A non-perturbative definition of the QCD energy-momentum tensor from a moving frame and its application to thermodynamics

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Introduction

The goal

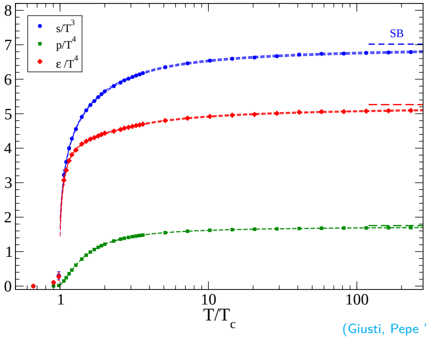
QCD equation of state (EoS)

$$s(T),\ p(T),\ \varepsilon(T)$$

Thermodynamics

$$s(T) = \frac{\partial p(T)}{\partial T}$$

$$Ts(T) = p(T) + \varepsilon(T)$$



(Giusti, Pepe '17)

SU(3) YM – $T_c \approx 300\,\mathrm{MeV}$

Why is this interesting?

- Fundamental property of QCD
- Heavy-ion collisions
- Cosmology
- ...

What do we know?

- EoS of $N_\mathrm{f} = 2+1$ QCD for $T \lesssim 500\,\mathrm{MeV}$ (Bazavov *et al.* '14; Borsanyi *et al.* '14; Bali *et al.* '14; ...)
- First results up to $T \approx 1-2\,\mathrm{GeV}$ (some for $N_\mathrm{f} = 2+1+1$) (Borsanyi *et al.* '16; Bazavov, Petreczky, Weber '18)
- Most results use variants of staggered fermions
  Wilson-quarks are catching up ... (tmfT Collab. '16; WHOT-QCD Collab. '18; MDB, Giusti, Pepe '18; ...)

Introduction

A non-perturbative problem

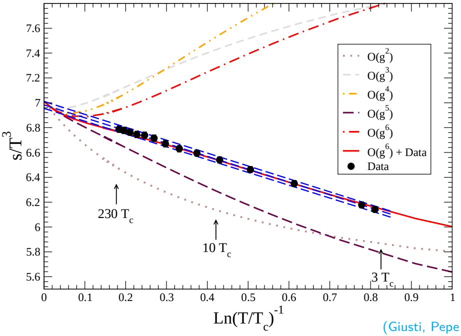
Asymptotic freedom

$$\alpha_s(\mu \approx T) \overset{T\to\infty}{\longrightarrow} 0$$

$\Rightarrow$ PT should work at large T

Free quarks & gluons gas

$$\frac{s_{SB}(T)}{T^3} = \frac{\pi^2}{45}\left(32 + 21 \times N_f\right) \overset{N_f=0}{\approx} 7.02$$



(Giusti, Pepe '17)

SU(3) YM – $T_c \approx 300\,\text{MeV}$

Problems

- PT at finite T shows very **poor convergence**
  - Works only up to a **finite** order: no matter how **small** α_s is! (Lindé '80)
  - The "O(g^6) + Data" term at $T \approx 68\,\text{GeV}$ is $\approx 50\%$ of the correction to free gas given by the other terms
- Resummation techniques seem to improve convergence but cf. (Andersen *et al.* '16)
  - Uncertainties are **hard** to reliably quantify within PT
  - Lindé problem is **not** solved

Introduction

A difficult non-perturbative problem

Free energy

$$f = -p = -\frac{T}{V}\ln\mathcal{Z}$$

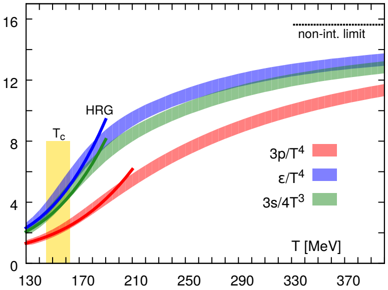
Trace anomaly

(Boyd *et al.* '96; Umeda *et al.* '09; ...)

$$\frac{I(T)}{T^4} \equiv \frac{\varepsilon - 3p}{T^4} = T\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{p}{T^4}\right)$$

Pressure

$$\frac{p(T)}{T^4} = \frac{p(T_0)}{T_0^4} + \int_{T_0}^{T}\mathrm{d}T'\,\frac{I(T')}{T'^5}$$



(HotQCD Bazavov *et al.* '14)

QCD with $N_\mathrm{f} = 2+1$ quarks

Lattice obs. $(\widehat{A} \equiv$ **lattice**$)$

$$\widehat{I}(T) = -\frac{T}{V}\frac{\mathrm{d}\ln\widehat{\mathcal{Z}}}{\mathrm{d}\ln a} = \frac{T}{V}\left(a\frac{\mathrm{d}\vec{b}}{\mathrm{d}a}\right)\left\langle\frac{\partial\widehat{S}_{\mathrm{QCD}}}{\partial\vec{b}}\right\rangle_T \qquad \vec{b} = \{g_0(a), m_{0,f}(a), \ldots\} \Leftarrow \mathrm{LCP}$$

Renormalization

$$I(T) = \lim_{a\to 0}\widehat{I}_R(T) = \lim_{a\to 0}\left[\widehat{I}(T) - \widehat{I}(0)\right]\Big|_{\vec{b}}$$

Problem

The renormalization **unnaturally** ties together two **separate** physical scales:

$$L^{-1} \ll T \ll a^{-1} \;\&\; L^{-1} \sim m_\pi \;\Rightarrow\; L/a = \mathrm{O}(100) \text{ for } T = \mathrm{O}(1\,\mathrm{GeV})$$

The energy-momentum tensor (EMT)

Back to basics

Thermodynamics & EMT

$$Ts(T) = p(T) + \varepsilon(T) \qquad p = \langle \mathcal{T}_{kk} \rangle_T \qquad \varepsilon = -\langle \mathcal{T}_{00} \rangle_T$$

EMT (continuum)

(Callan, Coleman, Jackiw '71; ...)

$$\mathcal{T}^R_{\mu\nu} = \mathcal{T}_{\mu\nu} = \mathcal{T}^F_{\mu\nu} + \mathcal{T}^G_{\mu\nu}$$

$$\mathcal{T}^F_{\mu\nu} = \frac{1}{4}\left\{\overline{\psi}\gamma_\mu \overleftrightarrow{D}_\nu \psi + \overline{\psi}\gamma_\nu \overleftrightarrow{D}_\mu \psi\right\} - \delta_{\mu\nu}\mathcal{L}^F \qquad \mathcal{T}^G_{\mu\nu} = \frac{1}{g_0^2} F^a_{\mu\alpha}F^a_{\nu\alpha} - \delta_{\mu\nu}\mathcal{L}^G$$

EMT (lattice)

(Caracciolo *et al.* '90 '91 '92)

$$\mathcal{T}^R_{\mu\nu} \neq \mathcal{T}_{\mu\nu} \qquad \mathcal{T}^{\{6\}}_{\mu\nu} = \mathcal{T}_{\mu\neq\nu} \qquad \mathcal{T}^{\{3\}}_{\mu\nu} = \mathcal{T}_{\mu\mu} - \mathcal{T}_{\nu\nu} \qquad \mathcal{T}^{\{1\}}_{\mu\nu} = \delta_{\mu\nu}\, \tfrac{1}{4}\sum_\alpha \mathcal{T}_{\alpha\alpha}$$

- $\mathcal{T}^{R,\{6,3\}}_{\mu\nu} = Z^{\{6,3\}}_F(g_0)\mathcal{T}^{F,\{6,3\}}_{\mu\nu} + Z^{\{6,3\}}_G(g_0)\mathcal{T}^{G,\{6,3\}}_{\mu\nu} \qquad \{6,3,1\} \in H(4)$
- $\langle \mathcal{T}^{\{1\}}_{\mu\mu} \rangle_T / T^4 \overset{a\to 0}{\propto} (aT)^{-4}$

Any ideas?

- Thermal QFT in a moving frame (Giusti, Meyer '13)
- Ward identities with flowed probes (Del Debbio, Patella, Rago '13)
- $\mathcal{T}^R_{\mu\nu}$ from small flow-time expansion (Suzuki '13)

Thermodynamics in a moving frame

The relativistic perfect fluid **[Minkowski space]**

Rest frame

$$\mathcal{T}_{\mu\nu} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

Moving frame

$$\mathcal{T}_{0k} = \gamma^2 (p+\varepsilon) v_k \qquad \boldsymbol{v} \equiv \text{velocity}, \ \gamma = \frac{1}{\sqrt{1-\boldsymbol{v}^2}}$$

$$\mathcal{T}_{00} = \gamma^2 (p+\varepsilon) - p \qquad \mathcal{T}_{kk} = \gamma^2 (p+\varepsilon) v_k^2 + p$$

Entropy density (using $Ts = p + \varepsilon$)

$$Ts = \frac{\mathcal{T}_{0k}}{\gamma^2 v_k} \qquad [\,T \equiv \text{temp. rest frame}\,]$$

Kinematic relations

$$\mathcal{T}_{0k} = \frac{v_k}{1+v_k^2} \left(\mathcal{T}_{00} + \mathcal{T}_{kk}\right) \qquad [\, v_k \neq 0 \,]$$

Thermodynamics in a moving frame

Shifted boundary conditions (continuum)

Thermal QCD path integral $[\,L=\infty\,]$

$$\mathcal{Z}(L_0',\theta_0) = \int [DA][D\overline{\psi}][D\psi]\, e^{-S_{\rm QCD}[A,\overline{\psi},\psi]} \qquad \begin{aligned} A_\mu(L_0',\boldsymbol{x}) &= A_\mu(0,\boldsymbol{x}) \\ \psi(L_0',\boldsymbol{x}) &= -e^{i\theta_0}\psi(0,\boldsymbol{x}) \end{aligned}$$

Euclidean boost / $\mathrm{SO}(4)$ **rotation** $\left[\,\boldsymbol{\xi} = -i\boldsymbol{v}\ ;\ \gamma^{-1} = \sqrt{1+\boldsymbol{\xi}^2}\,\right]$

$$\begin{aligned} &\mathcal{L}_{\rm QCD} & & & &\mathcal{L}_{\rm QCD} \\ A_\mu(L_0',\boldsymbol{x}) &= A_\mu(0,\boldsymbol{x}) & &\overset{\text{SO(4)}}{\longrightarrow} & A_\mu(L_0,\boldsymbol{x}) &= A_\mu(0,\boldsymbol{x}-\boldsymbol{\xi}L_0) \qquad [\,L_0' = L_0/\gamma\,] \\ \psi(L_0',\boldsymbol{x}) &= -e^{i\theta_0}\psi(0,\boldsymbol{x}) & & & \psi(L_0,\boldsymbol{x}) &= -e^{i\theta_0}\psi(0,\boldsymbol{x}-\boldsymbol{\xi}L_0) \end{aligned}$$

Partition function $[\,\mu_{\mathcal{I}} = -\theta_0/L_0\,]$

$$\mathcal{Z}(L_0,\boldsymbol{\xi},\theta_0) = \mathrm{Tr}\left\{e^{-L_0(\hat{H}-i\boldsymbol{\xi}\cdot\hat{\boldsymbol{P}}-i\mu_{\mathcal{I}}\hat{N})}\right\}$$

$\hat{\boldsymbol{P}} \equiv$ momentum $\quad \hat{N} \equiv$ quark num.

Free energy

$$f(L_0,\boldsymbol{\xi},\theta_0) = -\frac{1}{L_0 V}\ln\mathcal{Z}(L_0,\boldsymbol{\xi},\theta_0) \qquad f(L_0,\boldsymbol{\xi},\theta_0) \overset{V\to\infty}{=} f(L_0/\gamma,0,\theta_0)$$

Entropy density

$$Ts(T) = -\frac{\langle \mathcal{T}_{0k}\rangle_{\boldsymbol{\xi},\theta_0=0}}{\gamma^2\xi_k} \qquad T = \frac{\gamma}{L_0}$$

Thermodynamics in a moving frame

Shifted boundary conditions (lattice)

Thermal QCD path integral $[\,L=\infty\,]$

$$\widehat{\mathcal{Z}}(L_0',\theta_0) = \int [DU][D\overline{\psi}][D\psi]\, e^{-S_{\rm LQCD}[U,\overline{\psi},\psi]} \qquad \begin{aligned} U_\mu(L_0',\boldsymbol{x}) &= U_\mu(0,\boldsymbol{x}) \\ \psi(L_0',\boldsymbol{x}) &= -e^{i\theta_0}\psi(0,\boldsymbol{x}) \end{aligned}$$

Euclidean boost / $\mathrm{SO}(4)$ **rotation** $\left[\,\boldsymbol{\xi} = -i\boldsymbol{v}\ ;\ \gamma^{-1} = \sqrt{1+\boldsymbol{\xi}^2}\,\right]$

$$\begin{array}{c} \mathcal{L}_{\rm LQCD} \\ U_\mu(L_0',\boldsymbol{x}) = U_\mu(0,\boldsymbol{x}) \\ \psi(L_0',\boldsymbol{x}) = -e^{i\theta_0}\psi(0,\boldsymbol{x}) \end{array} \quad \xrightarrow{\text{SO(4)}}\!\!\!\!\!\!\!\times \quad \begin{array}{c} \mathcal{L}_{\rm LQCD} \\ U_\mu(L_0,\boldsymbol{x}) = U_\mu(0,\boldsymbol{x}-\boldsymbol{\xi}L_0) \\ \psi(L_0,\boldsymbol{x}) = -e^{i\theta_0}\psi(0,\boldsymbol{x}-\boldsymbol{\xi}L_0) \end{array} \quad [\,L_0' = L_0/\gamma\,]$$

Partition function $[\,\mu_{\mathcal{I}} = -\theta_0/L_0\,]$

$$\widehat{\mathcal{Z}}(L_0,\boldsymbol{\xi},\theta_0) = \mathrm{Tr}\left\{e^{-L_0(\hat{H}-i\boldsymbol{\xi}\cdot\hat{\boldsymbol{P}}-i\mu_{\mathcal{I}}\hat{N})}\right\}$$

$\hat{\boldsymbol{P}} \equiv$ momentum $\quad \hat{N} \equiv$ quark num.

Free energy

$$\widehat{f}(L_0,\boldsymbol{\xi},\theta_0) = -\frac{1}{L_0 V}\ln\widehat{\mathcal{Z}}(L_0,\boldsymbol{\xi},\theta_0) \qquad \widehat{f}(L_0,\boldsymbol{\xi},\theta_0) \overset{V\to\infty}{=} \widehat{f}(L_0/\gamma,0,\theta_0)$$

Entropy density

$$Ts(T) = \lim_{a\to 0} -\frac{\langle T^R_{0k}\rangle_{\boldsymbol{\xi},\theta_0=0}}{\gamma^2\xi_k} \qquad T = \frac{\gamma}{L_0}$$

Renormalization of the EMT

Ward identities (continuum)

Momentum identities

$$\langle \mathcal{T}_{0k}\rangle_{\boldsymbol{\xi},\theta_0} = -\frac{\partial}{\partial \xi_k} f(L_0, \boldsymbol{\xi}, \theta_0) \quad \overset{\theta_0=\theta_0^A,\theta_0^B}{\Longrightarrow} \quad \mathcal{T}_{0k}^R = \mathcal{T}_{0k}$$

$$L_0 \langle \overline{\mathcal{T}}_{0k}(x_0)\mathcal{O}\rangle_{\boldsymbol{\xi},\theta_0,c} = \frac{\partial}{\partial \xi_k}\langle \mathcal{O}\rangle_{\boldsymbol{\xi},\theta_0} \qquad \overline{\mathcal{T}}_{0k}(x_0) = \int_V \mathrm{d}\boldsymbol{x}\, \mathcal{T}_{0k}(x_0,\boldsymbol{x})$$

Baryon number identities

$$i\langle V_0\rangle_{\boldsymbol{\xi},\theta_0} = L_0 \frac{\partial}{\partial \theta_0} f(L_0, \boldsymbol{\xi}, \theta_0)$$

$$\langle \overline{V}_0(x_0)\mathcal{O}\rangle_{\boldsymbol{\xi},\theta_0,c} = i\frac{\partial}{\partial \theta_0}\langle \mathcal{O}\rangle_{\boldsymbol{\xi},\theta_0} \qquad V_0(x) = \overline{\psi}(x)\gamma_0\psi(x)$$

Other identities

$$\frac{\partial}{\partial \theta_0}\langle \mathcal{T}_{0k}\rangle_{\boldsymbol{\xi},\theta_0} = -\frac{i}{L_0}\frac{\partial}{\partial \xi_k}\langle V_0\rangle_{\boldsymbol{\xi},\theta_0}$$

$$\langle \mathcal{T}_{0k}\rangle_{\boldsymbol{\xi},\theta_0} = \frac{\xi_k}{1-\xi_k^2}\Big(\langle \mathcal{T}_{00}\rangle_{\boldsymbol{\xi},\theta_0} - \langle \mathcal{T}_{kk}\rangle_{\boldsymbol{\xi},\theta_0}\Big) + \text{finite } V \text{ terms}$$

Renormalization of the EMT

Ward identities (lattice)

Momentum identities

$$\langle \mathcal{T}^R_{0k} \rangle_{\boldsymbol{\xi},\theta_0} = -\frac{\partial}{\partial \xi_k} \widehat{f}(L_0, \boldsymbol{\xi}, \theta_0) + \mathrm{O}(a) \qquad \frac{\partial}{\partial \xi_k}\widehat{f}(\boldsymbol{\xi}) = \frac{L_0}{2a}\left[\,\widehat{f}(\boldsymbol{\xi} + \tfrac{a\hat{k}}{L_0}) - \widehat{f}(\boldsymbol{\xi} - \tfrac{a\hat{k}}{L_0})\,\right]$$

$$L_0 \langle \overline{\mathcal{T}}^R_{0k}(x_0) \mathcal{O} \rangle_{\boldsymbol{\xi},\theta_0,c} = \frac{\partial}{\partial \xi_k} \langle \mathcal{O} \rangle_{\boldsymbol{\xi},\theta_0} + \mathrm{O}(a)$$

Baryon number identities

$$i \langle \widetilde{V}_0 \rangle_{\boldsymbol{\xi},\theta_0} = L_0 \frac{\partial}{\partial \theta_0} \widehat{f}(L_0, \boldsymbol{\xi}, \theta_0)$$

$$\langle \overline{\widetilde{V}}_0(x_0) \mathcal{O} \rangle_{\boldsymbol{\xi},\theta_0,c} = i \frac{\partial}{\partial \theta_0} \langle \mathcal{O} \rangle_{\boldsymbol{\xi},\theta_0} \qquad \widetilde{V}_0(x) = \tfrac{1}{2}\left[\,\overline{\psi}(x)(\gamma_0 - 1) e^{i\theta_0} U_0(x) \psi(x + \hat{0}) + \text{c.c.}\,\right]$$

Other identities

$$\frac{\partial}{\partial \theta_0} \langle \mathcal{T}^R_{0k} \rangle_{\boldsymbol{\xi},\theta_0} = -\frac{i}{L_0} \frac{\partial}{\partial \xi_k} \langle \widetilde{V}_0 \rangle_{\boldsymbol{\xi},\theta_0} + \mathrm{O}(a)$$

$$\langle \mathcal{T}^R_{0k} \rangle_{\boldsymbol{\xi},\theta_0} = \frac{\xi_k}{1 - \xi_k^2} \Big(\langle \mathcal{T}^R_{00} \rangle_{\boldsymbol{\xi},\theta_0} - \langle \mathcal{T}^R_{kk} \rangle_{\boldsymbol{\xi},\theta_0} \Big) + \mathrm{O}(a) + \text{finite } V \text{ terms}$$

Renormalization of the EMT

Finite volume considerations

Spatial boundary conditions

$$A_\mu(x_0, \boldsymbol{x} + \hat{k}L_k) = A_\mu(x_0, \boldsymbol{x}) \qquad \psi(x_0, \boldsymbol{x} + \hat{k}L_k) = e^{i\theta_k}\psi(x_0, \boldsymbol{x})$$

Finite volume terms

The finite spatial extent is source of additional $\mathrm{SO}(4)$ symmetry **breaking**, e.g.,

$$\langle \mathcal{T}_{0k} \rangle_{\boldsymbol{\xi},\theta} + \frac{1+\xi_k^2}{1-\xi_k^2} \langle \mathcal{T}_{0k} \rangle_{V_k} = \frac{\xi_k}{1-\xi_k^2} \left(\langle \mathcal{T}_{00} \rangle_{\boldsymbol{\xi},\theta} - \langle \mathcal{T}_{kk} \rangle_{\boldsymbol{\xi},\theta} \right) \qquad \theta = (\theta_0, \boldsymbol{\theta})$$

V_k: system with shifted bc. along the k-direction and $\theta \to \tilde{\theta} = f_{\mathrm{ugly}}(\theta, \boldsymbol{\xi}, L_\mu)$

Special geometries

$$\frac{L_k \xi_k}{L_0(1+\xi_k^2)} \in \mathbb{Z}$$

$$b_0 = \boldsymbol{\xi} \cdot \boldsymbol{b} + \frac{b_k}{\xi_k} \quad \text{where} \quad b_\mu = \frac{\theta_\mu}{L_\mu}$$

- In general finite volume terms are **exponentially** suppressed for $L_k \to \infty$
- Special geometries, however, guarantee that, e.g., $\langle \mathcal{T}_{0k} \rangle_{V_k} = 0$ **exactly!**
  $\Rightarrow$ The above finite volume WI takes its **infinite** volume form
- Different constraints might be needed for different WIs

O(a)-improvement of the EMT

Massless case

Lattice action

O(a)-improved Wilson fermions

Lattice EMT

(Caracciolo *et al.* '90 '91 '92)

$$\mathcal{T}^G_{\mu\nu} = \frac{1}{g_0^2} \widehat{F}^a_{\mu\alpha} \widehat{F}^a_{\nu\alpha} - \delta_{\mu\nu} \widehat{\mathcal{L}}^G$$

$$\mathcal{T}^F_{\mu\nu} = \frac{1}{8} \left\{ \overline{\psi}\gamma_\mu \left[\overleftrightarrow{\nabla}^*_\nu + \overleftrightarrow{\nabla}_\nu \right] \psi + \overline{\psi}\gamma_\nu \left[\overleftrightarrow{\nabla}^*_\mu + \overleftrightarrow{\nabla}_\mu \right] \psi \right\} - \delta_{\mu\nu} \widehat{\mathcal{L}}^F$$

O(a)-counterterms $[\, i \in \{6, 3\} \,]$

$$\mathcal{O}_{1,\mu\nu} = \overline{\psi}\sigma_{\mu\alpha} F_{\nu\alpha} \psi \qquad \mathcal{O}_{2,\mu\nu} = \partial_\alpha \left(\overline{\psi}\sigma_{\mu\alpha} \overleftrightarrow{D}_\nu \psi \right) \qquad \mathcal{O}_{3,\mu\nu} = \partial_\mu \partial_\nu \left(\overline{\psi}\psi \right)$$

$$\mathcal{T}^{R,\{i\}}_{\mu\nu,I} = Z^{\{i\}}_G(g_0) \mathcal{T}^{G,\{i\}}_{\mu\nu} + Z^{\{i\}}_F(g_0) \left\{ \mathcal{T}^{F,\{i\}}_{\mu\nu} + a\delta\mathcal{T}^{F,\{i\}}_{\mu\nu} \right\}$$

$$\delta\mathcal{T}^{F,\{i\}}_{\mu\nu} = \sum_k c^{\{i\}}_k(g_0) \mathcal{O}^{\{i\}}_{k,\mu\nu}$$

Remarks

- A classical expansion of the EMT shows that: $c^{\{i\}}_k(g_0) = 0 + \mathrm{O}(g_0^2)$
- $\mathcal{O}_2$ and $\mathcal{O}_3$ can be neglected in forward matrix elements of the EMT
- If **NO** spontaneous chiral symmetry breaking $\Rightarrow \langle \delta\mathcal{T}^{F,\{i\}}_{\mu\nu} \rangle_{\boldsymbol{\xi},\theta} = \mathrm{O}(a)$
  This happens in a finite volume **AND in infinite volume if $T > T_c$!**

O(a)-improvement of the EMT

Mass-degenerate case

O(a)-improved parameters

(Lüscher *et al.* '92)

$$\tilde{g}_0^2 = g_0^2\big(1 + b_\mathrm{g}(g_0)am_\mathrm{q}\big) \qquad m_\mathrm{q} = m_0 - m_\mathrm{crit}$$

$$\widetilde{m}_\mathrm{q} = m_\mathrm{q}\big(1 + b_\mathrm{m}(g_0)am_\mathrm{q}\big)$$

O(am)-counterterms $[\, i \in \{6,3\}\,]$

$$\mathcal{O}^{\{i\}}_{4,\mu\nu} = m_\mathrm{q}\mathcal{T}^{G,\{i\}}_{\mu\nu} \qquad \mathcal{O}^{\{i\}}_{5,\mu\nu} = m_\mathrm{q}\mathcal{T}^{F,\{i\}}_{\mu\nu}$$

O(a)-improved EMT

$$\mathcal{T}^{R,\{i\}}_{\mu\nu,I} = Z^{\{i\}}_G(\tilde{g}_0)\mathcal{T}^{G,\{i\}}_{\mu\nu,I} + Z^{\{i\}}_F(\tilde{g}_0)\mathcal{T}^{F,\{i\}}_{\mu\nu,I}$$

$$\mathcal{T}^{G,\{i\}}_{\mu\nu,I} = \big(1 + b^{\{i\}}_G(g_0)am_\mathrm{q}\big)\mathcal{T}^{G,\{i\}}_{\mu\nu} \qquad b^{\{i\}}_G = 0 + \mathrm{O}(g_0^2)$$

$$\mathcal{T}^{F,\{i\}}_{\mu\nu,I} = \big(1 + b^{\{i\}}_F(g_0)am_\mathrm{q}\big)\big\{\mathcal{T}^{F,\{i\}}_{\mu\nu} + a\delta\mathcal{T}^{F,\{i\}}_{\mu\nu}\big\} \qquad b^{\{i\}}_F = 1 + \mathrm{O}(g_0^2)$$

Remarks

- Non-degenerate quarks require one additional b-contribution of O(g_0^4)
- At high temperature O(a)-effects are suppressed like: $(aT) \times (m_\mathrm{q}/T)$
  $\Rightarrow$ One expects $\lesssim 1\%$ effects from light quarks once $T \gtrsim 1\,\mathrm{GeV}$
- Perturbative estimates for the b-coefficients are, thus, likely **sufficient** at high temperature

A test in perturbation theory

Renormalization constants and improvement coefficients to 1-loop order

Ward identities

$$\langle \mathcal{T}^R_{0k,I}\rangle_{\boldsymbol{\xi},\theta} = -\frac{\partial}{\partial \xi_k}\widehat{f}(L_0,\boldsymbol{\xi},\theta) \qquad \langle \mathcal{T}^R_{0k,I}\rangle_{\boldsymbol{\xi},\theta} = \frac{\xi_k}{1-\xi_k^2}\Big(\langle \mathcal{T}^R_{00,I}\rangle_{\boldsymbol{\xi},\theta} - \langle \mathcal{T}^R_{kk,I}\rangle_{\boldsymbol{\xi},\theta}\Big)$$

Renormalization constants

$$Z_G^{\{i\}}(g_0) = 1 + g_0^2\Big[N Z_G^{\{i\}(1,N_c)} + \frac{1}{N} Z_G^{\{i\}(1,\frac{1}{N_c})} + N_{\rm f} Z_G^{\{i\}(1,N_{\rm f})}\Big] + \mathrm{O}(g_0^4)$$

$$Z_F^{\{i\}}(g_0) = 1 + g_0^2 C_F Z_F^{\{i\}(1,N_c)} + \mathrm{O}(g_0^4)$$

Results: Unimproved and O(a)-improved Wilson for $i \in \{6,3\}$

Perfect agreement with the literature (when available) (Caracciolo *et al.* '92; Capitani, Rossi '95)

O(a)-improvement coefficients

$$c_k^{\{i\}}(g_0) = 0 + \mathrm{O}(g_0^2)$$

$$b_F^{\{i\}}(g_0) = 1 + g_0^2 C_F b_F^{\{i\}(1,N_c)} + \mathrm{O}(g_0^4) \qquad b_G^{\{i\}}(g_0) = 0 + g_0^2 N_{\rm f} b_G^{\{i\}(1,N_{\rm f})} + \mathrm{O}(g_0^4)$$

Results: O(a)-improved Wilson for $i \in \{6,3\}$

For the geeks

- $L_0/a = 4-32$, $R = L/L_0 = 5-15$, several $\boldsymbol{\xi}$ and θ values
- Gauge zero-mode removal $\Rightarrow$ $R \to \infty$ extrapolations
- Coordinate space calculation based on FFT

Towards the EoS at high temperature

General strategy

Master equation

$$\frac{s(T)}{T^3} = \lim_{a\to 0} -\frac{L_0^4 \langle \mathcal{T}^R_{0k,I} \rangle_{\boldsymbol{\xi},\theta=0}}{\gamma^6 \xi_k} \qquad T = \frac{\gamma}{L_0}$$

Lattice set-up

- $N_\mathrm{f} = 3$ O(a)-improved Wilson quarks with shifted bc., $\boldsymbol{\xi} = (1,0,0)$

Lines of constant physics

(ALPHA Collab. MDB *et al.* '16, '18; ALPHA Collab. Campos *et al.* '18)

- 8 values of $T \approx 2.8 - 80\,\mathrm{GeV}$ fixed by $\bar{g}^2_\mathrm{SF}(\mu = T/\gamma)$
- $L_0/a = 4, 6, 8, 10$ and $L/a = 288 \Rightarrow TL \approx 50 - 20$
- $m_{\mathrm{q},R} = \mathrm{O}(a^2)$, i.e., massless quarks
- PT values improvement coefficients

Systematic effects

- **Mass effects:** $s(T)|_m = s(T)|_{m=0} + \mathrm{O}(m^2/T^2)$
  Expected to be quite small for light quarks for $T \gtrsim 2.8\,\mathrm{GeV}$
  Actual size needs to be estimated at the smaller T's
- **Finite size effects:** $s(T)|_L = s(T)|_{L=\infty} + \mathrm{O}(e^{-mL})$ w/ $m = \mathrm{O}(T)$ (Giusti, Meyer '13)
  $L/a = 96$ simulations and measure of $m(T)$ to estimate actual size
- $\mathrm{O}(a)$-**effects**: Monitor size of $\mathrm{O}(a)$-counterterms

Towards the EoS at high temperature

Renormalization strategy

Renormalization condition

$$\langle \mathcal{T}_{0k}^R \rangle_{\boldsymbol{\xi},\theta} = -\frac{\partial}{\partial \xi_k} \widehat{f}(L_0, \boldsymbol{\xi}, \theta) \qquad \theta = \theta^A, \theta^B$$

General strategy

1. At fixed L_0/a, L/L_0, and $\boldsymbol{\xi}$, estimate

$$\frac{\partial}{\partial \xi_k} \widehat{f}(L_0, \boldsymbol{\xi}, \theta^*)\Big|_{g_0^*} \equiv F_*$$

2. Evaluate

$$\frac{\mathrm{d}}{\mathrm{d}g_0^2} \frac{\partial}{\partial \xi_k} \widehat{f}\Big|_{\theta} = \frac{\partial}{\partial \xi_k} \left[\frac{-1}{g_0^2} \langle S_G \rangle_{\boldsymbol{\xi},\theta} + \left(\frac{\partial m_0}{\partial g_0^2} \right) \langle \overline{\psi} \psi \rangle_{\boldsymbol{\xi},\theta} + \left(\frac{\partial c_{\mathrm{sw}}}{\partial g_0^2} \right) \langle \overline{\psi} \tfrac{i}{4} \sigma_{\mu\nu} \widehat{F}_{\mu\nu} \psi \rangle_{\boldsymbol{\xi},\theta} \right]$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta_\mu} \frac{\partial}{\partial \xi_k} \widehat{f}\Big|_{g_0^2} = \frac{i}{L_\mu} \frac{\partial}{\partial \xi_k} \langle \widetilde{V}_\mu \rangle_{\boldsymbol{\xi},\theta}$$

for a range of values of (g_0, θ) to obtain $\frac{\partial}{\partial \xi_k} \widehat{f}(L_0, \boldsymbol{\xi}, \theta)\Big|_{g_0}$

Initial condition

- For small enough g_0^*, F_* can be estimated in PT (Giusti, Pepe '15)
- Else F_* could be computed explicitly (not easy!)

Towards the EoS at high temperature

Renormalization strategy

Renormalization condition

$$\langle Z_F \mathcal{T}^F_{0k} + Z_G \mathcal{T}^G_{0k} \rangle_{\boldsymbol{\xi},\theta} = \frac{1}{2aV} \ln \left[\frac{\widehat{\mathcal{Z}}(L_0, \boldsymbol{\xi} - \frac{a\hat{k}}{L_0}, \theta)}{\widehat{\mathcal{Z}}(L_0, \boldsymbol{\xi} + \frac{a\hat{k}}{L_0}, \theta)} \right] \qquad \theta = \theta^A, \theta^B$$

General strategy

1. At fixed L_0/a, L/L_0, and $\boldsymbol{\xi}$, estimate

$$\frac{\partial}{\partial \xi_k} \widehat{f}(L_0, \boldsymbol{\xi}, \theta^*) \Big|_{g_0^*} \equiv F_*$$

2. Evaluate

$$\frac{\mathrm{d}}{\mathrm{d}g_0^2} \frac{\partial}{\partial \xi_k} \widehat{f} \Big|_{\theta} = \frac{\partial}{\partial \xi_k} \left[\frac{-1}{g_0^2} \langle S_G \rangle_{\boldsymbol{\xi},\theta} + \left(\frac{\partial m_0}{\partial g_0^2} \right) \langle \overline{\psi} \psi \rangle_{\boldsymbol{\xi},\theta} + \left(\frac{\partial c_{\mathrm{sw}}}{\partial g_0^2} \right) \langle \overline{\psi} \tfrac{i}{4} \sigma_{\mu\nu} \widehat{F}_{\mu\nu} \psi \rangle_{\boldsymbol{\xi},\theta} \right]$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta_\mu} \frac{\partial}{\partial \xi_k} \widehat{f} \Big|_{g_0^2} = \frac{i}{L_\mu} \frac{\partial}{\partial \xi_k} \langle \widetilde{V}_\mu \rangle_{\boldsymbol{\xi},\theta}$$

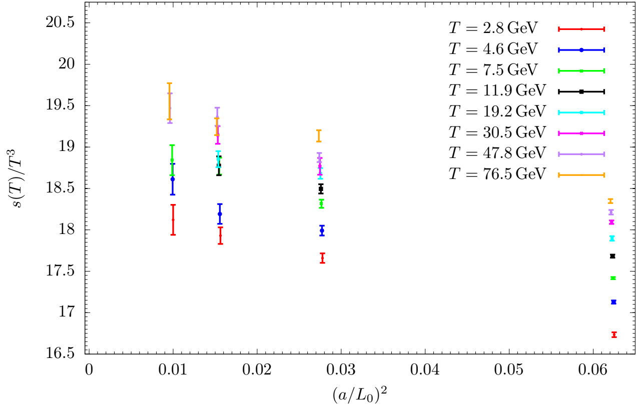
for a range of values of (g_0, θ) to obtain $\frac{\partial}{\partial \xi_k} \widehat{f}(L_0, \boldsymbol{\xi}, \theta) \Big|_{g_0}$

Initial condition

- For small enough g_0^*, F_* can be estimated in PT (Giusti, Pepe '15)
- Else F_* could be computed explicitly (not easy!)

Towards the EoS at high temperature

Some preliminary results. Perturbative Z's have been used for illustration!
Vertical scale should not be taken at face value!



Remarks

- $N_{\rm ms} = 50, 100, 250, 450$ for $L_0/a = 4, 6, 8, 10$
- $\mathrm{Var}(s(T))/s(T)^2 \propto (L_0/a)^8$; $\tau_{\rm int} \lesssim 2\,\mathrm{MDUs}$
- About 1% error for $L_0/a = 10$
- Small discretization errors?

Conclusions & Outlook

Conclusions

- QCD in a moving frame is a **powerful** framework for thermodynamic studies
- Offers alternative ways for computing thermodynamic quantities
- Provides many Ward identities for the non-perturbative **renormalization** of the EMT
- The framework passes with flying colours an analysis at 1-loop order in PT
- Preliminary results for the bare entropy are **encouraging**

Outlook

- The non-perturbative renormalization of the EMT is on its way
- Accurate determination of the EoS of $N_{\mathrm{f}} = 3$ QCD in a totally **unexplored** temperature range
- How accurate is PT in this regime?
- Heavy-quark effects?


Thanks