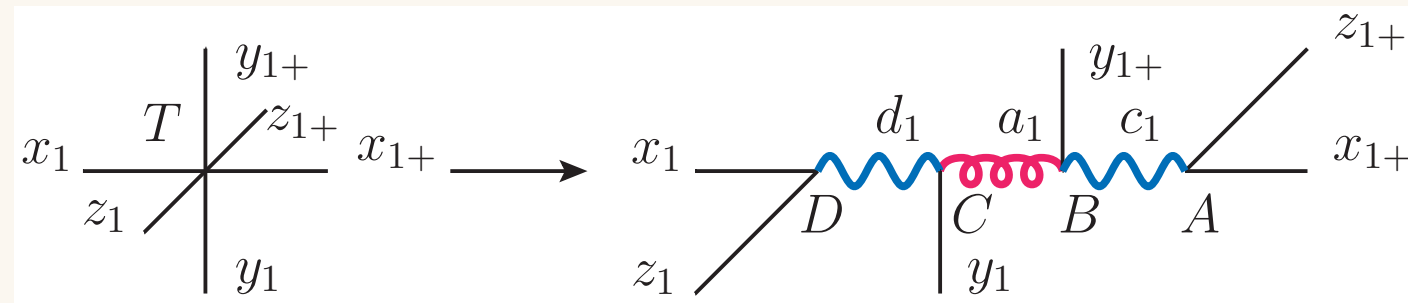
[D. Kadoh and K.N. arXiv:1912.xxxxx]

$$O(D^{4\text{dim}-1}) \rightarrow O(D^{\text{dim}+3})$$

● Triad RG

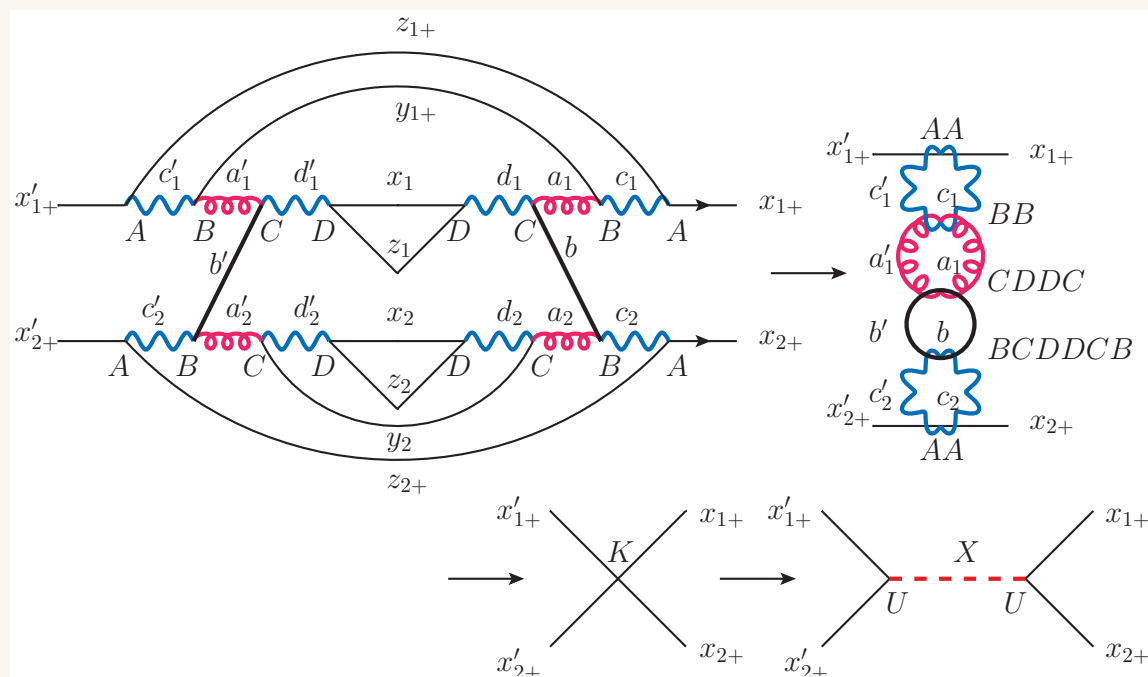
[D. Kadoh and K.N. arXiv:1912.xxxxx]

◇ Using the Triad (Rank-3) tensor as a fundamental tensor

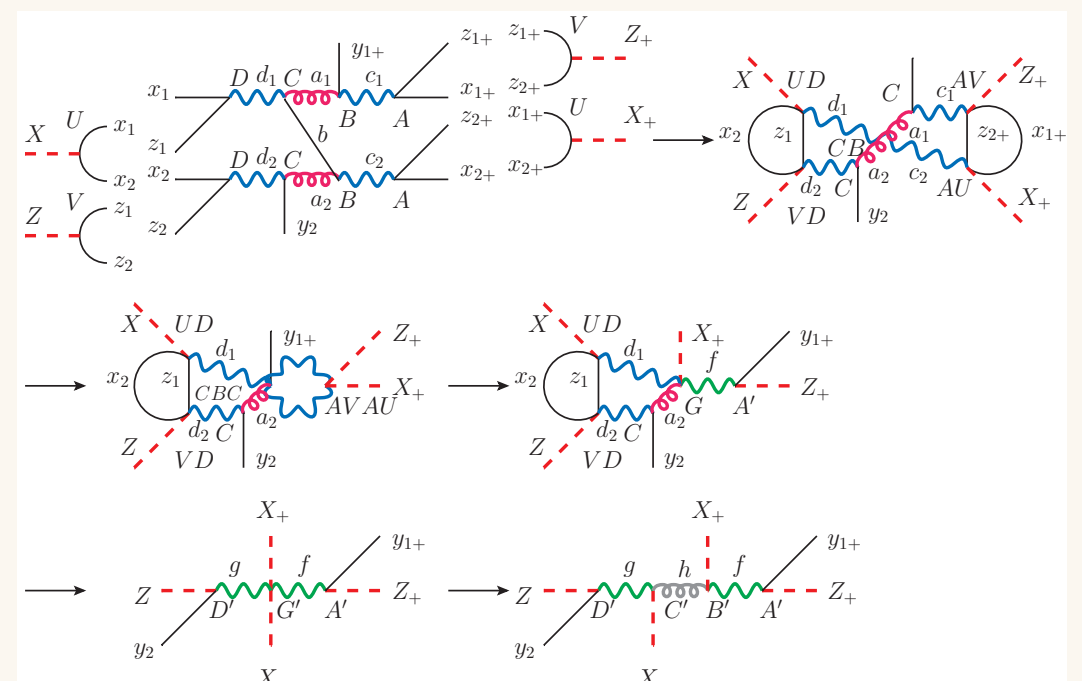


→ We apply HOTRG-like procedure to Triad tensor rep.

◇ projection operator U



◇ contraction part



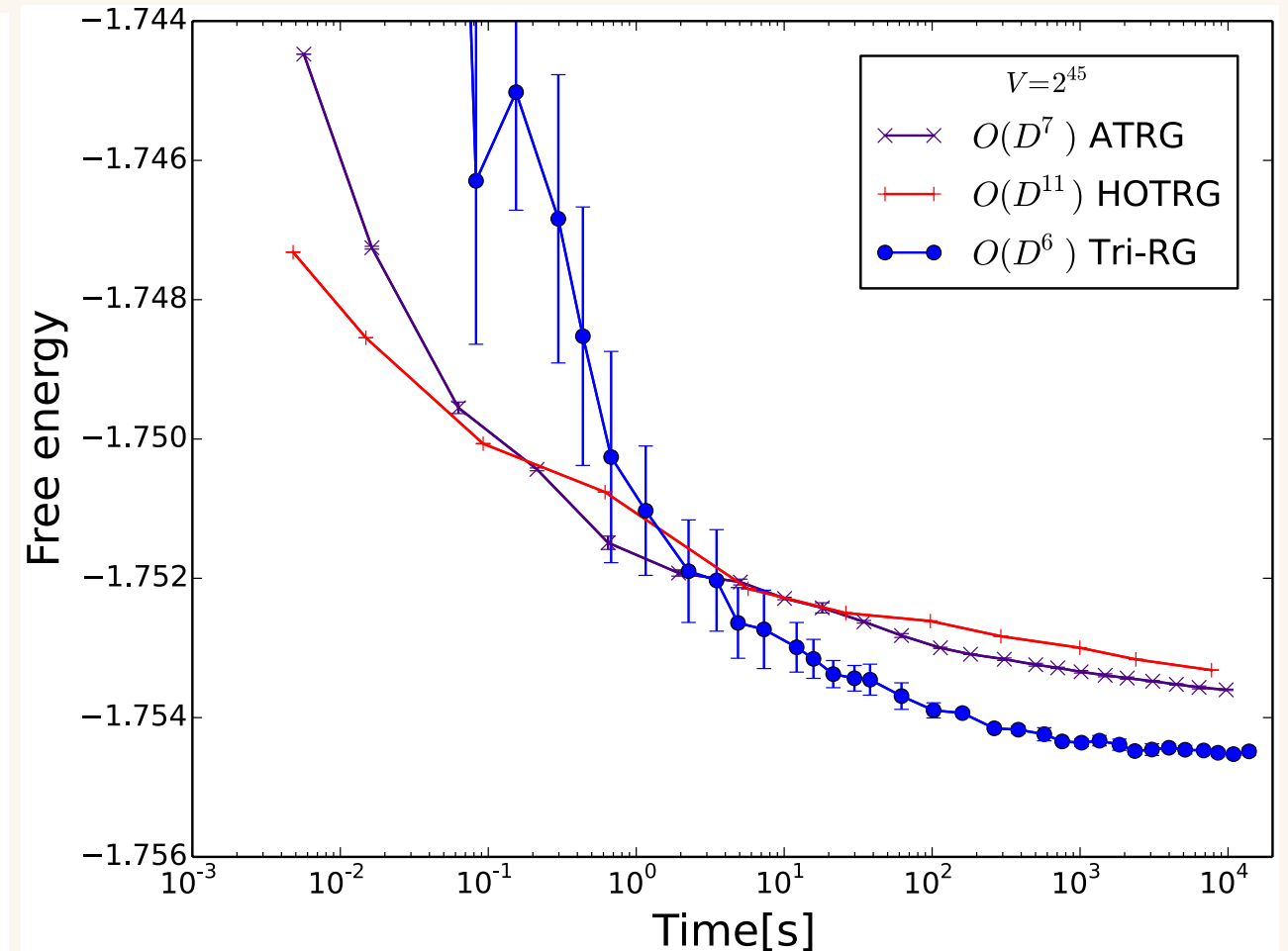
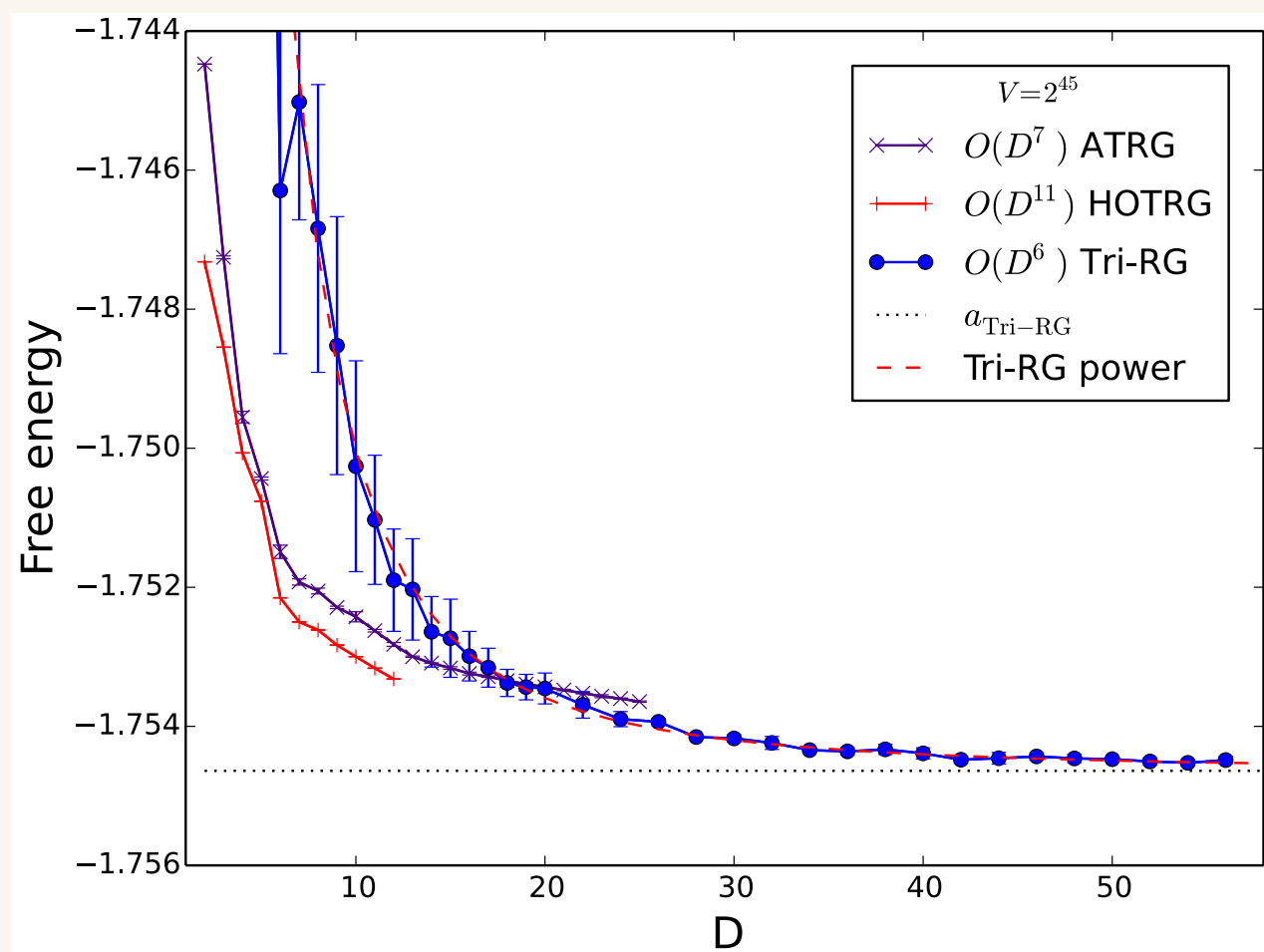
● Triad RG

[D. Kadoh and K.N. arXiv:1912.xxxxx]

◇ Triad rep. reduces the cost without loss of precision

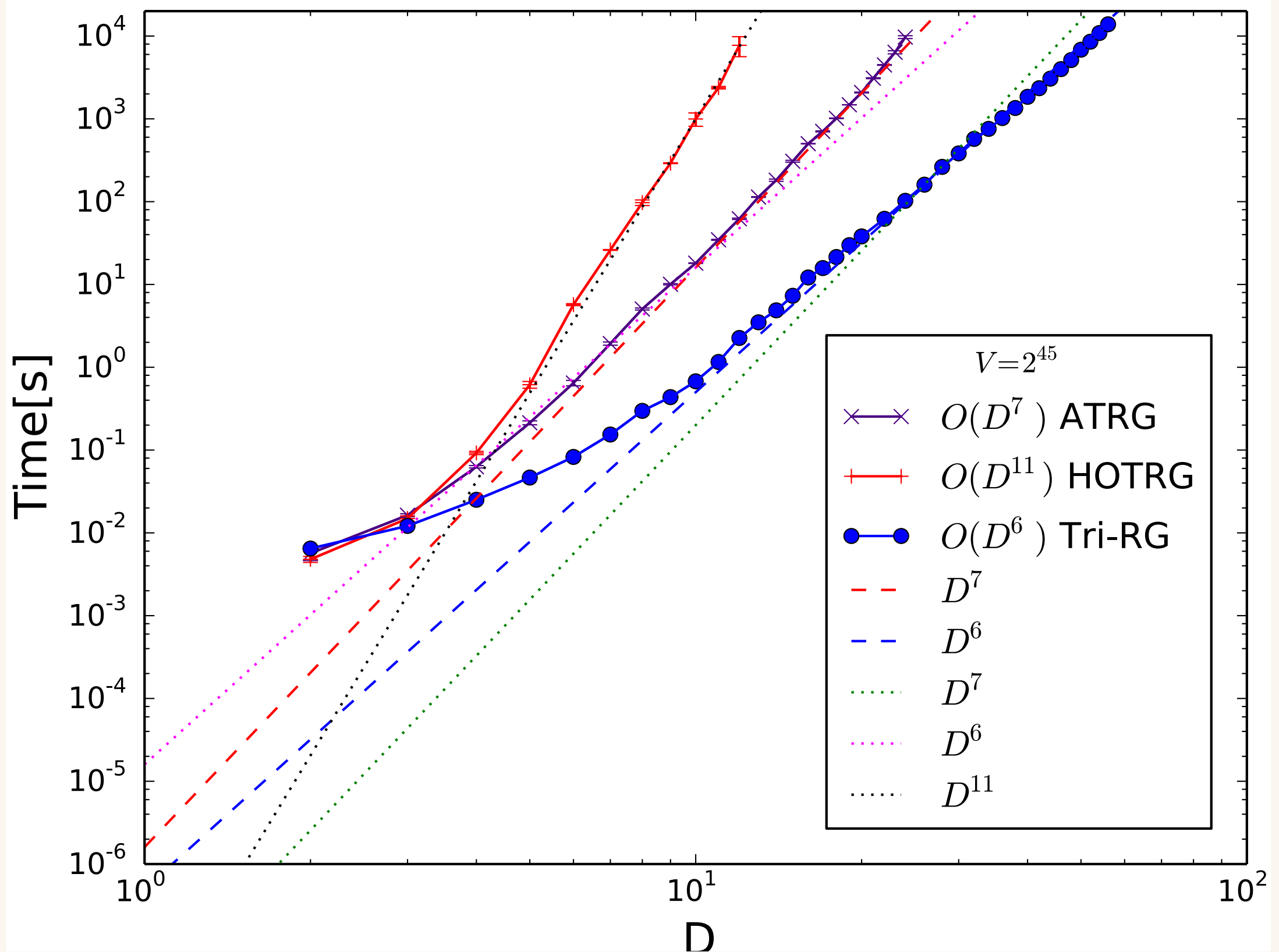
$$O(D^{4\text{dim}-1}) \rightarrow O(D^{\text{dim}+3})$$

◇ 3-dim Ising



● Triad RG

[D. Kadoh and K.N. arXiv:1912.xxxxx]



● Summary for TRG part

- ◆ How to construct the tensor network rep. is one important step to apply TRG method to the statistical system.
- ◆ For 1 or 2 dimension, TRG method provide the precise determination of the physical quantities without sign-problem
- ◆ Now TRG methods are improved to apply to the higher dim. Low-rank tensor is a key-idea.
- ◆ We need theory of the systematic error of cutoff parameter. For more realistic calculation, (maybe) we should introduce some stochastic method such as Monte-Carlo simulation.