On the spectral theory of Painlevé kernels

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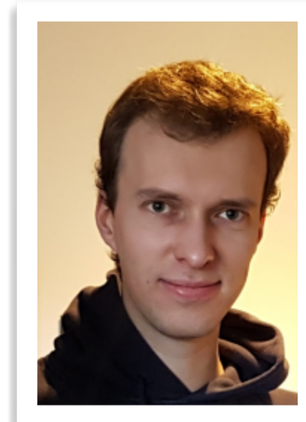
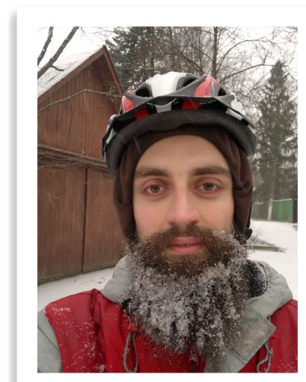
Mostly based on:

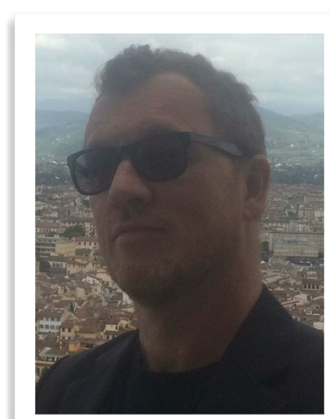
ArXiv **2310.09262**
with M.François

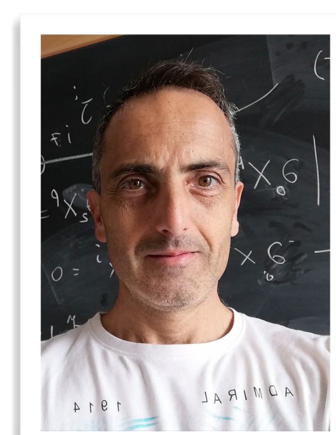


ArXiv **2304.11027** with
P.Gavrylenko & Q.Hao



ArXiv **1603.01174** with
G.Bonelli & A.Tanzini





Introduction

String theory:

- originally motivated by the goal of unifying the theory of general relativity with quantum mechanics

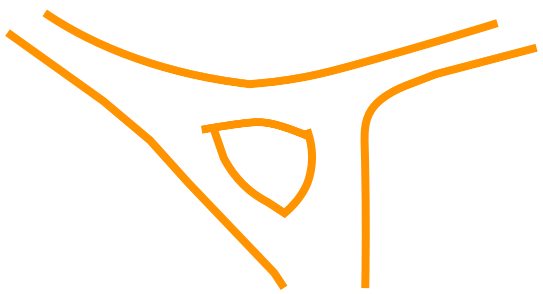
- has generated several tools leading to amazing new results and applications in various fields. In particular in mathematical physics

These new results are often a consequence of string dualities: highly non-trivial relations which arise within a string theory context

Introduction

Example: Mirror symmetry *[Candelas-de la Ossa-Green-Parkes, Chiang-Klemm-Yau-Zaslow, ….]*

manifold X $\longleftrightarrow$ manifold $\widetilde{X}$



Canonical bundle over $\mathbb{CP}^2$

(local $\mathbb{CP}^2$)

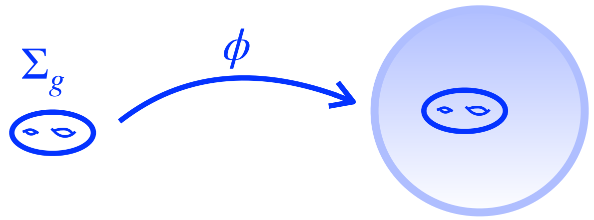
Underlying intuition: string propagation in both spaces is identical.

→ Application: difficult computations on one manifold can be mapped into simpler problems on its mirror partner.

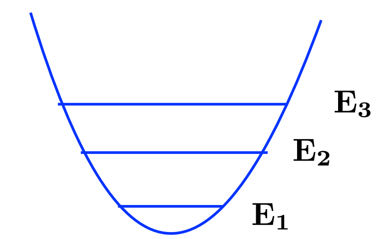
Introduction

Example: the TS/ST correspondence [AG-Hatsuda-Mariño, …..]

topological string theory
on toric CY manifolds



spectral theory of QM
operators on the real line



- concrete toy model to study non-perturbative effects in string theory and the corresponding notion of quantum geometry
- surprising relation between enumerative geometry and spectral theory

Summary

There have been many applications and tests of the TS/ST correspondence but eventually we would like to give a mathematical proof of it.

In a series of papers we have unveiled the role played by isomonodromic deformation equations toward a proof of this duality.

[Bonelli-AG-Tanzini 2016-2017, Gavrylenko-Hao-AG 2023, WIP]

In addition, this led to new results in the spectral theory of Painlevé kernels and in the study of four dimensional $\mathcal{N} = 2$ theories at strong coupling.

[François-AG 2023]

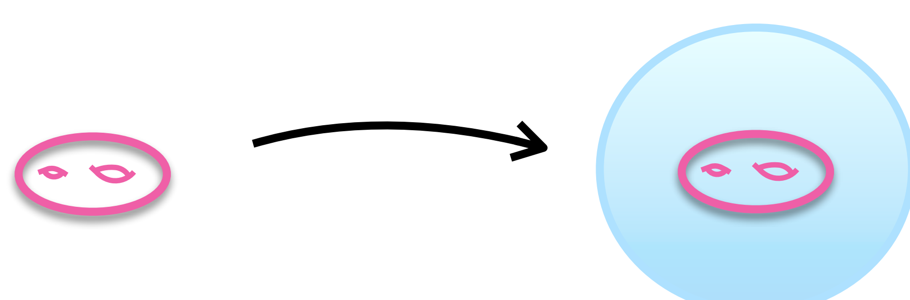
Topological string theory

Topological string theory is a simplified model of string theory which we often use in mathematical applications. The free energies of this model F_g encode in a precise way the enumerative geometry of the target space X

$$F_g(t) = \sum_{d \geq 1} N_g^d \, \mathrm{e}^{-dt}$$

N_g^d are the Gromov-Witten (GW) invariants: “count” holomorphic maps

$$\phi : \Sigma_g \to X$$



t: Kähler parameter of X

Special functions

An interesting aspect of topological string theory is that it geometrically engineers $\mathcal{N} = 2$ supersymmetric gauge theories in four dimension. This interplay led to the discovery of new classes of special functions that today permeate many areas of theoretical and mathematical physics: from general relativity to integrable systems.

Examples:

- Gopakumar-Vafa functions
- Nekrasov functions
- Nekrasov-Shatashvili functions

$\cdots$

Special functions

Example 1: the Nekrasov partition function of 4-dim SU(2) $\mathcal{N} = 2$ SYM at $\epsilon_1 + \epsilon_2 = 0$

$$Z(\sigma, t) = t^{\sigma^2} \frac{\mathscr{B}(\sigma, t)}{G(1 + 2\sigma)G(1 - 2\sigma)}$$

where $\mathscr{B}(\sigma, t) = 1 + \sum_{n \geq 1} c_n(\sigma) t^n$ with $c_1(\sigma) = \frac{1}{2\sigma^2}$, $c_2(\sigma) = \frac{8\sigma^2 + 1}{4\sigma^2(4\sigma^2 - 1)}$, $\ldots$

Comments:

- Let $2\sigma \neq \mathbb{Z}$, then $\mathscr{B}(\sigma, t)$ is convergent *[Its-Lisovyy-Tykhyy 2014]*
- Physically $t \sim \mathrm{e}^{-1/g^2_{YM}}$ is the instanton counting parameter in SYM whereas $\sigma \sim$ vev of scalar in the vector multiplet
- ϵ_1, ϵ_2 the Ω background parameters. We can set without loss of generality $\epsilon_1 = 1 = -\epsilon_2$
- Via the AGT correspondence this makes contact with Liouville CFT at $c = 1$

Special functions

Example 2: the Nekrasov-Shatashvili free energy the for 4-dim SU(2) $\mathcal{N} = 2$ SYM

$$F^{\mathrm{NS}}(t,\sigma) = -\psi^{(-2)}(1-2\sigma) - \psi^{(-2)}(1+2\sigma) + \sigma^2 \log(t) + \sum_{n\geq 1} t^n b_n(\sigma)$$

where $$b_1(\sigma) = \frac{2}{4\sigma^2 - 1}, \quad b_2(\sigma) = \frac{\left(20\sigma^2 + 7\right)}{\left(4\sigma^2 - 1\right)^3 \left(4\sigma^2 - 4\right)}, \quad \cdots$$

$\psi^{(-2)}(x)$: polygamma function of order -2

Comments:
- $\sum_{n\geq 1} t^n b_n(\sigma)$ is convergent
- This corresponds to setting $\epsilon_2 = 0,\ \epsilon_1 = 1$ in the Ω background
- Via AGT this makes contact with Liouville theory at $c \to \infty$

Special functions

Example 3: the Nekrasov partition function the for 4-dim SU(2) $\mathcal{N} = 2$ SYM at $\epsilon_1 + \epsilon_2 = 0$ in presence of a half-BPS surface defect

$$Z^D(y,t,\sigma) = \Gamma\left(-\mathrm{i}y - \sigma + \frac{1}{2}\right)\Gamma\left(-\mathrm{i}y + \sigma + \frac{1}{2}\right)Z(t,\sigma)\left(1 + t\frac{1+2iy}{4\sigma^4 + (-2y+i)^2\sigma^2} + \mathcal{O}(t^2)\right)$$

[Drukker et al, Alday et al, Gaiotto et al, Kozlowski et al,Gorsky et al , Bullimore et al, Sciarappa, Jeong et al..]

Comments:

- This partition function can be realized via 4d/2d systems
- In AGT, the Fourier transform $\int \mathrm{e}^{\mathrm{i}2xy} Z^D(y,t,\sigma)$ has a natural interpretation in Liouville CFT @ $c = 1$ as 5-point of four primaries with a degenerate field insertion (the $\Phi_{2,1}(y)$ field)

Special functions and spectral theory

Today we will see that these special functions play a central role in the in spectral theory of a certain class of quantum mechanical operators

Spectral theory is a branch of mathematics that is concerned with the study of the spectrum of operators, for example by finding their eigenvalues and eigenfunctions.

Example: Schrödinger operators

$$\left(-\hbar^2\partial_x^2 + V(x)\right)\varphi(x) = E\varphi(x)$$

Exact analytic solutions for spectral theory are rare

➡ need new ideas and new tools

Special functions and spectral theory

Fruitful guideline: think of spectral theory geometrically. *[Balian, Parisi, Voros]*

Geometry allows us to make contact with supersymmetric gauge theory and topological string theory

➡ new class of solvable spectral problems

Special functions and spectral theory

Today we will see that these special functions play a central role in the in spectral theory of a certain class of quantum mechanical operators

I. Quantum mirror curves of toric Calabi-Yau manifolds (brief)

Exact solutions via special functions starting from 2014

[AG-Hatsuda-Mariño 2014+…..]

II. Four dimensional quantum Seiberg-Witten curves (brief)

Exact solutions via special functions starting from 2009

[Nekrasov-Shatashvili 2009+…..]

III. Painlevé kernels

New

Special functions and spectral theory

To maintain concreteness and simplicity, today we will mainly focus on one illustrative example

I. Quantum mirror curves of **local** $\mathbb{CP}_1 \times \mathbb{CP}_1$

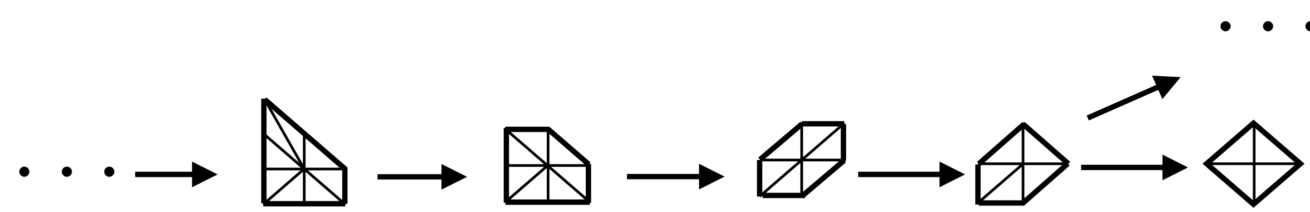
II. Four dimensional quantum Seiberg-Witten curve of SU(2) $\mathcal{N} = 2$ SYM

III. **Painlevé** III_3 kernels

Special functions and spectral theory

However is good to keep in mind that what we will discuss is part of a broader picture

I. Quantum mirror curves of toric CYs



II. Four dimensional quantum SW curves

$$\mathcal{N}_f = 4 \longrightarrow \mathcal{N}_f = 3 \longrightarrow N_f = 2 \longrightarrow \mathcal{N}_f = 1 \longrightarrow \text{SYM}$$

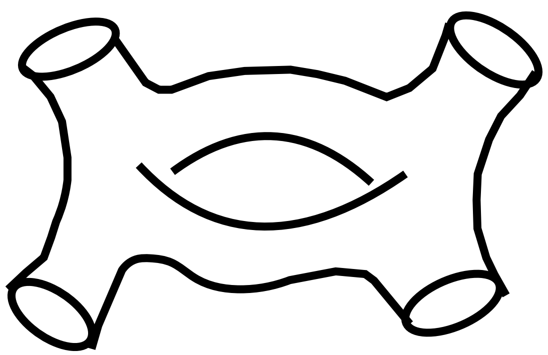
III. Painlevé kernels

$$\mathrm{VI} \longrightarrow \mathrm{V} \longrightarrow \mathrm{III}_1 \longrightarrow \mathrm{III}_2 \longrightarrow \mathrm{III}_3$$

Quantum mirror curves

An example: local $\mathbb{P}_1 \times \mathbb{P}_1$

Thanks to mirror symmetry we can relate such geometry to mirror curve



$$m\left(\mathrm{e}^{x}+\mathrm{e}^{-x}\right)+\mathrm{e}^{p}+\mathrm{e}^{-p}+\kappa=0$$

$x, p \in \mathbb{C}$

m, κ complex moduli

[Batyrev, Hori-Vafa, Katz-Klemm-Vafa,…]

The quantization of this curve leads to $\mathcal{O}=m\left(\mathrm{e}^{\hat{x}}+\mathrm{e}^{-\hat{x}}\right)+\mathrm{e}^{\hat{p}}+\mathrm{e}^{-\hat{p}}$ $\quad [\hat{\mathrm{x}}, \hat{\mathrm{p}}]=\mathrm{i}\hbar$

where $\mathrm{e}^{\hat{\mathrm{p}}}\phi(x)=\phi(x-\mathrm{i}\hbar)$ and $\mathrm{e}^{\hat{\mathrm{x}}}\phi(x)=\mathrm{e}^{x}\phi(x)$ *[Dijkgraaf et al, Mironov-Morozov. . .].*

Terminology: $\mathcal{O}$ is the quantum mirror curve to local $\mathbb{P}_1 \times \mathbb{P}_1$

Quantum mirror curves

$$\mathrm{Tr}\rho^N = \sum_{n\geq 0} E_n^{-N} < \infty$$

Theorem: The operator $\rho = \mathcal{O}^{-1}$ has a discrete spectrum $\{E_n^{-1}\}_{n\geq 0}$ and it is of trace class on $L^2(\mathbb{R})$ *[AG-Hatsuda-Mariño, Kashaev-Mariño, Laptev-Schwimmer-Takhtajan]*

The kernel of ρ is

[Kashaev-Mariño]

$$\rho(x,y) = \frac{\mathrm{e}^{-u(x,m,b)-u(y,m,b)}}{\cosh\left(\frac{x-y}{2}\right)} \qquad \hbar = \pi b^2$$

where the potential is $u(x,m,b) = -\frac{xb^2}{4} - \log\left|\frac{\phi_{\mathrm{b}}\left(\frac{bx}{2\pi} + \frac{1}{2\pi b}\log m + ib/4\right)}{\phi_{\mathrm{b}}\left(\frac{bx}{2\pi} - \frac{1}{2\pi b}\log m - ib/4\right)}\right| + \frac{1}{8b^2}\log m$

and ϕ_{b} is the Faddeev quantum dilogarithm

If Im(b)>0 it reduces to $\Phi_b(x) = \frac{(\mathrm{e}^{2\pi \mathrm{b}(x+c_b)}, \mathrm{e}^{2i\pi b^2})_\infty}{(\mathrm{e}^{2\pi \mathrm{b}^{-1}(x+c_b)}, \mathrm{e}^{-2i\pi b^{-2}})_\infty}$ $\qquad 2c_b = i(b+b^{-1})$

Quantum mirror curves

Theorem: The operator $\rho = \mathcal{O}^{-1}$ has a discrete spectrum $\{E_n^{-1}\}_{n\geq 0}$ and it is of trace class on $L^2(\mathbb{R})$

Can we compute spectral quantities such as eigenvalues, Fredholm determinant, eigenfunctions explicitly?

Yes, by using (refined) topological strings partition functions.

Let us start with the spectrum. A convenient way to encode the spectrum is via the Fredholm determinant

$$\det\left(1+\kappa\rho\right) = \prod_{n\geq 0}\left(1+\frac{\kappa}{E_n}\right)$$

Topological String and Spectral Theory

Claim: let $\rho = \mathcal{O}^{-1}$ the (inverse) quantum mirror curve to local $\mathbb{P}_1 \times \mathbb{P}_1$. Then

$$\det\left(1+\kappa\rho\right) = \underbrace{\sum_{n\in\mathbb{Z}} \exp\left[\mathsf{J}\left(t_1+2\pi i n, t_2, \hbar\right)\right]}_{\text{generalized theta function}}$$

[AG-Hatsuda-Mariño]

where: $\mathsf{J}(t_1, t_2, \hbar) = \sum_{i=1}^{2} \frac{t_i}{2\pi}\frac{\partial}{\partial t_i} F_{\rm NS}(t_1, t_2, \hbar) + \frac{\hbar^2}{2\pi}\frac{\partial}{\partial \hbar}\left(\frac{F_{\rm NS}(t_1, t_2, \hbar)}{\hbar}\right) + F_{\rm GV}\left(\frac{2\pi}{\hbar}t_1, \frac{2\pi}{\hbar}t_2, \frac{4\pi^2}{\hbar}\right)$

$F_{\rm GV}$: Goparkumar-Vafa function of local $\mathbb{P}_1 \times \mathbb{P}_1$

$F_{\rm NS}$: Nekrasov-Shatashvili function of local $\mathbb{P}_1 \times \mathbb{P}_1$

examples of topological string functions

$t_1 \equiv t_1(\kappa, \hbar, m)$ quantum mirror map

$t_2 \equiv 2\log m$

Topological String and Spectral Theory

$$\det\left(1+\kappa\rho\right)=\sum_{n\in\mathbb{Z}}\exp\left[\mathsf{J}\left(t_1+2\pi\mathrm{i}n,t_2,\hbar\right)\right]$$

This result has many practical applications and conceptual consequences. For example it gives a new interpretation of GW invariants from the point of view of spectral theory as well as a non-perturbative definition of the topological string partition function from quantum mirror curves.

$$Z(N,m,\hbar)=\oint \mathrm{d}\kappa\,\det(1+\kappa\rho)\kappa^{-1-N}$$

$$\log Z(N,m,\hbar)\ \xrightarrow[t_1=\frac{N}{\hbar}\,,\ t_2=\frac{\log m}{\hbar}\quad \text{fixed}]{\hbar\ ,\ N\ ,\ m\to\infty}\ \sum_{g\geq 0} F_g(t_1,t_2)\ \hbar^{2-2g}$$

F_g : genus g free energy of topological string on local $\mathbb{P}_1\times\mathbb{P}_1$

Topological String and Spectral Theory

$$\det\left(1+\kappa\rho\right) = \sum_{n\in\mathbb{Z}} \exp\left[\mathsf{J}\left(t_1 + 2\pi \mathrm{i} n, t_2, \hbar\right)\right] \qquad (\star)$$

This result has many practical applications and conceptual consequences.

For example it gives a new interpretation of GW invariants from the point of view of spectral theory: they emerge from the spectral traces in the limit $\hbar \to \infty$.

Can we prove $(\star)$?

First limit

Let us start by looking at some simple but non-trivial limit.

[Katz-Klemm-Vafa, Klemm-Lerche-Mayr-Vafa -Warner, Iqbal-Kashani Poor,..]

Limit 1 (standard four dimensional limit)

Set $t_1 = \beta 2\pi\sigma$

$$t_2 = -\log \beta^4 t$$

$$\hbar = \beta$$

mirror map $\longleftrightarrow$

$$m^{-1} = \sqrt{t}\beta^2$$

$$\kappa = -\frac{2}{\sqrt{t}\beta^2} + \frac{E(\sigma)}{\sqrt{t}}$$

and send $\beta \to 0$. In this limit we have

$$\left(m\left(\mathrm{e}^{\hat{x}} + \mathrm{e}^{-\hat{x}}\right) + \mathrm{e}^{\hat{p}} + \mathrm{e}^{-\hat{p}} + \kappa\right)\psi(x) = 0 \quad \xrightarrow{\beta \to 0} \quad \left(-\partial_x^2 + \sqrt{t}\cosh(x) + E\right)\varphi(x) = 0$$

quantum mirror curve to local $\mathbb{P}_1 \times \mathbb{P}_1$

four dimensional quantum Seiberg-Witten curve of SU(2) SYM

First limit

$$\det\left(1+\kappa\rho\right) = \sum_{n\in\mathbb{Z}} \exp\left[\mathrm{J}\left(t_1 + 2\pi i n, t_2, \hbar\right)\right]$$

Limit 1

AG-Gu-Mariño

$$\det\left(1+E\,\mathrm{O}_{\mathrm{Ma}}^{-1}\right) = \frac{\sinh\left(\partial_\sigma F^{\mathrm{NS}}(t,\sigma)\right)}{\sin\left(2\pi\sigma\right)}$$

$$E = -\,t\partial_t F^{\mathrm{NS}}\left(t,\sigma\right)$$

Matone relation

$$\mathrm{O}_{\mathrm{Ma}}\;\varphi(x) = \left(-\partial_x^2 + \sqrt{t}\cosh\left(x\right)\right)\varphi(x)$$

F_{NS}: Nekrasov-Shatashvili functions for 4-dim SU(2) $\mathcal{N}=2$ SYM

In this limit we make contact with the work of Nekrasov and Shatashvili relating the modified Mathieu operator to 4d $\mathcal{N}=2$ SYM in the NS phase of the Ω background. Many aspect of this have been studied in details and proven.

Many Many References

First limit

The energy spectrum is determined by the zeroes of the determinant

$$\det\left(1 + E\,\mathrm{O}_{\mathrm{Ma}}^{-1}\right) = \frac{\sinh\left(\partial_\sigma F^{\mathrm{NS}}(t,\sigma)\right)}{\sin(2\pi\sigma)}$$

$$E = -\,t\partial_t F^{\mathrm{NS}}(t,\sigma)$$

Matone relation

$$\longrightarrow \quad E_n = -\,t\partial_t F^{\mathrm{NS}}\left(t,\sigma_n\right)$$

where σ_n is determined by

$$\partial_\sigma F^{\mathrm{NS}}(t,\sigma) = \mathrm{i}\pi n, \quad n = 0,1,2,\cdots$$

Nekrasov-Shatashvili, Kozlowski-Teschner, …

Second limit

In this first limit 1 we make contact with well known results.

There is another scaling limit which we can take obtain new results.

$$\mathrm{J}(t_1, t_2, \hbar) = \sum_{i=1}^{2} \frac{t_i}{2\pi} \frac{\partial}{\partial t_i} F_{\mathrm{NS}}(t_1, t_2, \hbar) + \frac{\hbar^2}{2\pi} \frac{\partial}{\partial \hbar} \left(\frac{F_{\mathrm{NS}}(t_1, t_2, \hbar)}{\hbar} \right) + F_{\mathrm{GV}} \left(\frac{2\pi}{\hbar} t_1, \frac{2\pi}{\hbar} t_2, \frac{4\pi^2}{\hbar} \right)$$

Limit 2:

Set

$$\frac{2\pi t_1}{\hbar} = \beta 2\pi\sigma \qquad \qquad \kappa = \log\left(1 + \mathrm{e}^{4\pi \mathrm{i}\sigma}\right) - \frac{4\pi}{\beta} \log\left(\beta^4 t\right)$$

$$\frac{2\pi t_2}{\hbar} = -\log \beta^4 t \qquad \xleftrightarrow{\text{mirror map}} \qquad \log m = -\frac{2\pi}{\beta} \log \beta^4 t$$

$$\frac{4\pi^2}{\hbar} = \beta$$

And sent $\beta \to 0$.

Second limit

$$\det\left(1+\kappa\rho\right) = \sum_{n\in\mathbb{Z}} \exp\left[\mathsf{J}\left(t_1+2\pi \mathrm{i} n, t_2, \hbar\right)\right]$$

$$\Big\downarrow \text{ Limit 2}$$

[Bonelli-AG-Tanzini]

$$\boxed{\det\left(1+\frac{\cos(2\pi\sigma)}{2\pi}K\right) = \mathrm{e}^{4\sqrt{t}}t^{-1/16}\sum_{k\in\mathbb{Z}} Z(\sigma+k,t)} \qquad (\star)$$

$$K(x,y) = \frac{\mathrm{e}^{-t^{1/4}\cosh x - t^{1/4}\cosh y}}{2\cosh\left(\frac{x-y}{2}\right)}$$

$$Z(\sigma,t) = t^{\sigma^2}\frac{\mathscr{B}(\sigma,t)}{G(1+2\sigma)G(1-2\sigma)}$$

Nekrasov partition function of 4-dim SU(2) $\mathcal{N}=2$ SYM at $\epsilon_1+\epsilon_2=0$

We can prove (⋆) using an underlying connection with the theory of Painlevé equations

Second limit

$$\det\left(1+\frac{\cos(2\pi\sigma)}{2\pi}K\right) = \mathrm{e}^{4\sqrt{t}}t^{-1/16}\sum_{k\in\mathbb{Z}} Z(\sigma+k,t)$$

By combining results from the '70s [McCoy et al., Widom] with more recent developments [Gamayun-Iorgov-Lisovyy, Iorgov-Lisovyy-Teschner, Bershtein-Shchechkin], we can prove that both sides solve the Painlevé III_3 equation with a particular choice of initial conditions. *[Bonelli-AG-Tanzini]*

This is one specific example (the starting point is local $\mathbb{P}_1 \times \mathbb{P}_1$) but it can be generalized to other geometries. For instance the so called $Y^{N,0}$ geometries that make contact with SU(N) gauge theories and non-autonomous Toda equations. *[Gavrylenko-AG-Hao]*

It would be great to prove the TS/ST in its full generality without taking any limit, but for this more is needed

[Bonelli-AG-Tanzini, Gavrylenko-AG-Hao + work in progress]

Painlevé kernels and strong coupling

From the point of view of gauge theory & Nekrasov functions the identity

$$\det\left(1+\frac{\cos(2\pi\sigma)}{2\pi}K\right) = e^{4\sqrt{t}}t^{-1/16}\sum_{k\in\mathbb{Z}} Z(\sigma+k,t) \qquad (\star)$$

is interesting because the lhs is exact in t (the instant counting parameter in Nekrasov function) and it can be used to extract the strong coupling version of Nekrasov function. We can write the determinant as:

$$\det\left(1+\frac{\cos(2\pi\sigma)}{2\pi}K\right) = \sum_{N\geq 0}\left(\frac{\cos(2\pi\sigma)}{2\pi}\right)^N Z(N,t)$$

where

$$Z(N,t) = \frac{1}{N!}\int_{\mathbb{R}}\prod_{i=1}^{N} dx_i e^{-8t^{1/4}\cosh x_i}\prod_{j<i}\tanh^2\left(\frac{x_i-x_j}{2}\right)$$

Painlevé kernels and strong coupling

By expanding the matrix model at large t we get

$$Z(N,t) = t^{N^2/8} G\,(1+N)\left(1 + \sum_{\ell \geq 1} \frac{D_\ell(N)}{\left(t^{1/4}\right)^{\ell}}\right)$$

where D_ℓ is a polynomial of degree 2ℓ. This is the analogous of Nekrasov function at strong coupling upon identification

$$\frac{a_D}{\epsilon} \sim N \qquad \frac{\Lambda}{\epsilon} \sim t^{1/4}$$

In the small ϵ expansion the first few coefficients agree with [Klemm et al, D'Hoker-Phong,..]

[Bonelli-AG-Tanzini, Gavrylenko-Marshakov-Stoyan]

Rmk: analogous results hold for higher rank gauge theories, in such case we have a multi-cut matrix model *[Gavrylenko-AG-Hao]*

Spectral theory of Painlevé kernels

Let us now look at the spectral theory of the Painlevé III_3 kernel

$$K(x,y) = \frac{\mathrm{e}^{-t^{1/4}\cosh x - t^{1/4}\cosh y}}{4\pi \cosh\left(\frac{x-y}{2}\right)}$$

K is of trace class on $L^2(\mathbb{R})$. It has a discrete spectrum with square integrable eigenfunctions

$$\int_{\mathbb{R}} \mathrm{d}y\ K\left(x,y\right)\varphi_n\left(y,t\right) = E_n \varphi_n\left(x,t\right)$$

Can we compute spectrum and eigenfunctions explicitly using Nekrasov functions?

Yes: the relevant Nekrasov functions will be specialized to the $\epsilon_1 = -\epsilon_2$ phase of the Ω background (i.e. $c=1$ form the Liouville point of view).

Spectral theory of Painlevé kernels

Let us start from the spectrum. The Fredholm determinant identity

$$\det\left(1+\frac{\cos(2\pi\sigma)}{2\pi}K\right)=\mathrm{e}^{4\sqrt{t}}t^{-1/16}\sum_{k\in\mathbb{Z}}Z(\sigma+k,t)$$

implies that the spectrum of K is

$$E_n^{-1}=-\frac{1}{2\pi}\cos\left(2\pi\left(\frac{1}{2}+\mathrm{i}\sigma_n\right)\right)$$

where $\sigma_n\in\mathbb{R}$ are solutions to

$$\sum_{k\in\mathbb{Z}}Z\left(t,k+\frac{1}{2}+\mathrm{i}\sigma_n\right)=0$$

Nekrasov partition function of 4-dim SU(2) $\mathcal{N}=2$ SYM at $\epsilon_1+\epsilon_2=0$

[Bonelli-AG-Tanzini]

Spectral theory of Painlevé kernels

$$K(x,y) = \frac{\mathrm{e}^{-t^{1/4}\cosh x - t^{1/4}\cosh y}}{4\pi\cosh\left(\frac{x-y}{2}\right)}$$

$$\mathrm{O_M} = \left(-\partial_x^2 + \sqrt{t}\cosh(x)\right)$$

$$\int_{\mathbb{R}} \mathrm{d}y\ K(x,y)\,\varphi_n(y,t) = E_n^{(K)}\varphi_n(x,t)$$

$$\mathrm{O_M}\varphi_n(x) = E_n^{(M)}\varphi_n(x)$$

$$E_n^{-1} = -\frac{1}{2\pi}\cos\left(2\pi\left(\frac{1}{2}+\mathrm{i}\sigma_n\right)\right)$$

$$E_n = -\,t\partial_t F^{\mathrm{NS}}(t,\sigma_n)$$

$$\sum_{k\in\mathbb{Z}} Z\left(t, k+\frac{1}{2}+\mathrm{i}\sigma_n\right) = 0 \quad \longleftrightarrow \quad \sinh\left(\partial_\sigma F^{\mathrm{NS}}(t,\sigma_n)\right) = 0$$

1-1 correspondence
via blowup equations

[Huang-Sun-Wang 2016, AG-Gu 2016]

Spectral theory of Painlevé kernels

What about eigenfunctions of $K(x,y)$? We found that these are computed by the Nekrasov partition function of 4-dim SU(2) $\mathcal{N}=2$ SYM at $\epsilon_1+\epsilon_2=0$ in presence of a surface defects.

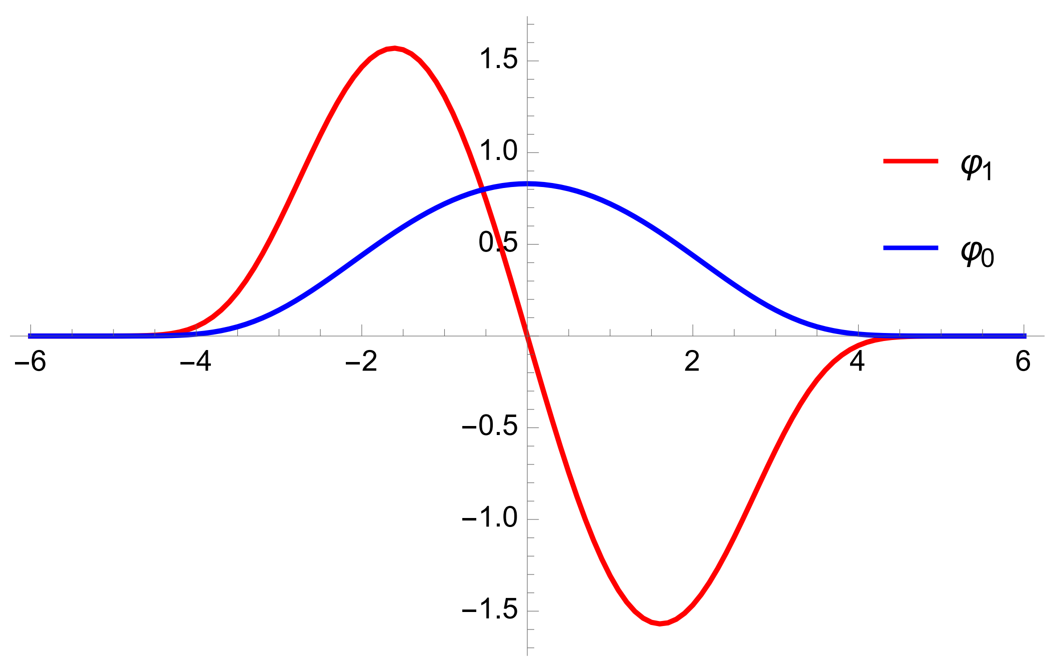
Recall that

$$Z^D(y,t,\sigma)=\Gamma\left(-\mathrm{i}y-\sigma+\frac{1}{2}\right)\Gamma\left(-\mathrm{i}y+\sigma+\frac{1}{2}\right)\left(1+t\frac{1+2iy}{4\sigma^4+(-2y+i)^2\sigma^2}+\mathcal{O}(t^2)\right)$$

Spectral theory of Painlevé kernels

More precisely we find that the eigenfunctions are

$$\varphi_n(x,t) = \frac{e^{4\sqrt{t}}}{t^{3/16}} \int_{\mathbb{R}} dy\ e^{i2yx} \sum_{k\in\mathbb{Z}} \left(Z^{D}\left(y, t, k + \frac{1}{2} + i\sigma_n\right) + Z^{D}\left(-y - \frac{i}{2}, t, k + i\sigma_n\right)\right)$$



[François-AG]

Spectral theory of Painlevé kernels

One can write explicit expression using matrix models *[François-AG]*

$$\int_{\mathbb{R}+i\sigma_*} d\sigma \frac{\tan(2\pi\sigma)}{\left(2\cos(2\pi\sigma)\right)^N} \int_{\mathbb{R}} dy e^{i2yx} \left(Z^{\mathrm{D}}\left(y,t,\sigma\right) + Z^{\mathrm{D}}\left(-y-\frac{i}{2}, t, \sigma+\frac{1}{2}\right)\right)$$

$$= i\frac{t^{3/16}}{e^{4\sqrt{t}}} e^{-4t^{1/4}\cosh x} e^{x/2} \Psi_N\left(e^x, t\right)$$

$$\Psi_N(z,t) = \frac{1}{N!}\int_{\mathbb{R}_+} \prod_{i=1}^{N} \frac{dz_i}{z_i} \frac{z - z_i}{z + z_i} e^{-4t^{1/4}\left(z_i + z_i^{-1}\right)} \prod_{j<i} \left(\frac{z_i - z_j}{z_i + z_j}\right)^2$$

→ as before, such expression can be used to extract the strong coupling version of Nekrasov function with a surface defect

Outlook & Future Directions

Mathematics and physics have a long history of cross-fertilization, and string dualities currently play a major role in this interaction

Today we focus on example of string duality: TS/ST correspondence.

While working towards a proof, we discovered an intriguing connection to the theory of Painlevé equations. This connection has interesting consequences both on the Painlevé and the gauge/string theory sides.

Example:

- New results for the spectral theory of Painlevé kernels
- New results for 4 dim $\mathcal{N} = 2$ theories at strong coupling

Outlook & Future Directions

Some questions directly related to what we discussed

- Painlevé kernels as for dimensional quantum curves with a different quantization scheme?
- Connection with topological recursion?
- $\det\left(1+\frac{\cos(2\pi\sigma)}{2\pi}K\right)$ is the tau function of Painlevé III_3. What is the role of the eigenfunctions of K in the context of Painlevé equations? Relation with the linear system?
- Combinatorial formula at strong coupling?

…

Thank you!