17/10/23

BLACK HOLES

DESY WORKSHOP

GRAVITATIONAL COLLAPSE - The case for BHs

Spherical stars

To study BHs we first look at the end of a star's life.

- General stars, including our Sun, rely on nuclear reactions inside their cores to counteract gravitational collapse by generating thermal pressure
- However, stars eventually run out of nuclear "fuel", a critical question: how do stars end their lives when nuclear reactions are no longer sufficient to counteract gravitational forces?
- To maintain stability, a new source of pressure is required when nuclear reactions cease. This pressure sources must be non-thermal because stars cool over time. A non-thermal pressure source emerges from Pauli exclusion principle: the degeneracy pressure makes a gas of cold fermions resistant to compression
- White Dwarfs (WD): Stars like our Sun, where gravity is balanced out by electron degeneracy pressure, ultimately transform into "white dwarfs".
  These are much denser than normal stars
  E.g. if $m_{WD} \sim m_\odot \Rightarrow r_{WD} \sim \mathcal{O}(10^{-2})\, r_\odot$
  Using Newtonian gravity it can be shown that a white dwarf cannot exceed a certain mass limit ("Chandrasekhar limit"): if $m_{star} > 1.4\, m_\odot$ the star cannot end its life as a WD.
- Neutron stars (NS): When the density of matter reaches nuclear density, neutron degeneracy becomes significant.
  NS are supported by this pressure and are very compact: if $m_{NS} \simeq m_\odot \Rightarrow r_{NS} \sim \mathcal{O}(10^{-4})\, r_\odot$.

Gravity (encoded in the Newtonian gravitational potential Φ) is very strong, Newtonian gravity is no longer valid, and GR becomes necessary.

GR predicts a max. mass for NS: $m_{max} \simeq 3\, m_\odot$

A hot star with mass $> m_{max}$ will undergo full gravitational collapse and form a Black Hole.

Spacetime symmetries


SO(3)

- Sphere: $ds^2 = dr^2 + r^2 d\Omega$
  Isometries of the spacetime belong to the group $SO(3)$

A spacetime is spherically symmetric if it contains an $SO(3)$ subgroup within the isometry group of the spacetime whose orbits are 2-spheres. The symmetry is manifest in the spacetime geometry as the orbits are spherical regions in spacetime.

Obs: Isometry group orbits are defined as the set of points obtained by acting on a reference point with all the diffeomorphism (coordinate trnsf.) generated by the $SO(3)$ subgroup.
In essence, these orbits are the regions within the spacetime where the spherical symmetry is preserved

To work with spherical symmetry we define the area-radius function (r) as

$$r(p) = \sqrt{\frac{A(p)}{4\pi}} \quad \text{w/ } A(p) = \text{area of the } S^2 \text{ orbit through } p$$

Obs: S^2 passing through p induces metric $r(p)^2 d\Omega^2$

- A spacetime is stationary when it admits a time like killing vector field : k^a s.t. $g_{ab} k^a k^b > 0$
  One can always introduce a corresponding time coordinate t, such that $x^\mu = (t, \vec{x})$ and k^a corresponds to the time derivative $\partial/\partial t$

As a result, the metric is independent of time

$$ds^2 = g_{00}\, dt^2 + g_{0i}\, dt\, dx^i + g_{ij}\, dx^i dx^j \qquad \begin{matrix} g_{00} < 0 \\ g_{00}, g_{0i}, g_{ij} \not\ni t \end{matrix}$$

Note: $\begin{matrix} t \to t' = t + f(\vec{x}) \\ \vec{x} \to \vec{x}' = x \end{matrix} \Rightarrow \partial_t = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} + \frac{\partial x^{i'}}{\partial t}\partial_{i'} = \partial_{t'}$

- Static spacetimes: they are stationary and one can chose t s.t. they are invariant under $t \to -t + \text{const}$.

  ↳ $ds^2 = g_{00}(\vec{x})\, dt^2 + g_{ij}(\vec{x})\, dx^i dx^j$

  Equivalently, a stationary spacetime is static if k^a $(\partial/\partial t)$ is orthogonal to a family of space-like surfaces parametrized by t and along which we can use coordinates x^i.

  Static spacetimes "do not evolve" while stationary spacetimes evolve always in the same way.

Let us make this a little bit more mathematically precise.

Take Σ hypersurface nowhere tangent to k^a w/ x^i coord. on Σ
We can assign $(t, \vec{x})$ coord. to a point on Σ parametrised by the distance t along the integral curve through the point p on Σ with coord. x^i.

Now, take Σ specified by $f(x) = 0$ w/ $f: M \to \mathbb{R}$ w/ $df \neq 0$ on Σ
$\Rightarrow$ the 1-form df is normal to Σ
(take t^a tangent vector to Σ $\Rightarrow df(t) = t(f) = t^\mu \partial_\mu f = 0$ since f is constant on Σ □)

All other normal 1-forms can be written as $n = g\,df + f n'$ w/ $g \neq 0$ smooth fnct. on Σ and n' smooth 1-form.
$\Rightarrow dn = dg \wedge df + df \wedge n' + f\, dn'$ and so $(dn)|_\Sigma = (dg - n') \wedge df$
Thus, if n is normal to Σ $\Rightarrow$ $(n \wedge dn)|_\Sigma = 0$

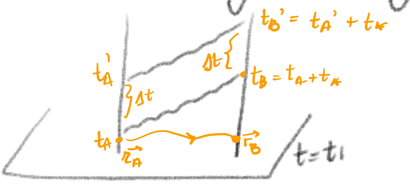
Theorem (Frobenius): if n is a non-zero 1-form s.t. $n \wedge dn = 0$ everywhere $\Rightarrow \exists$ f, g functions, s.t. $n = g\,df$ and n is normal to surfaces of const. f.
n is hypersurfaces-orthogonal.

Spacetime is static if it admits an hypersurface-orthogonal timelike killing vector field.
$\to (t, x^i)$ on Σ, where $t=0$, with normal dt.

$t=0$ $\quad k_\mu \sim \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $k^i = 0$ but $k_i = g_{0i} = 0$

↳ Back to the physics. In a static spacetime we can give a natural definition of symultaneous event: same t in static coordinates.

Furthermore, the static time t can be identified with the time obtained syncronizing static clock rates through light signals



Null trajectory: $-g_{00}(\vec{x})\,dt^2 = g_{ij}(\vec{x})\,dx^i dx^j = dl^2$

$\Rightarrow t_* = \int_{r_A}^{r_B} \frac{dl}{\sqrt{g_{00}(\vec{x})}}$ depends only on the spatial trajectory and not on time

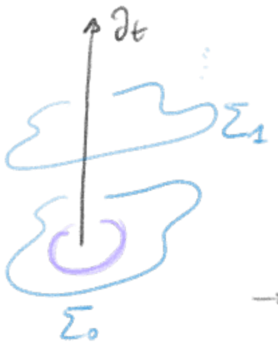
$\rightarrow$ All static clocks see each other ticking at the same rate

Obs: In stationary spacetimes the stationary cloks are similarly rate syncronized but there is no unique def. of global killing time

$\rightarrow$ We consider STATIC and SPHERICALLY SYMMETRIC spacetimes

$\downarrow$

Foliate spacetimes w/ surfaces Σ_t orthogonal to k^a.



The orbit of $SO(3)$ through $p \in \Sigma_t$ lies on Σ_t.

$\rightarrow$ Sperical polar coord. $\begin{pmatrix} r \\ \vartheta \\ \phi \end{pmatrix}$ on S^2 on Σ_0, then extend the def of $\begin{pmatrix} \vartheta \\ \phi \end{pmatrix}$ to the rest of Σ_0 by taking them to be const. along geodesic normal to S^2.

r is the area-radius.

$\rightarrow$ On Σ_0 $\quad ds^2 = e^{2\psi(r)}dr^2 + r^2 d\Omega^2$

Now we define (t, r, ϑ, ϕ)

$$\rightarrow ds^2 = e^{-2\phi(r)}dt^2 + e^{2\psi(r)}dr^2 + r^2 d\Omega^2$$

- Matter in the star described by perfect fluid:

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab}$$ w/ u^a 4-velocity of the fluid (unit timelike $g_{ab}u^a u^b = -1$)

and ρ, p are energy density and pressure measured in the fluid's local rest frame.

In a time-independent and spherically symmetric spacetime we assume the fluid is at rest, so u^a is time-like

$$u^a = e^{-\phi}\left(\frac{\partial}{\partial t}\right)^a$$

Staticity + spherical symm. imply that ρ, p depend on r.

If R is the area (radius) of the star $\Rightarrow \rho = p = 0$ for $r > R$.

Tolman - Oppenheimer - Volkoff equations

One can obtain the EoM from Einstein equations, and since the Einstein tensor inherits the symmetries of the metric, there will be only three non-trivial components: tt, rr, $\theta\theta$.

By defining $m(r)$ via $\underbrace{e^{2\Psi(r)}}_{>0} = \left(1 - \frac{2m(r)}{r}\right)^{-1}$ ✱ $\Rightarrow m(r) < r/2$

(tt) $\frac{dm}{dr} = 4\pi r^2 \rho$ (rr) $\frac{d\Phi}{dr} = \frac{m + 4\pi r^3 p}{r(r-2m)}$ ($\theta\theta$) $\frac{dp}{dr} = -(p+\rho)\frac{(m+4\pi r^3 p)}{r(r-2m)}$

+ eq. of state $p = p(\rho)$ we assume $p, \rho > 0$ and $\frac{dp}{d\rho} > 0$

otherwise fluid is unstable
(fluctuation increas ρ decreas p would cause the fluid to move into that region and increase ρ
→ fluctuations would grow)

The Schwarzschild sol.

1) Outside the star

$r > R \rightarrow p = \rho = 0$ and $m(r) = M$, const. (tt)

From (rr) $\phi = \frac{1}{2}\log\left(1 - \frac{2M}{r}\right) + \phi_0$, $\phi_0 = \text{const.}$

Redefine $t = e^{\phi_0 t}$ and set $\phi_0 = 0$ without loss of generality
s.t. $g_{00} \rightarrow 1$ as $r \rightarrow \infty$

↳ Schwarzschild sol.

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

Singular at $r = 2M$ schwarzschild radius ($g_{00} \rightarrow 0$, $g_{rr} \rightarrow \infty$)
therefore $r = 2M$ must be inside the BH
$\rightarrow R > 2M$

<u>Obs</u>: Is $r_s = 2M$ a proper or a coord. singularity?
we'll see that $r_s = 2M$ will be interpreted as horizon radius, beyond which no signal can escape the gravitational potential and the sol. will be interpreted as a BH.

2) Interior solution

(tt) gives: $m(r) = 4\pi \int_0^r \rho(r) r'^2 dr' + m_*$, $m_* = \text{Const.}$

The spacetime should be smooth at the center of the star, which means that measurements in a small region near the center should resemble those in Euclidian space.
The proper radius of a sphere with area-radius r is

$$\int_0^r e^{\Psi(r')} dr' \simeq e^{\Psi(0)} r \quad \text{for small } r$$

In Eucl space, the proper radius of a sphee with an area-radius r is simply r

$$\Rightarrow \Psi(0) = 0 \quad \overset{\star}{\Rightarrow} m(0) = 0 \quad \Rightarrow m_* = 0$$

Let us match the interior to the exterior at $r = R$
for $r > R$, $m(r) = M = \text{const}$

$$M = 4\pi \int_0^R \rho(r) r^2 dr \quad \star\star$$

Difference with Newtonian gravity:

Euclidian vol element on a surface of const. t
$r^2 \sin\theta \, dr \wedge d\theta \wedge d\varphi$
$\Rightarrow M = E$

GR vol element on Σ_t
$e^{\Psi} r^2 \sin\theta \, dr \wedge d\theta \wedge d\varphi$
$E = 4\pi \int_0^R \rho e^{\Psi} r^2 dr$

Since $e^{\Psi} > 1$ (for $m > 0$) $\Rightarrow E > M$

Obs: $\star$ implies $\frac{m(r)}{r} < \frac{1}{2}$ $\forall r$ $\rightsquigarrow$ $r = R$ $R > 2M$ already seen

Let us improve: $(\theta\theta)$ implies $\frac{dp}{dr} \leq 0$ and $\frac{d\rho}{dr} \leq 0$

$\rightarrow$ one can show $\frac{m(r)}{r} < \frac{2}{9}\left[1 - 6\pi r^2 p(r) + (1 + 6\pi r^2 p(r))^{1/2}\right] \star\star\star$

For $r = R$, $p = 0$ Buchdahl inequality

$$R > \frac{9}{4} M$$

The TOV eq. can be solved numerically, given the central density ρ_c, of the star, these eq. can be solved to determine R, M, and $\Phi(r)$ in the star.
$\hookrightarrow$ In the context of static sph. symm cold stars the sol. form a one parameter family of solutions characterized by ρ_c

Maximum mass of a cold star

When one numerically evolves the TOV equations he will find:

↳ The mass of cold, spherically symmetric stars, increases up to m_{max} a maximum value as the central density increases; beyond this point, the mass decreases.

The value of m_{max} depends on the equation of state of cold matter e.g. if you consider eq. of state of white dwarf matter, you obtain Chandrasekhar bound.

However, GR predicts a bound on the mass independent on the form of the equation of state at high densities $\sim 5 M_\odot$.

Def: core of the star the region with $\rho > \rho_0$, where ρ_0 is the value of the density up to which we experimentally know the equation of state ($\sim$ nuclear density)
Here we have $r < r_0$

envelope the region w/ $\rho < \rho_0$ where we know the eq. of state.
Here $r_0 < r < R$

The mass of the core $m_0 = m(r_0)$ $(\star\star)$ $m_0 \geq \frac{4}{3}\pi r_0^3 \rho_0$ A
and from $(\star\star)$ at $r = r_0$

$$\frac{m_0}{r_0} < \frac{2}{9}\left[1 - 6\pi r_0^2 p_0 + \left(1 + 6\pi r_0^2 p_0\right)^{1/2}\right] \quad \text{w/ } p_0 = p(r_0)$$

For $p_0 = 0 \to m_0 < \frac{4}{9} r_0$ B

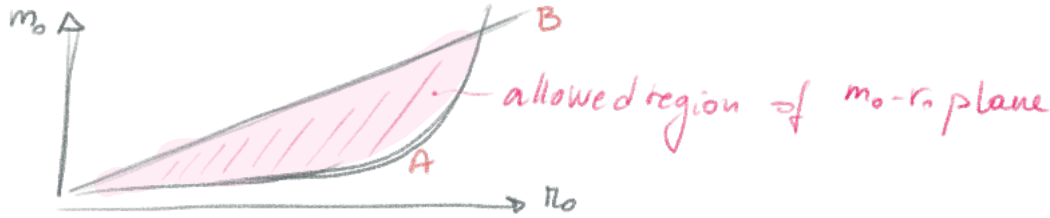
A and B define a finite region of the $m_0 - r_0$ plane

↳ Upper bound on the mass of the core: $m_0 < \sqrt{\frac{16}{243\pi\rho_0}}$

Even if in the core we do not know ρ_0,
we know m_0 cannot be indefinitely large.
Largest ρ_0 we know is that of nuclear matter $\rho_0 = 5\cdot 10^{14}\, g/cm^3$

Therefore $m_0 < 5 M_\odot$

↳ Given a core w/ m_0 and r_0, the envelope can be determined solving (tt) (θ,θ) using the known eq. of state at $\rho<\rho_0$



Investigating, numerically, the behaviour of this function in m_0-r_0 plane it is found that M is max at maximum of m_0.
At this max, the contribution of the envelope to the total max is less than 1%. So the maximum M is almost identical to m_0.

Schwarschild BH

GR tells us a cold star with $m \gtrsim 5 M_\odot$ will collapse to form a BH. The simplest sol. is the Schwarschild BH.
We will investigate the geometry of spacetime assuming that the Schwarschild sol. is valid everywhere, not just in the exterior.

Birkhoff theorem:

In Schwarschild coord, the Schwarschild sol. is

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

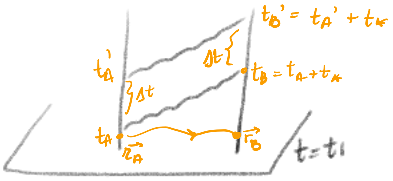
We extend the sol to $r<2M$.

In Schwarschild, the static time t can then be interpreted as the time obtained by syncronising static clocks with the proper time of the observer at $r=\infty$

In fact, one can prove that Schwarschild is the unique spherically symmetric vacuum solution, without assuming staticity or stationarity.
Hence, any spherically symmetric process eventually ends up in Schwarschild in the vacuum regions.

This is Birkhoff's theorem, and is a first ex. of "no-hair" th. for BHs

Gravitational redshift



The proper time between the photons emitted by A measured by A will be

$$\Delta\tau_A = \sqrt{1-\frac{2M}{r_A}}\,\Delta t$$

$$\Delta\tau_B = \sqrt{1-\frac{2M}{r_B}}\,\Delta t$$

$$\rightarrow \quad \frac{\Delta\tau_B}{\Delta\tau_A} = \sqrt{\frac{1-2M/r_B}{1-2M/r_A}} > 1$$

$\rightsquigarrow$ since $\Delta\tau = \lambda$ (for $c=1$) $\Rightarrow \lambda_B > \lambda_A \rightsquigarrow$ light undergoes a redshift

If B is s.t. $r_B \gg 2M$

$$1+z := \frac{\lambda_B}{\lambda_A} = \sqrt{\frac{1}{1-2M/r_A}}$$

Obs: $1+z \rightarrow \infty$ as $r_A \rightarrow 2M$. We showed that a spherical star must have $R > 9M/4$
$\Rightarrow$ max. redshift is at $z=2$