Schwarzschild Black Hole & a whiff of singularity theorems

1. Birkhoff's theorem

Schwarzschild solution in (t, r, θ, ϕ) coordinates:

$$ds^2 = -c^2\left(1 - \frac{2}{r}\frac{GM}{c^2}\right)dt^2 + \left(1 - \frac{2}{r}\frac{GM}{c^2}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

time coordinate that for $r > 2M$ is measured by a clock located at $r \to \infty$ from the massive body and stationary with respect to it.

$G = 1 = c$

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2 \qquad (1)$$

This is a solution of the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} = 0$$

For $r > 2M$ but also for $0 < r < 2M$. At $r = 2M$ there is a singularity (in these coordinates) and

$r_H = 2M$: Schwartzschild radius

Birkhoff's theorem: Any spherically symmetric solution of the vacuum Einstein equation is isometric to the Schwartzschild solution.

The proof in Hawking-Ellis assumes that there is a spherical symmetry and that

$$\frac{\partial}{\partial t} \text{ is a hypersurface-orthogonal Killing vector field}$$

[See Margherita's notes: Frobenius theorem]

Since we saw that for $r > 2M$, $\frac{\partial}{\partial t}$ is timelike, then for $r > 2M$ Schwarzschild solution is static

Consequence: The spacetime outside any spherical body is described by the time-independent (exterior) Schwarzschild solution (even if the body itself is time-dependent)

1.1 A first comment on singularities

In the Schwarzschild solution we see two places where the metric diverges:

$$r = 0 \qquad \text{and} \qquad r = 2M$$

However the solution is a coordinate dependent quantity, so it's possible that these are "coordinate singularities", which means that in another coordinate system we can eliminate them.

We will see, indeed that $r = 2M$ is not a physical singularity but just a kind of coordinate singularity.

Usually a coordinate-independent criterion that would tell us if the geometry is not under control is to check if the curvature is infinite.

However, since the curvature is a tensor, it's difficult to say if it's infinite or not and one looks at scalars that are by definition coordinate independent:

$$R = g^{\mu\nu} R_{\mu\nu} \, , \quad R^{\mu\nu} R_{\mu\nu} \, , \quad R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \, , \quad \dots$$

This is a sufficient but not necessary criterion. For instance, the Kretschmann invariant $R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48 M^2}{r^6}$ for Schwarzschild finding that $r = 2M$ is not a singularity, while $r = 0$ is.

2. Gravitational redshift

Let us compute the redshift in the Schwarzschild geometry. We consider two observers O_A and O_B stuck at fixed spatial coordinates:

$$O_A : (r_A, \theta_A, \phi_A) \qquad O_B : (r_B, \theta_B, \phi_B)$$

using (1) the proper time for $i = A, B$ is given by

$$\frac{d\tau_i}{dt} = \left(1 - \frac{2M}{r_i}\right)^{1/2}$$

1. Suppose that O_A emits a light pulse to O_B, such that O_A measures the time between two successive crests of the lightwave to be $\Delta\tau_A$. Each crest follow the same path to O_B but they are separeted by a coordinate time

$$\Delta t = \left(1 - \frac{2M}{r_A}\right)^{-1/2} \Delta\tau_A$$

2. The second observer measures a time between succesive crests given by

$$\Delta\tau_B = \left(1 - \frac{2M}{r_B}\right)^{1/2} \Delta t = \left(\frac{1 - \frac{2M}{r_B}}{1 - \frac{2M}{r_A}}\right)^{1/2} \Delta\tau_A$$

3. Since $\lambda_i = \Delta\tau_i \quad (c=1)$ then

$$\frac{\lambda_B}{\lambda_A} = \left(\frac{1 - \frac{2M}{r_B}}{1 - \frac{2M}{r_A}}\right)^{1/2}$$

4. Assuming $r_B > r_A$, then

$$\frac{\lambda_B}{\lambda_A} = \left(\frac{1 - \frac{2M}{r_B}}{1 - \frac{2M}{r_A}} \right)^{1/2} > 1$$

So we have that $\lambda_B > \lambda_A$ and the light undergoes a <u>redshift</u> as it climbs out of the gravitational field

5. Defining

$$1 + z = \frac{\lambda_B}{\lambda_A} \xrightarrow{r_B >> 2M} \frac{1}{\sqrt{1 - \frac{2M}{r_A}}}$$

since you found that a spherical star must have radius $R > \frac{9M}{4}$ at $r_A = R = \frac{9M}{4}$ we have $z = 2$ as maximum possible redshift of light emitted from a surface of a spherical star

3. Geodesics of the Schwarzschild Solution

Let us consider the geodesic equation

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \qquad (2)$$

where λ is the parameter for the curve, and it can be interpreted as the proper time τ. We can compute $\Gamma^\mu_{\rho\sigma}$ using the Mathematica notebook attached to these notes and we will find (2) to be quite complicated. Instead we notice that there are Killing vectors that lead to a constant of the motion for a free particle:

$$K_\mu \frac{dx^\mu}{d\lambda} = \text{constant}$$

(Short proof at the end of the notes)

Moreover, there is another constant of the motion for geodesics:

$$\sigma = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \qquad (3)$$

If $\lambda = \tau$, then $\sigma = -g_{\mu\nu} U^\mu U^\nu$ with $\sigma = \underset{\text{spacelike}}{-1}, \underset{\text{null}}{0}, \underset{\text{timelike}}{1}$

One Killing vector is invariance under time translation, $K = \frac{\partial}{\partial t}$, leading to conservation of the energy:

$$K_\mu = \left(-\left(1-\frac{2M}{r}\right), 0, 0, 0\right) \Rightarrow E = -K_\mu \frac{dx^\mu}{d\lambda} = \left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda}$$

The other Killing vectors are invariance under rotations, $L = \frac{\partial}{\partial \phi}$, so

$$L_\mu = \left(0, 0, 0, r^2 \sin^2\theta\right) \Rightarrow \qquad L = L_\mu \frac{dx^\mu}{d\lambda} = \left(r^2 \sin^2\theta\right)\frac{d\phi}{d\lambda}$$

moreover one can rotate the system such that it happens on the plane

$$\theta = \frac{\pi}{2}$$

so the final conserved quantities are

$$\left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} = E \qquad\qquad r^2 \frac{d\phi}{d\lambda} = L \qquad\qquad (4)$$

We can then expand (3) to obtain

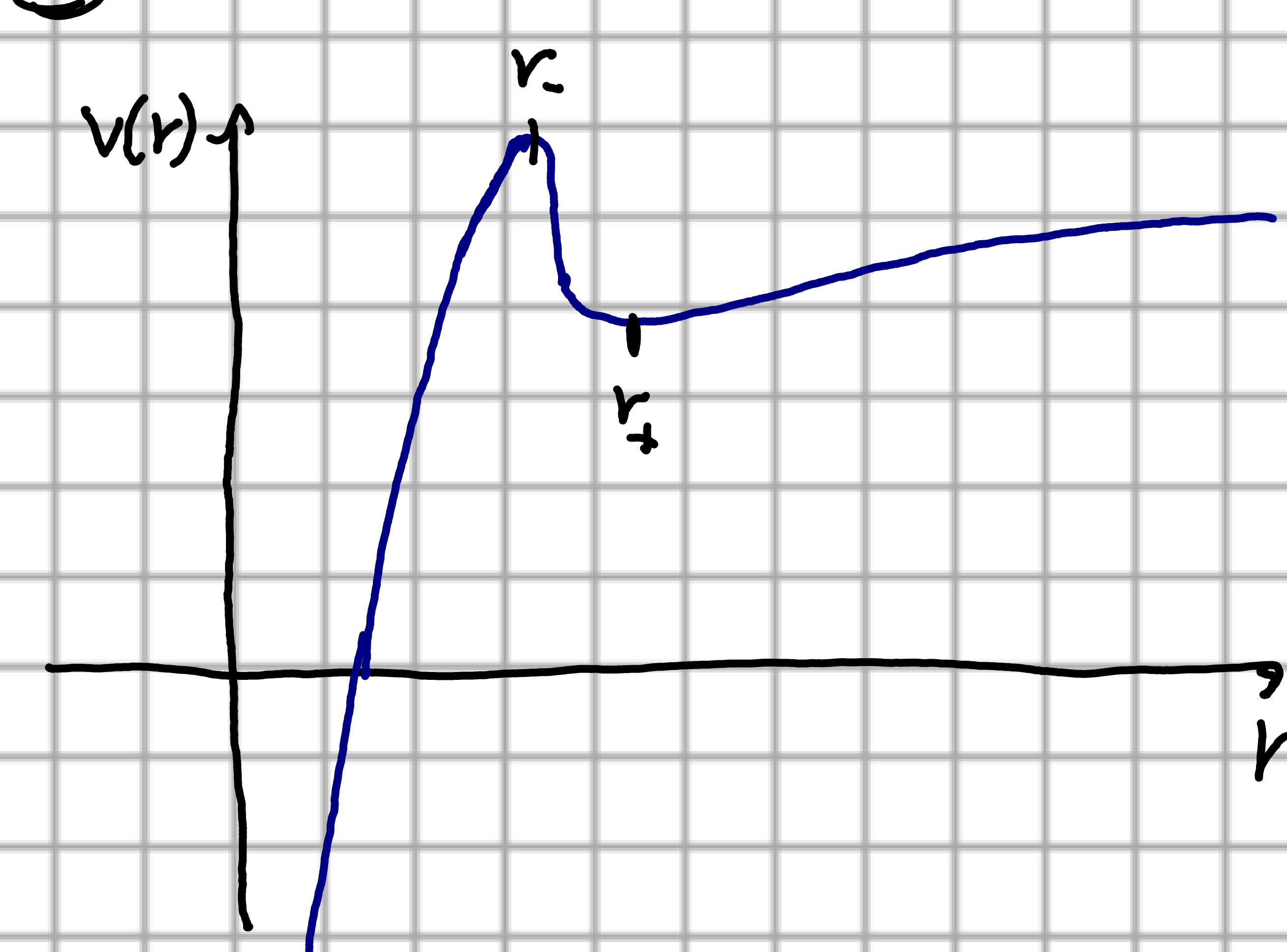
$$\sigma = \left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2$$

Using (4) and multiplying for $\left(1 - \frac{2M}{r}\right)$ one gets

$$-E^2 + \left(\frac{dr}{d\lambda}\right)^2 + \left(1 - \frac{2M}{r}\right)\left(\frac{L^2}{r^2} + \sigma\right) = 0$$

Inner most stable circular orbit

$3M < r_- < 6M < r_+$



or analogously

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \frac{1}{2}E^2$$

with

$$V(r) = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\sigma + \frac{L^2}{r^2}\right)$$

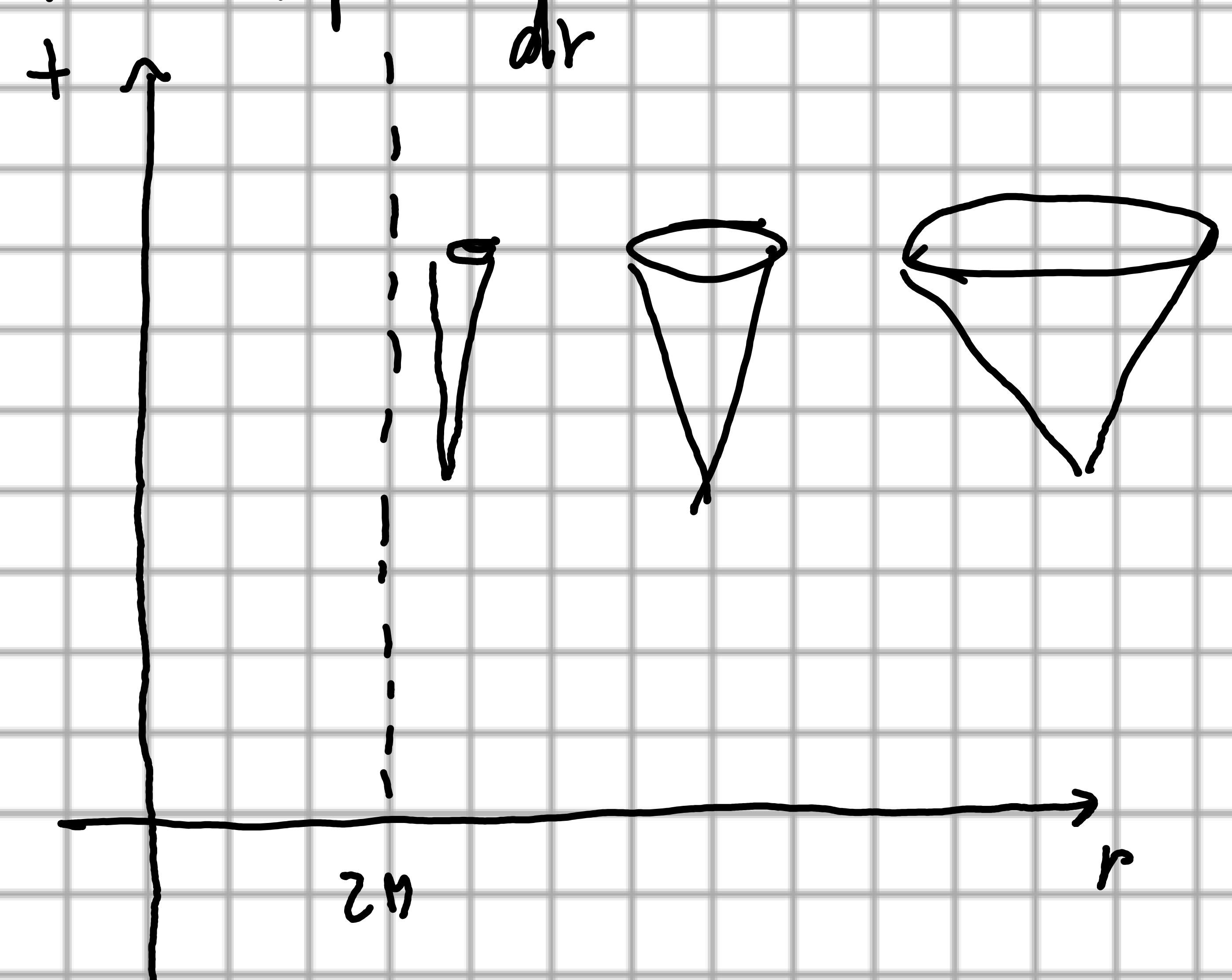
describing a classical particle of unit mass and energy $\frac{1}{2}E^2$ moving in a 1d potential $V(r)$

3. Change of coordinates

Let us see better the behavior at $r=2M$. First we can consider radial null curves at fixed (θ, ϕ):

$$ds^2 = 0 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 \Rightarrow \frac{dt}{dr} = \pm\left(1-\frac{2M}{r}\right)^{-1} \quad (5)$$

so for $r \to 2M$ the slope $\frac{dt}{dr} \to \pm\infty$ so



The light seems to asymptote to $r=2M$ without reaching it

Remember the redshift computation: if we stay outside and we look at an observer going towards the BH, we will see their signals reaching us here and more slowly. We would never see the astronaut crossing $r=2M$.

What's the meaning of this? We can understanding it by solving (5)

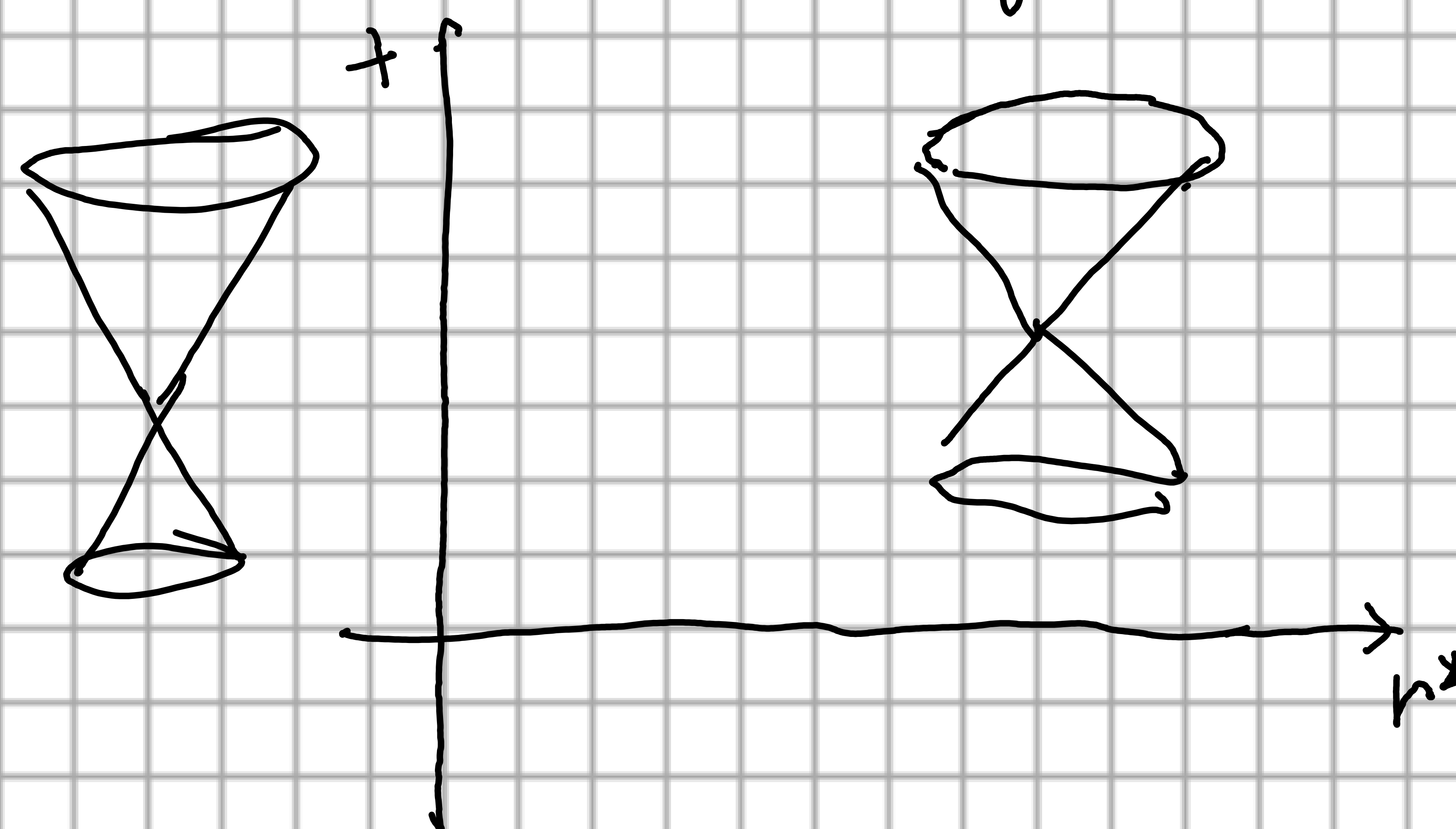
$$t = \pm r^* + \text{constant} \quad \text{with} \quad r^* = r + 2M \ln\left(\frac{r}{2M} - 1\right) \quad \text{for } r \geq 2M$$

In these coordinates:

not infinite at $r = 2M$

$$ds^2 = \left(1 - \frac{2M}{r(r^*)}\right)\left(-dt^2 + dr^{*2}\right) + r^2(r^*)\,d\Omega^2$$

Now the light cones are not closing up



and $r = 2M$ corresponds to $r^* \to -\infty$.

Now we define

$$v = t + r^* \qquad \text{and} \qquad u = t - r^*$$

Infalling radial null geodesics have $v = \text{const.}$, while outgoing ones have $u = \text{const.}$ Replacing t with v in the original coordinates:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dv^2 + \left(dv\,dr + dr\,dv\right) + r^2 d\Omega^2$$

Eddington-Finkelstein coordinates

These coordinates are smooth for all $r > 0$ and smooth for $r = 2M$

In these coordinates, the radial null curves is solved by

$$\frac{dv}{dr} = \begin{cases} 0 & \text{infalling} \\ 2\left(1-\frac{2M}{r}\right)^{-1} & \text{outgoing} \end{cases}$$

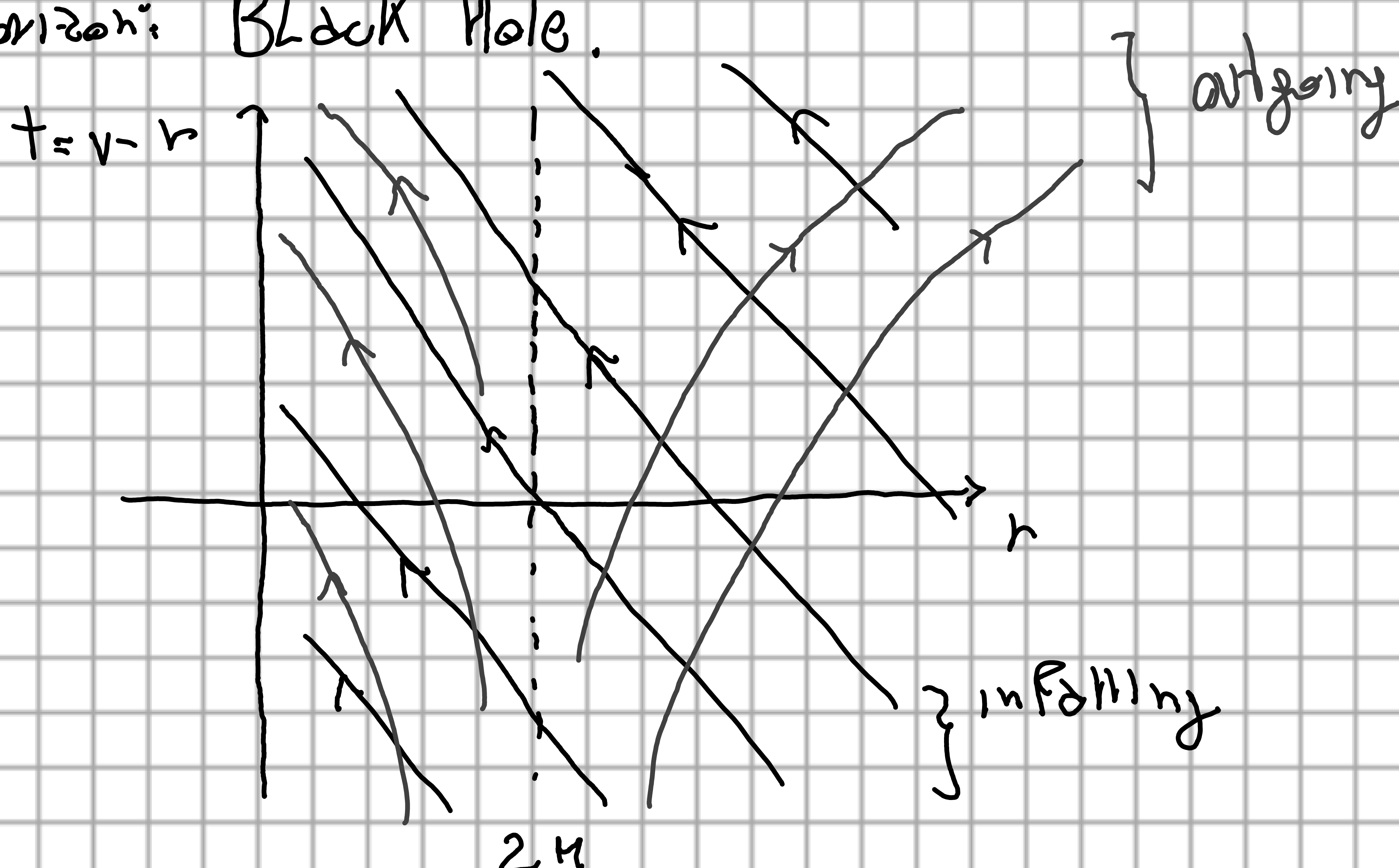
$r=2M$ is well-behaving and at finite coordinate value

However for $r<2M$ all future directed paths are in the direction of decreasing r.

$r=2M$ is a point of no return: once the particle goes below it, it can never come back

For this reason $r=2M$ is called event horizon: no event at $r \leq 2M$ can influence an event at $r>2M$

Such event horizon is a null surface and nothing can escape the event horizon: Black Hole.



4. Black Hole region

Causal vector: a timelike or null vector.

Causal curve: a curve whose tangent vector is everywhere causal

Time-orientable spacetime: a spacetime that admits a time-orientation
a causal vector field T^a.

X^a is said future-directed if it lies in the same lightcone of T^a, and past-directed otherwise.

For $r > 2M$ we chose $k = \frac{\partial}{\partial t}$ as time-orientation but for $r < 2M$ $k = \frac{\partial}{\partial v}$ which is spacelike for $r < 2M$.
However we can use $\pm\frac{\partial}{\partial r}$ as time-orientation, because $\frac{\partial}{\partial r}$ is null.
We define $-\frac{\partial}{\partial r}$ as time orientation for $r > 0$, which is tangent to infalling radial null geodesics.

Proposition: Let $x^\mu(\lambda)$ be any future-directed causal curve.
Assume $r(\lambda_0) \leq 2M$ then $r(\lambda) \leq 2M$ for $\lambda \geq \lambda_0$.

Proof in the lecture notes.

So the spacetime for $r \leq 2M$ lies inside the BH, while for $r > 2M$ it is possible to find a future-directed causal curve to $r \to \infty$, so the spacetime is not inside the BH.

5. White holes

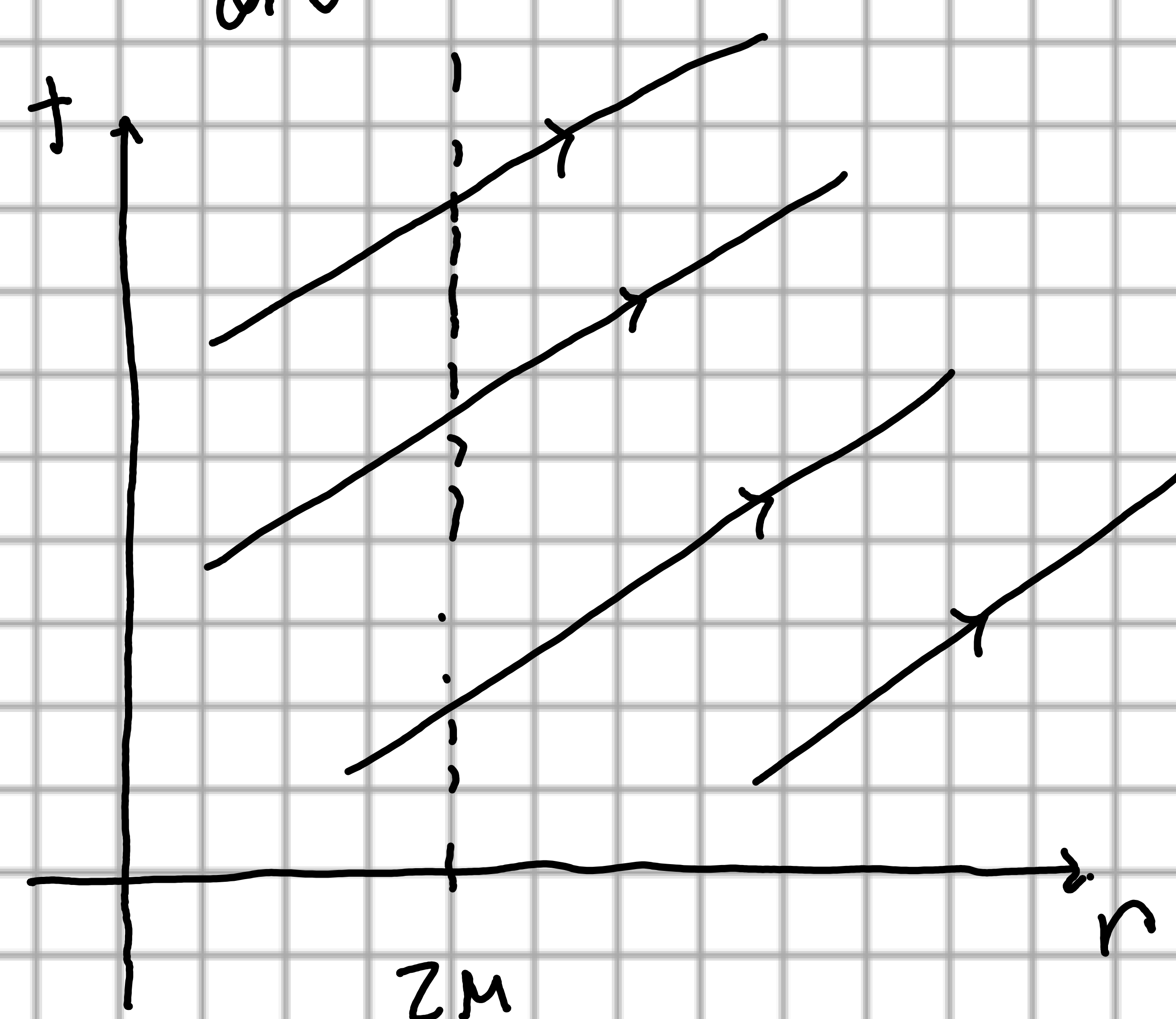
What if instead of v, we would have chosen $u = t - r_*$. In this case

$$ds^2 = -\left(1 - \frac{2M}{r}\right)du^2 - \left(du\,dr + dr\,du\right) + r^2 d\Omega^2$$

We have extended the metric to $r>0$, with smooth $r=2M$. However, $r<2M$ is not the same as before. Note that now

$$\frac{dr}{d\tau} = 1$$

so



which means that starting from $r=0$ one can go through $r=2M$ to $r>2M$ in finite time: in contradiction with Black Hole region. So any time like curve from $r<2M$ goes through $r=2M$ and no signal can be sent from $r>2M$ to $r<2M$: White Hole.

They are the time reversal of the Black Holes $(v \to -u)$ and believed to unphysical and unstable

6. Kruskal extension

Can we relate black and white holes regions? Yes

We define

$$U = -\exp\left(-\frac{u}{4M}\right) \quad , \quad V = \exp\left(\frac{v}{4M}\right)$$

such that $U<0$ and $V>0$, moreover

$$UV = -\exp\left(\frac{r_*}{2M}\right) = -\exp\left(\frac{r}{2M}\right)\left(\frac{r}{2M}-1\right)$$

$$\frac{V}{U} = -\exp\left(\frac{t}{2M}\right)$$

So one can express

$$U = \left(\frac{r}{2M}-1\right)^{1/2} \exp\left(\frac{r-t}{4M}\right)$$

$$V = \left(\frac{r}{2M}-1\right)^{1/2} \exp\left(\frac{r+t}{4M}\right)$$

so that

$$ds^2 = -\frac{16M^3}{r(U,V)} \exp\left(-\frac{r}{2M}\right)\left(dU\,dV + dV\,dU\right) + r^2 d\Omega^2$$

Non-singular at $r=2M$!

It is somehow better to work with coordinates where one is timelike and the rest is spacelike (U, V are null coordinates), so we define:

$$\tilde{V} = \frac{1}{2}(V - U) = \left(\frac{r}{2M} - 1\right)^{1/2} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right)$$

$$\tilde{U} = \frac{1}{2}(V + U) = \left(\frac{r}{2M} - 1\right)^{1/2} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right)$$

such that

$$ds^2 = \frac{32 M^3}{r(\tilde{U},\tilde{V})} e^{-\frac{r(\tilde{U},\tilde{V})}{2M}} \left(-d\tilde{U}^2 + d\tilde{V}^2\right) + r^2 d\Omega^2$$

where $r(\tilde{U},\tilde{V})$ is defined by

$$\left(\tilde{V}^2 - \tilde{U}^2\right) = \left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}}$$

These coordinates are Kruskal-Szekeres coordinates

6. Kruskal diagram (in the lecture notes it uses U, V)

The KS coordinates have nice features.

1. radial null curves are flat:

$$\tilde{U} = \pm \tilde{V} + \text{const.}$$

2. $r = 2M$ is at

$$\tilde{U} = \pm \tilde{V}$$

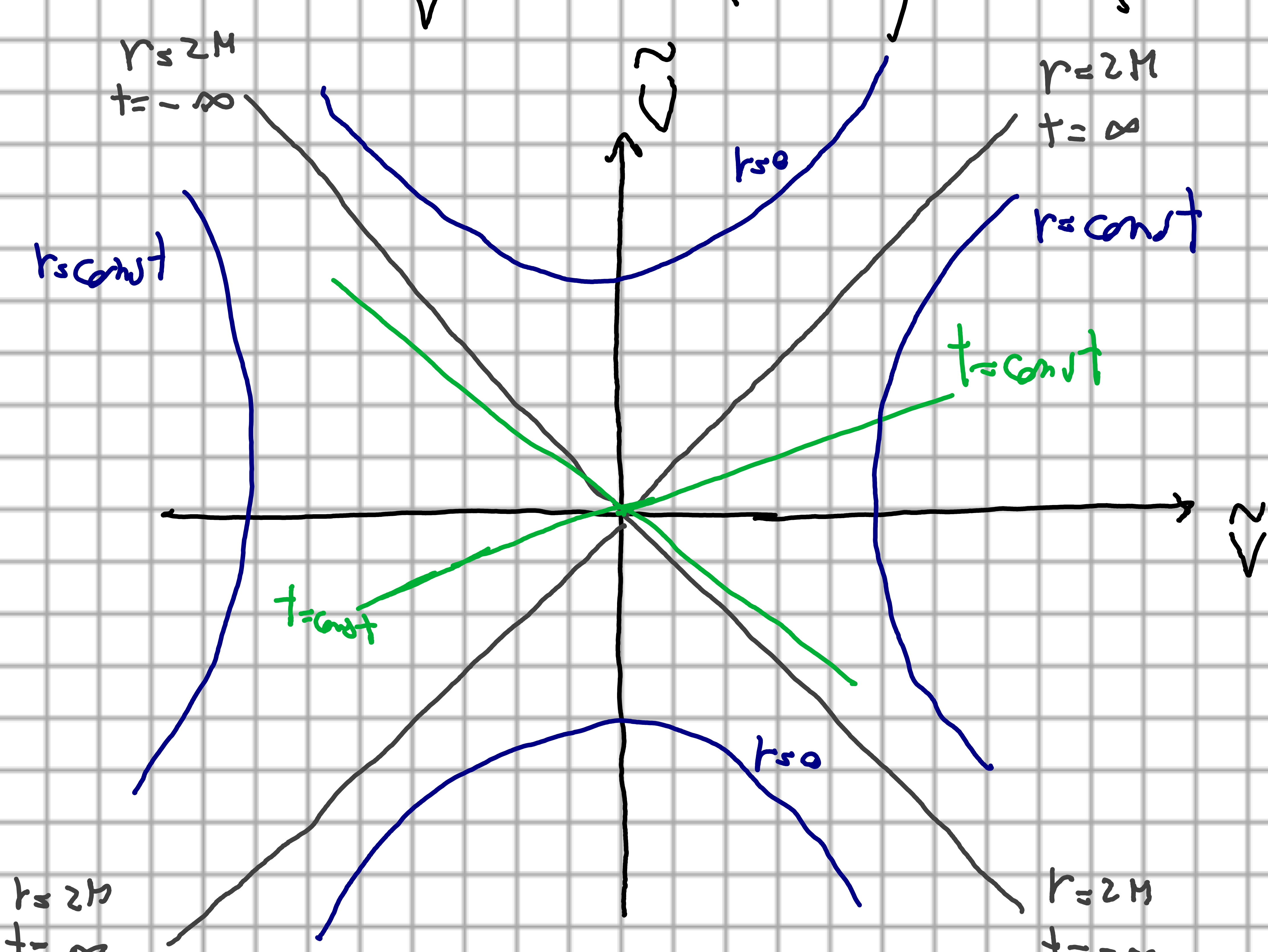
3. Any $r = \text{const.}$ surface are such that

$$\tilde{V}^2 - \tilde{U}^2 = \text{const.} \quad \left(\text{hyperboles in } (\tilde{V}, \tilde{U})\text{-plane}\right)$$

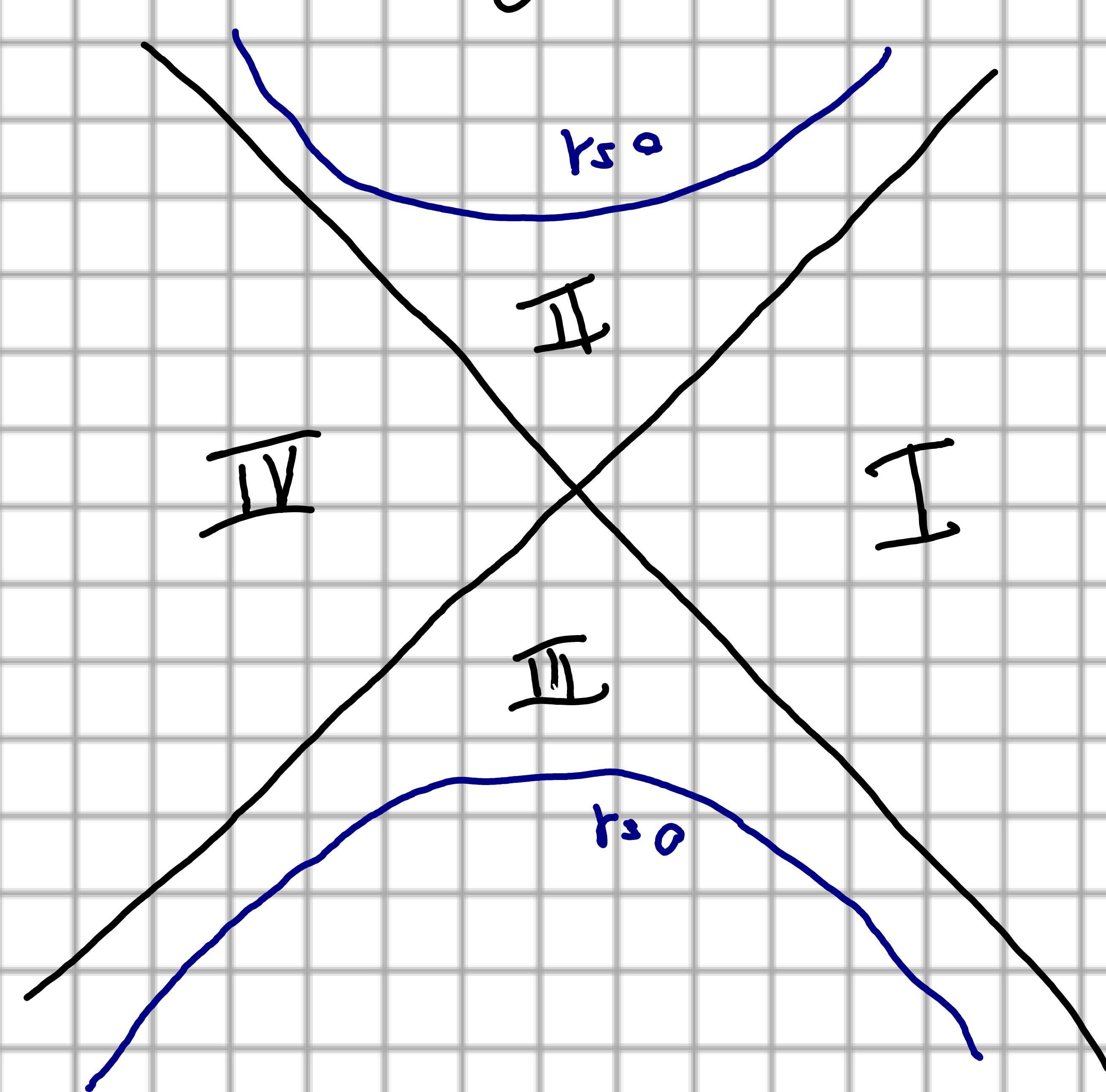
4. $t = \text{const}$ are

slope

$$\frac{\tilde{U}}{\tilde{V}} = \tanh\left(t/(4M)\right) \quad \left(\text{lines on } (\tilde{V}, \tilde{U})\text{-plane}\right)$$



We have then 4 regions



We started from I and with future directed null rays we reached II by past-directed null rays we reached III. By moving with spacelike geodesics we would have reached IV.

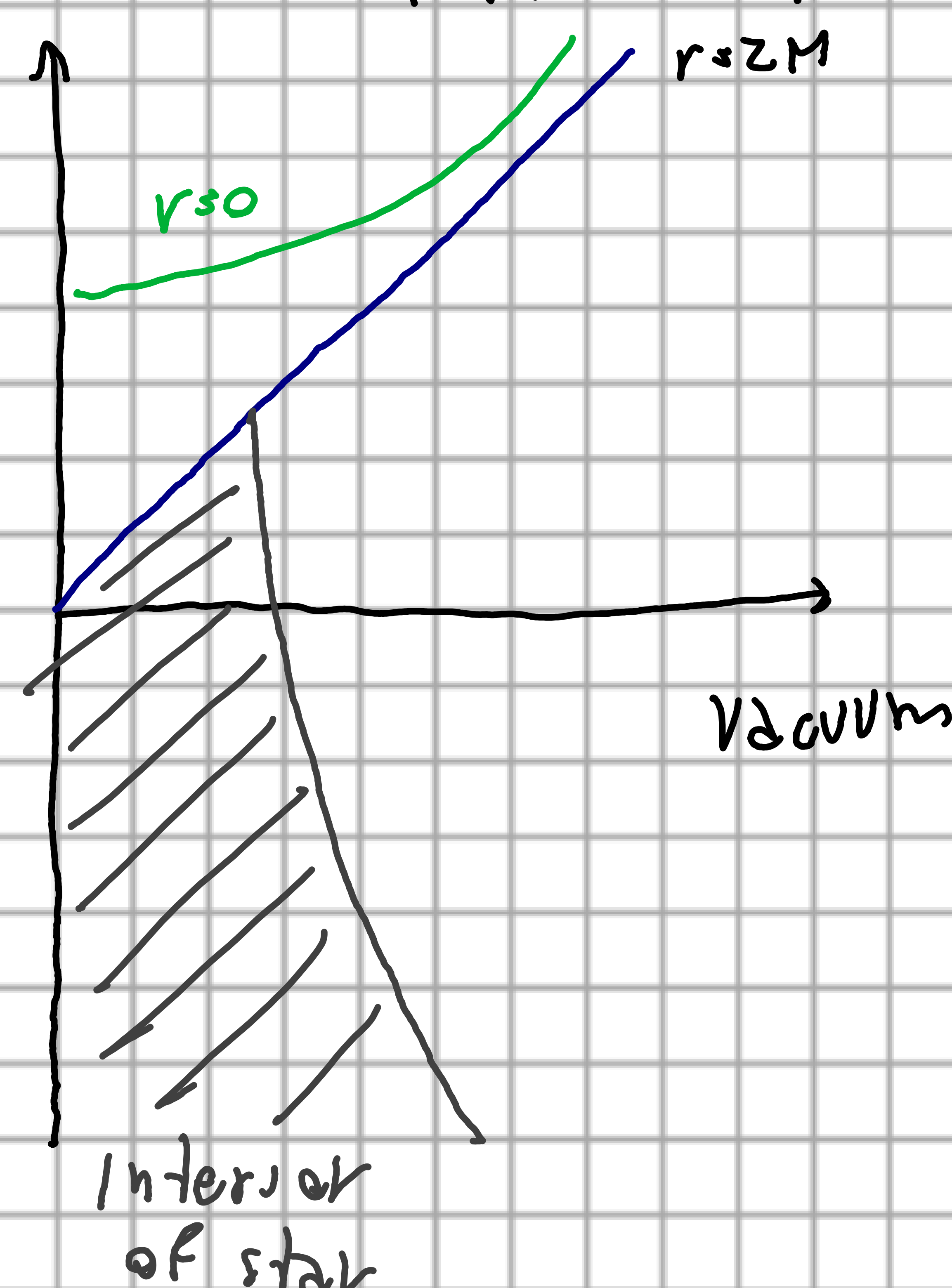
Region II is the Black Hole: once we travel from I to II we can't go back.

Not only you can't escape back to I but you must follow a path that decreases r.

Region III is the White hole; you can't reach it but things can escape to us.

Region IV can't be reached from I either forward or backword in time. It's a mirror image of our region. One can think to go from I to IV via a wormhole as we'll see in Section 7, however most of Kruskal diagram is unphysical.

Remember that Schwarzschild metric is only valid in vacuum, e.g. outside a star. If the star has a radius $r > 2M$ we never need to worry about any event horizon. If we include the timelike worldline corresponding to the surface of a collapsing star and we replace the region to the left of this line with the (non-vacuum) spacetime corresponding to the star's interior, from Kruskal diagram we are left with



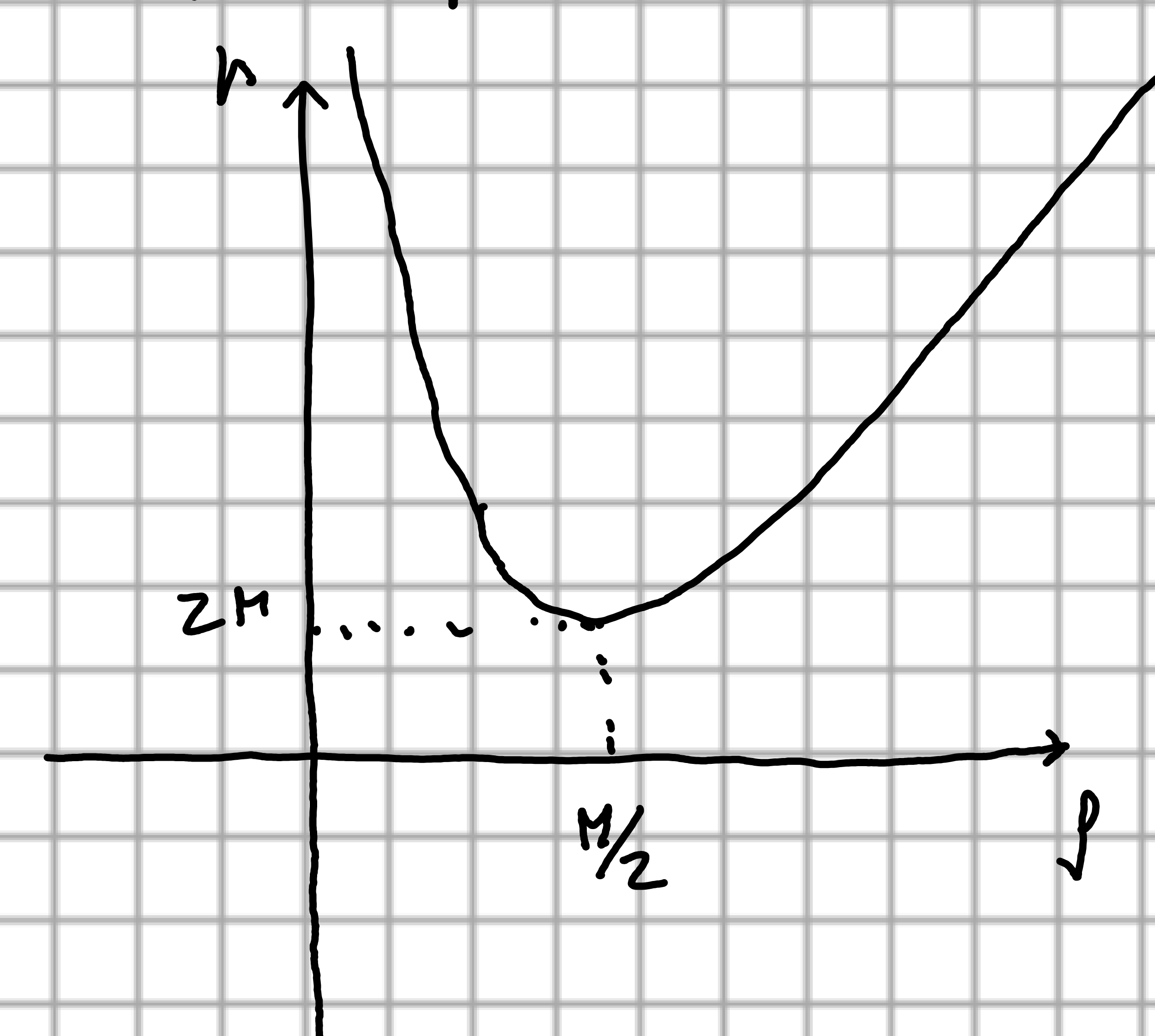
The interior is not described by Schwarzschild so there is no reason to think about white holes and wormholes.

7. Einstein-Rosen Bridge

Recall that

$$\frac{\tilde{U}}{\tilde{V}} = \tanh\left(\frac{t}{4M}\right)$$

So in the $(\tilde{U}, \tilde{V})$-plane surface at constant t are lines. We can think to go from region I to IV in this way.



Let us define

$$r = \rho + M + \frac{M^2}{4\rho}$$

and we choose region I for $\rho > M/2$
and region IV for $0 < \rho < M/2$

Then

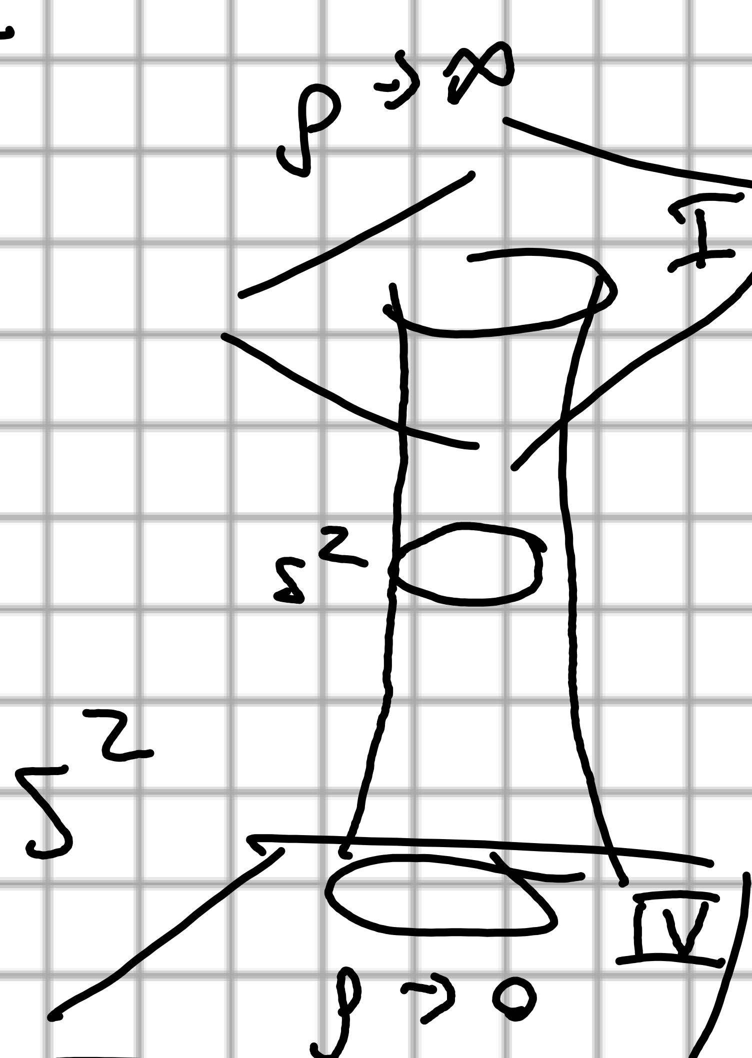
$$ds^2 = -\left(\frac{1-\frac{M}{2\rho}}{1+\frac{M}{2\rho}}\right)^2 dt^2 + \left(1+\frac{M}{2\rho}\right)^4 \left(d\rho^2 + \rho^2 d\Omega^2\right)$$

isotropic coordinates

If $\rho \to \frac{M^2}{4\rho}$ we exchange region I with region IV

At constant t:

$$ds^2 = \left(1+\frac{M}{2\rho}\right)^4 \left(d\rho^2 + \rho^2 d\Omega^2\right)$$



which is a Riemannian 3-manifold with topology $\mathbb{R} \times S^2$ and the throat has minimal radius at $\rho = \frac{M}{2}$, equal to $r = 2M$

8. Extendibility

A spacetime (M, g) is extendible if it is isometric to a proper subset of another spacetime (M', g') called extension of (M, g)

(M, g) is Schwarzschild solution for $r > 2M$, (M', g') is the Kruskal solution, which is inextendible (not extendible)

9. More on singularities

We saw in section 1.1 that $r=0$ is a curvature singularity, while from section 3 we realized that $r=2M$ is only a coordinate singularity.

In general there can be other kind of singularities, such as conical or Taub-NUT etc. so we need a way to define what we mean by singular. First

Definition: $p \in M$ is a future endpoint of a future-directed causal curve $\gamma : (a, b) \to M$ if for any neighbourhood Θ of p, there exists t_0 such that $\gamma(t) \in \Theta$ $\forall t > t_0$.

We say that γ is future-inextendible if it has no future endpoint. (analogously for past endpoints and inextendibility)

γ is inextendible if it is both past- and future-inextendible

Definition: A geodesic is complete if an affine parameter for the geodesic extends to $\pm\infty$. A spacetime is geodesically complete if all inextendible causal geodesics are complete

Consequence: Minkowski is geodesically complete, as is the spacetime describing a static spherical star.

Kruskal spacetime is geodesically incomplete because some geodesics have $r \to 0$ in finite affine parameter and cannot be extended to infinite affine parameter.

So a spacetime is singular if it is both geodesically incomplete and inextendible.

A. Showing the formula in page 6

Recall the Lie derivative for vectors

$$\mathcal{L}_V U^\mu = [V, U]^\mu$$

and for 1-form

$$\mathcal{L}_V \omega_\mu = V^\nu \partial_\nu \omega_\mu + (\partial_\mu V^\nu) \omega_\nu$$

We find that the Lie derivative of the metric is

$$\mathcal{L}_V g_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu = 2 \nabla_{(\mu} V_{\nu)}$$

A symmetry of some tensor T generated by some vector field V^μ is such that

$$\mathcal{L}_V T = 0$$

The symmetries of the metric are called isometries and V^μ is called Killing vector field. Let's call it K^μ then:

$$\mathcal{L}_K g_{\mu\nu} = 0 \Rightarrow \nabla_{(\mu} K_{\nu)} = 0 \qquad \text{(A)}$$

Killing's equation

Now consider a conserved quantity for a free particle with tangent vector $\frac{dx^\mu}{d\lambda}$ then:

$$\frac{dx^\nu}{d\lambda} \nabla_\nu \left(K_\mu \frac{dx^\mu}{d\lambda} \right) = \underbrace{\frac{dx^\nu}{d\lambda} \frac{dx^\mu}{d\lambda} \nabla_\nu K_\mu}_{\text{symmetrize and use (A)}} + \underbrace{K_\mu \frac{dx^\nu}{d\lambda} \nabla_\nu \left(\frac{dx^\mu}{d\lambda} \right)}_{\text{geodesic equation } = 0}$$

$$= 0$$

$$\Rightarrow K_\mu \frac{dx^\mu}{d\lambda} = \text{const.}$$