

# Low- $x$ evolution of parton densities

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It is shown that a Bessel-like behaviour of the structure function  $F_2$  at small  $x$ , obtained for a flat initial condition in the DGLAP evolution equations, leads to good agreement with the deep inelastic scattering experimental data from HERA.

## 1 Introduction

The fairly reasonable agreement between HERA data [1]-[4] and the next-to-leading-order (NLO) approximation of perturbative QCD has been observed for  $Q^2 \geq 2 \text{ GeV}^2$  (see reviews in [5] and references therein) and, thus, perturbative QCD can describe the evolution of  $F_2$  and its derivatives down to very low  $Q^2$  values.

The standard program to study the  $x$  behaviour of quarks and gluons is carried out comparing the experimental data with the numerical solution of the DGLAP equations [6] by fitting the QCD energy scale  $\Lambda$  and the parameters of the  $x$ -profile of partons at some initial  $Q_0^2$  [7, 8]. However, to investigate exclusively the small- $x$  region, there is the alternative of doing the simpler analysis by using some of the existing analytical solutions of DGLAP in the small- $x$  limit [9]-[12]. It was pointed out in [9] that the HERA small- $x$  data can be well interpreted in terms of the so-called doubled asymptotic scaling (DAS) phenomenon related to the asymptotic behaviour of the DGLAP evolution discovered many years ago [13].

The study of [9] was extended in [10]-[12] to include the finite parts of anomalous dimensions (ADs) of Wilson operators and Wilson coefficients<sup>1</sup>. This has led to predictions [11, 12] of the small- $x$  asymptotic form of parton distribution functions (PDFs) in the framework of the DGLAP dynamics, which were obtained starting at some  $Q_0^2$  with the flat function

$$f_a(Q_0^2) = A_a \quad (\text{hereafter } a = q, g), \quad (1)$$

where  $f_a$  are PDFs multiplied by  $x$  and  $A_a$  are unknown parameters to be determined from the data.

We refer to the approach of [10]-[12] as *generalized* DAS approximation. In this approach the flat initial conditions, Eq. (1), determine the basic role of the AD singular parts as in the standard DAS case, while the contribution from AD finite parts and from Wilson coefficients can be considered as corrections which are, however, important for better agreement with experimental data.

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<sup>1</sup>In the standard DAS approximation [13] only the AD singular parts were used.

The use of the flat initial condition, given in Eq. (1), is supported by the actual experimental situation: low- $Q^2$  data [15, 16, 3] are well described for  $Q^2 \leq 0.4 \text{ GeV}^2$  by Regge theory with Pomeron intercept  $\alpha_P(0) \equiv \lambda_P + 1 = 1.08$ , close to the adopted ( $\alpha_P(0) = 1$ ) one. The small rise of HERA data [1, 4, 16, 17] at low  $Q^2$  can be explained, for example, by contributions of higher twist operators (see [12]).

The purpose of this paper is to demonstrate a good agreement [14] between the predictions of the generalized DAS approach [11] and the HERA experimental data [1] (see Fig. 1 below) for the structure function (SF)  $F_2$ . We also compare the result of the slope  $\partial \ln F_2 / \partial \ln(1/x)$  calculation with the H1 and ZEUS data [2, 3]. Looking at the H1 data [2] points shown in Fig. 2 one can conclude that  $\lambda(Q^2)$  is independent on  $x$  within the experimental uncertainties for fixed  $Q^2$  in the range  $x < 0.01$ . The rise of  $\lambda(Q^2)$  linearly with  $\ln Q^2$  could be traced in strong non-perturbative way (see [18] and references therein), i.e.,  $\lambda(Q^2) \sim 1/\alpha_s(Q^2)$ . The analysis [19], however, demonstrated that this rise can be explained naturally in the framework of perturbative QCD.

The ZEUS and H1 Collaborations have also presented [3] the preliminary data for  $\lambda(Q^2)$  at quite low values of  $Q^2$ . The ZEUS value for  $\lambda(Q^2)$  is consistent with a constant  $\sim 0.1$  at  $Q^2 < 0.6 \text{ GeV}^2$ , as it is expected under the assumption of single soft Pomeron exchange within the framework of Regge phenomenology. It was important to extend the analysis of [19] to low  $Q^2$  range with a help of well-known infrared modifications of the strong coupling constant. We used the ‘‘frozen’’ and analytic versions (see [14]).

## 2 Generalized DAS approach

The flat initial condition (1) corresponds to the case when PDFs tend to some constant value at  $x \rightarrow 0$  and at some initial value  $Q_0^2$ . The main ingredients of the results [11, 12], are:

- Both the gluon and quark singlet densities<sup>2</sup> are presented in terms of two components (‘‘+’’ and ‘‘-’’) which are obtained from the analytic  $Q^2$ -dependent expressions of the corresponding (‘‘+’’ and ‘‘-’’) PDF moments.
- The twist-two part of the ‘‘-’’ component is constant at small  $x$  at any values of  $Q^2$ , whereas the one of the ‘‘+’’ component grows at  $Q^2 \geq Q_0^2$  as

$$\sim e^\sigma, \quad \sigma = 2\sqrt{\left[\hat{d}_+ s - \left(\hat{D}_+ + \hat{d}_+ \frac{\beta_1}{\beta_0}\right) p\right] \ln\left(\frac{1}{x}\right)}, \quad \rho = \frac{\sigma}{2 \ln(1/x)}, \quad (2)$$

where  $\sigma$  and  $\rho$  are the generalized Ball–Forte variables,

$$s = \ln\left(\frac{a_s(Q_0^2)}{a_s(Q^2)}\right), \quad p = a_s(Q_0^2) - a_s(Q^2), \quad \hat{d}_+ = \frac{12}{\beta_0}, \quad \hat{D}_+ = \frac{412}{27\beta_0}. \quad (3)$$

Hereafter we use the notation  $a_s = \alpha_s/(4\pi)$ . The first two coefficients of the QCD  $\beta$ -function in the  $\overline{\text{MS}}$ -scheme are  $\beta_0 = 11 - (2/3)f$  and  $\beta_1 = 102 - (114/9)f$  with  $f$  being the number of active quark flavors.

Note here that the perturbative coupling constant  $a_s(Q^2)$  is different at the leading-order (LO) and NLO approximations. Hereafter we consider for simplicity only the LO approximation<sup>3</sup>, where the variables  $\sigma$  and  $\rho$  are given by Eq. (2) when  $p = 0$ .

<sup>2</sup>The contribution of valence quarks is negligible at low  $x$ .

<sup>3</sup>The NLO results may be found in [11].

## 2.1 Parton distributions and the structure function $F_2$

The SF  $F_2$  and PDFs have the following form

$$F_2(x, Q^2) = e f_q(x, Q^2), \quad f_a(x, Q^2) = f_a^+(x, Q^2) + f_a^-(x, Q^2), \quad (a = q, g) \quad (4)$$

where  $e = (\sum_1^f e_i^2)/f$  is the average charge square.

The small- $x$  asymptotic results for PDFs  $f_a^\pm$  are

$$\begin{aligned} f_g^+(x, Q^2) &= \left( A_g + \frac{4}{9} A_q \right) \tilde{I}_0(\sigma) e^{-\bar{d}_+(1)s} + O(\rho), & f_q^+(x, Q^2) &= \frac{f}{9} \frac{\rho \tilde{I}_1(\sigma)}{\tilde{I}_0(\sigma)} + O(\rho), \\ f_g^-(x, Q^2) &= -\frac{4}{9} A_q e^{-d_-(1)s} + O(x), & f_q^-(x, Q^2) &= A_q e^{-d_-(1)s} + O(x), \end{aligned} \quad (5)$$

where  $d_-(1) = 16f/(27\beta_0)$  and  $\bar{d}_+(1) = 1 + 20f/(27\beta_0)$  is the regular part of AD  $d_+(n)$  in the limit  $n \rightarrow 1^4$ . Here  $n$  is the variable in Mellin space. The functions  $\tilde{I}_\nu$  ( $\nu = 0, 1$ ) are related to the modified Bessel function  $I_\nu$  and to the Bessel function  $J_\nu$  by:

$$\tilde{I}_\nu(\sigma) = \begin{cases} I_\nu(\sigma), & \text{if } s \geq 0 \\ i^{-\nu} J_\nu(i\sigma), & i^2 = -1, \text{ if } s \leq 0 \end{cases} \quad (6)$$

## 2.2 Effective slopes

As it has been shown in [11], the behaviour of PDFs and  $F_2$  given in the Bessel-like form by generalized DAS approach can mimic a power law shape over a limited region of  $x$  and  $Q^2$

$$f_a(x, Q^2) \sim x^{-\lambda_a^{\text{eff}}(x, Q^2)} \quad \text{and} \quad F_2(x, Q^2) \sim x^{-\lambda_{F_2}^{\text{eff}}(x, Q^2)}.$$

The effective slopes  $\lambda_a^{\text{eff}}(x, Q^2)$  and  $\lambda_{F_2}^{\text{eff}}(x, Q^2)$  have the form:

$$\begin{aligned} \lambda_{F_2}^{\text{eff}}(x, Q^2) &= \lambda_g^{\text{eff}}(x, Q^2) = \frac{f_g^+(x, Q^2)}{f_g(x, Q^2)} \rho \frac{\tilde{I}_1(\sigma)}{\tilde{I}_0(\sigma)} \approx \rho - \frac{1}{4 \ln(1/x)}, \\ \lambda_q^{\text{eff}}(x, Q^2) &= \frac{f_q^+(x, Q^2)}{f_q(x, Q^2)} \rho \frac{\tilde{I}_2(\sigma)}{\tilde{I}_1(\sigma)} \approx \rho - \frac{3}{4 \ln(1/x)}, \end{aligned} \quad (7)$$

where the symbol  $\approx$  marks the approximation obtained in the expansion of the modified Bessel functions, when the “-” component is negligible. These approximations are accurate only at very large  $\sigma$  values (i.e. at very large  $Q^2$  and/or very small  $x$ ).

## 3 Comparison with experimental data

Using the results of previous section we have analyzed HERA data for  $F_2$  [1] and the slope  $\partial \ln F_2 / \partial \ln(1/x)$  [2, 3] at small  $x$  from the H1 and ZEUS Collaborations. In order to keep the analysis as simple as possible, we fix  $f = 4$  and  $\alpha_s(M_Z^2) = 0.1166$  (i.e.,  $\Lambda^{(4)} = 284$  MeV) in agreement with the recent ZEUS results in [1].

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<sup>4</sup>We denote the singular and regular parts of a given quantity  $k(n)$  in the limit  $n \rightarrow 1$  by  $\hat{k}(n)$  and  $\bar{k}(n)$ , respectively.

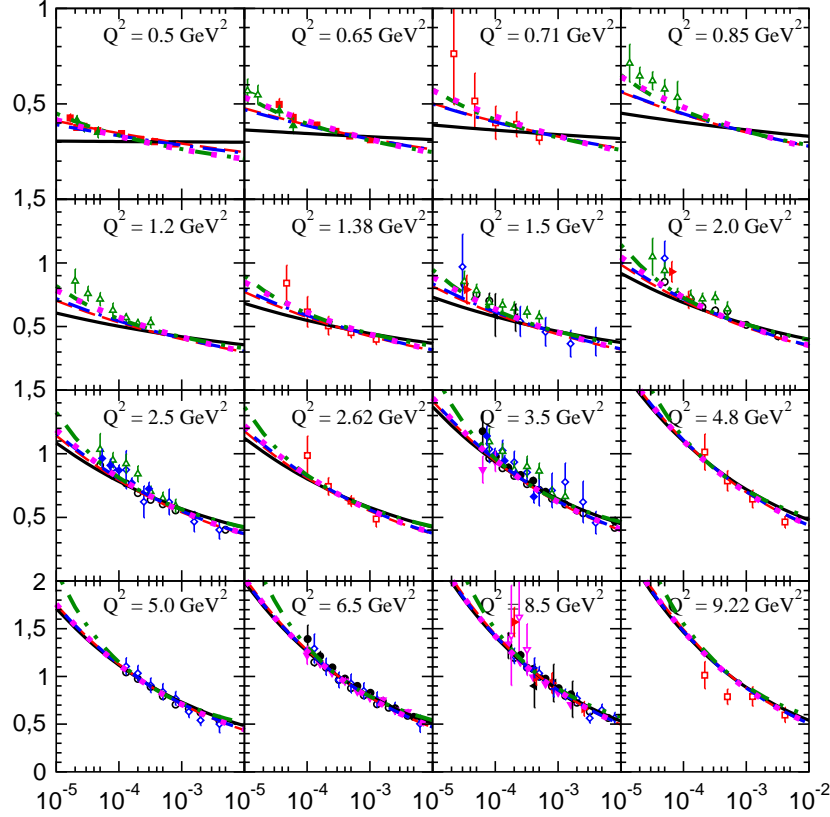


Figure 1:  $x$  dependence of  $F_2(x, Q^2)$  in bins of  $Q^2$ . The experimental data from H1 (open points) and ZEUS (solid points) [1] are compared with the NLO fits for  $Q^2 \geq 0.5 \text{ GeV}^2$  implemented with the canonical (solid lines), frozen (dot-dashed lines), and analytic (dashed lines) versions of the strong-coupling constant.

As it is possible to see in Figs. 1 and 2, the twist-two approximation is reasonable at  $Q^2 \geq 2 \div 4 \text{ GeV}^2$ . At smaller  $Q^2$ , some modification of the approximation should be considered. In Ref. [12] we have added the higher twist corrections. For renormalon model of higher twists, we have found a good agreement with experimental data at essentially lower  $Q^2$  values:  $Q^2 \geq 0.5 \text{ GeV}^2$  (see Figs. 4 and 5 in [12]).

In Ref. [14], to improve the agreement at small  $Q^2$  values, we modified the QCD coupling constant. We consider two modifications.

In one case, which is more phenomenological, we introduce freezing of the coupling constant by changing its argument  $Q^2 \rightarrow Q^2 + M_\rho^2$ , where  $M_\rho$  is the  $\rho$ -meson mass (see [14] and references therein). Thus, in the formulae of the Section 2 we should do the following replacement:

$$a_s(Q^2) \rightarrow a_{\text{fr}}(Q^2) \equiv a_s(Q^2 + M_\rho^2) \quad (8)$$

The second possibility incorporates the Shirkov–Solovtsov idea [20] about analyticity of the coupling constant that leads to the additional its power dependence. Then, in the formulae of

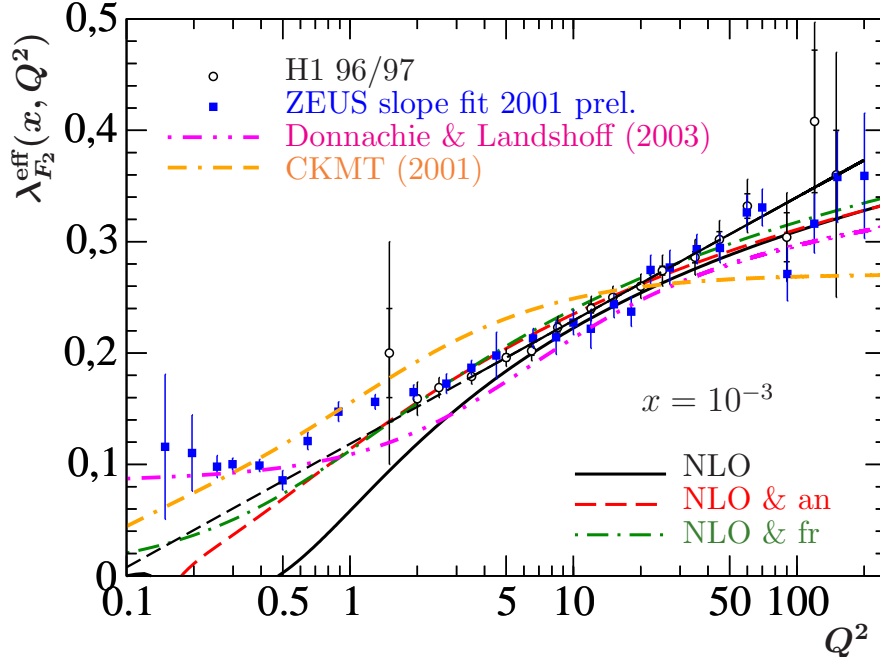


Figure 2: As in Fig. 1 but for the  $Q^2$  dependence of  $\lambda_{F_2}^{\text{eff}}(x, Q^2)$  for an average small- $x$  value of  $x = 10^{-3}$ . The linear rise of  $\lambda_{F_2}^{\text{eff}}(x, Q^2)$  with  $\ln Q^2$  [2] is indicated by the straight dashed line. For comparison, also the results obtained in the phenomenological models by Kaidalov et al. [22] (dash-dash-dotted line) and by Donnachie and Landshoff [23] (dot-dot-dashed line) are shown.

the previous section the coupling constant  $a_s(Q^2)$  should be replaced as follows: ( $k = 1$  and  $2$  at LO and NLO)

$$a_{\text{an}}(Q^2) = a_s(Q^2) - \frac{1}{k\beta_0} \frac{\Lambda^2}{Q^2 - \Lambda^2} + \dots, \quad (9)$$

where the symbol  $\dots$  stands for terms which are zero and negligible at  $Q \geq 1$  GeV [20] at LO and NLO, respectively.

Figure 2 shows the experimental data for  $\lambda_{F_2}^{\text{eff}}(x, Q^2)$  at  $x \sim 10^{-3}$ , which represents an average of the  $x$ -values of HERA experimental data. The top dashed line represents the aforementioned linear rise of  $\lambda(Q^2)$  with  $\ln(Q^2)$ . The Figs. 1 and 2 demonstrate that the theoretical description of the small- $Q^2$  ZEUS data for  $\lambda_{F_2}^{\text{eff}}(x, Q^2)$  by NLO QCD is significantly improved by implementing the “frozen” and analytic coupling constants  $\alpha_{\text{fr}}(Q^2)$  and  $\alpha_{\text{an}}(Q^2)$ , respectively, which in turn lead to very close results (see also [21]).

Indeed, the fits for  $F_2(x, Q^2)$  in [12] yielded  $Q_0^2 \approx 0.5\text{--}0.8$  GeV $^2$ . So, initially we had  $\lambda_{F_2}^{\text{eff}}(x, Q_0^2) = 0$ , as suggested by Eq. (1). The replacements of Eqs. (8) and (9) modify the value of  $\lambda_{F_2}^{\text{eff}}(x, Q_0^2)$ . For the “frozen” and analytic coupling constants  $\alpha_{\text{fr}}(Q^2)$  and  $\alpha_{\text{an}}(Q^2)$ , the value of  $\lambda_{F_2}^{\text{eff}}(x, Q_0^2)$  is nonzero and the slopes are quite close to the experimental data at  $Q^2 \approx 0.5$  GeV $^2$ . Nevertheless, for  $Q^2 \leq 0.5$  GeV $^2$ , there is still some disagreement with the data, which needs additional investigation.

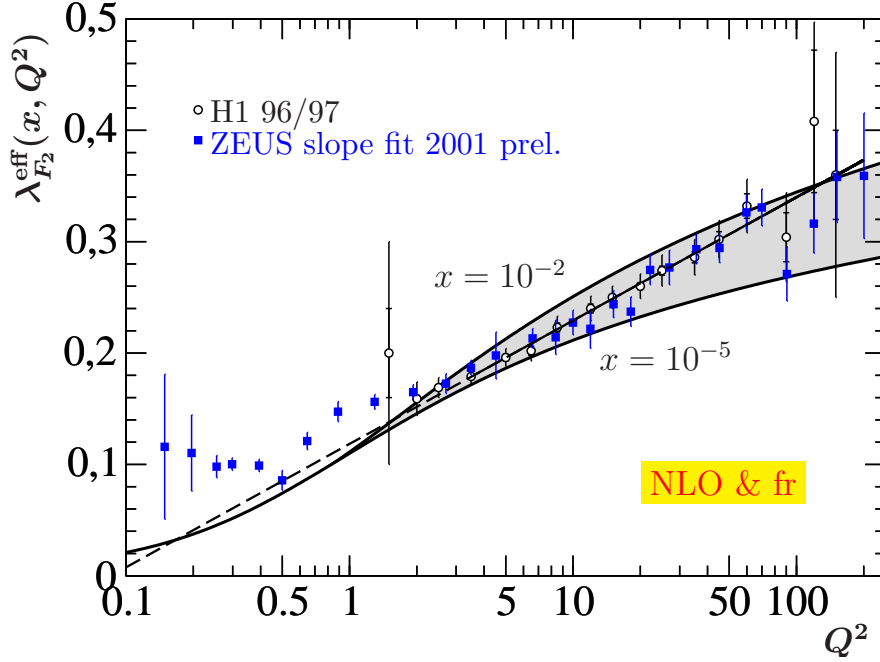


Figure 3: The values of effective slope  $\lambda_{F_2}^{\text{eff}}$  as a function of  $Q^2$ . The experimental points are same as in Fig. 2. The dashed line represents the fit from [2]. The solid curves represent the NLO fits with “frozen” coupling constant at  $x = 10^{-2}$  and  $x = 10^{-5}$ .

For comparison, we display in Fig. 2 also the results obtained by Kaidalov et al. [22] and by Donnachie and Landshoff [23] adopting phenomenological models based on Regge theory. While they yield an improved description of the experimental data for  $Q^2 \leq 0.4 \text{ GeV}^2$ , the agreement generally worsens in the range  $2 \text{ GeV}^2 \leq Q^2 \leq 8 \text{ GeV}^2$ .

The results of fits in [12, 14] have an important property: they are very similar in LO and NLO approximations of perturbation theory. The similarity is related to the fact that the small- $x$  asymptotics of the NLO corrections are usually large and negative (see, for example,  $\alpha_s$ -corrections [24] to BFKL approach [25]<sup>5</sup>). Then, the LO form  $\sim \alpha_s(Q^2)$  for some observable and the NLO one  $\sim \alpha_s(Q^2)(1 - K\alpha_s(Q^2))$  with a large value of  $K$ , are similar because  $\Lambda \gg \Lambda_{\text{LO}}$ <sup>6</sup> and, thus,  $\alpha_s(Q^2)$  at LO is considerably smaller than  $\alpha_s(Q^2)$  at NLO for HERA  $Q^2$  values.

In other words, performing some resummation procedure (such as Grunberg’s effective-charge method [26]), one can see that the NLO form may be represented as  $\sim \alpha_s(Q_{\text{eff}}^2)$ , where  $Q_{\text{eff}}^2 \gg Q^2$ . Indeed, from different studies [27], it is well known that at small- $x$  values the effective argument of the coupling constant is higher than  $Q^2$ .

In the generalized DAS approach the small effect of the NLO corrections can be explained by separated contributions of the singular and regular AD parts. Indeed, the singular parts modify the argument of the Bessel functions (see Eq.(2)) and the regular parts contribute to

<sup>5</sup>It seems that it is a property of any processes in which gluons, but not quarks, play a basic role.

<sup>6</sup>The equality of  $\alpha_s(M_Z^2)$  at LO and NLO approximations, where  $M_Z$  is the  $Z$ -boson mass, relates  $\Lambda$  and  $\Lambda_{\text{LO}}$ :  $\Lambda^{(4)} = 284 \text{ MeV}$  (as in ZEUS paper on [1]) corresponds to  $\Lambda_{\text{LO}} = 112 \text{ MeV}$  (see [12]).

the front of Bessel functions [11].

Figure 3 shows the  $x$ -dependence of the slope  $\lambda_{F_2}^{\text{eff}}(x, Q^2)$ . One observes good agreement between the experimental data and the generalized DAS approach for a broad range of small- $x$  values. The absence of a variation with  $x$  of  $\lambda_{F_2}^{\text{eff}}(x, Q^2)$  at small  $Q^2$  values is related to the small values of the variable  $\rho$  there.

From Figs. 2 and 6 in [12], one can see that HERA experimental data exists at  $x \sim 10^{-4} \div 10^{-5}$  for  $Q^2 = 4 \text{ GeV}^2$  and at  $x \sim 10^{-2}$  for  $Q^2 = 100 \text{ GeV}^2$ . Indeed, the correlations between  $x$  and  $Q^2$  in the form  $x_{\text{eff}} = a \times 10^{-4} \times Q^2$  with  $a = 0.1$  and 1 lead to a modification of the  $Q^2$  evolution which starts to resemble  $\ln Q^2$ , rather than  $\ln \ln Q^2$  as is standard [19].

## 4 Conclusions

We have shown the  $Q^2$ -dependence of the SF  $F_2$  and the slope  $\lambda_{F_2}^{\text{eff}} = \partial \ln F_2 / \partial \ln(1/x)$  at small- $x$  values in the framework of perturbative QCD. Our twist-two results are in very good agreement with the precise HERA data at  $Q^2 \geq 2 \text{ GeV}^2$ , where the perturbative theory is applicable. The application of the “frozen” and analytic coupling constants  $\alpha_{\text{fr}}(Q^2)$  and  $\alpha_{\text{an}}(Q^2)$  improves the agreement for smaller  $Q^2$  values, down to  $Q^2 \geq 0.5 \text{ GeV}^2$ .

As a next step of investigations, we plan to fit the H1&ZEUS data [4] and to extend the generalized DAS approach to evaluate the double PDFs which are very popular now (see [28] and references therein). Also we plan to use our approach to analyze the cross sections of processes studied at LHC by analogy with our investigations [29] of the total cross section of ultrahigh-energy deep-inelastic neutrino-nucleon scattering.

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