III From Kruskal to Penrose

7) Penrose Diagrams

Goal: Understand asymptotic behavior of dynamics

Idea: Use conformal compactification:

i) Conformal transformation of metric:

$$g \to \bar{g} = \Omega^2 g \,, \qquad \Omega(x) \text{ real}$$

$\longrightarrow$ leaves causal structure invariant, i.e.

$$\text{sign}\left(ds^2_g\right) = \text{sign}\left(ds^2_{\bar{g}}\right)$$

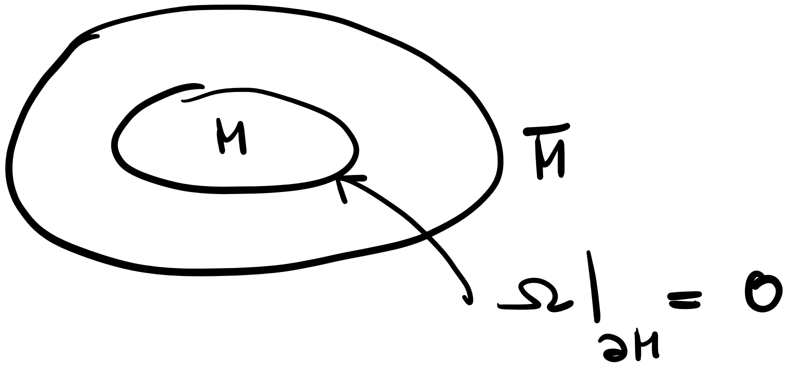
ii) Pick conformal factor s.t.

$\Omega(x) \longrightarrow 0$ at points at infinity w.r.t. g

$\Rightarrow$ Bring ∞ to finite distance in $\bar{g}$

iii) Embed $(M, \bar{g}) \subset (\bar{M}, \bar{g})$ "extension"

s.t. ∂M = loci at ∞ in g



Note: $\bar{g}$ is "unphysical", only g satisfies Einstein eqn. in general

<u>Example:</u> Flat Minkowski $\mathbb{R}^{1,3}$

- Starting point. Polar coordinates

$$ds_g^2 = -dt^2 + dr^2 + \underbrace{r^2\, d\omega^2}_{S^2 \text{ of radius } r}$$

Infinity: at $t = \pm\infty$, $r = \infty$

- Define: $\boxed{u = t - r, \qquad v = t + r}$

$$\Rightarrow \quad -\infty < u \leq v < \infty$$

$$ds_g^2 = - \, du\, dr + \frac{1}{4}(u-v)^2\, d\omega^2$$

- Goal: Bring ∞ to finite coordinate values:

Def: $\boxed{u = \tan p, \qquad v = \tan q}$

$$-\frac{\pi}{2} < p \leq q < \frac{\pi}{2}$$

"Infinity" at $|p| = \frac{\pi}{2}$ or $|q| = \frac{\pi}{2}$

$$ds^2_g = \underbrace{(2\cos p\cos q)^{-2}}_{\to\infty \text{ at } |p|=\frac{\pi}{2},\ |q|=\frac{\pi}{2}} \underbrace{\left(-4\,dp\,dq + \sin^2(p-q)\,dw^2\right)}_{=:\ ds^2_{\bar g}}$$

- Conformal trafo to unphysical metric

$$\bar g = \Omega^2 g \quad , \quad \Omega = (2\cos p\cos q)$$

$$ds^2_{\bar g} = -4\,dp\,dq + \sin^2(p-q)\,dw^2$$

- Back to "diagonal metric":

i.e.

$$t+r = \tan\left(\frac{1}{2}(T+\chi)\right)$$

$$t-r = \tan\left(\frac{1}{2}(T-\chi)\right).$$

$$\boxed{\begin{aligned} T &= q+p \\ \chi &= q-p \end{aligned}} \quad \begin{aligned} &\in (-\pi,\pi) \\ &\in [0,\pi) \end{aligned}$$ important!

$$\boxed{ds^2_{\bar g} = -dT^2 + d\chi^2 + \sin^2\chi\, dw^2}$$

- Where is "infinity"

i) Timelike infinity

$i^\pm$: $t \to \pm\infty$, r finite

$\iff u, v \to \pm\infty$, $u = v$

$\iff p, q \to \pm\pi/2$, $p = q$

$$\boxed{i^\pm : \quad T = \pm\pi, \quad \chi = 0}$$

ii) Spacelike infinity

i^0 : $r \to \infty$, t finite

$\Longleftrightarrow p = -\frac{\pi}{2}, \quad q = +\frac{\pi}{2}$

$$\boxed{i^0 : \quad T = 0, \quad \chi = \pi}$$

iii) Null infinity

$\mathcal{J}^\pm$: $\left\{\begin{matrix}\text{future}\\ \text{past}\end{matrix}\right\}$ null hypersurfaces on which radial null geodesics $(\theta, \varphi = \text{const})$ end

$\mathcal{J}^+$: future null infinity: $t + r \to \infty$, $t - r = \text{const}$.

$\Longleftrightarrow \quad v \to \infty, \quad u = \text{const}$

$\Longleftrightarrow \quad q = \pi/2, \quad -\frac{\pi}{2} < p < \pi/2$

$$\boxed{\mathcal{J}^+ : T + \chi = +\pi}$$

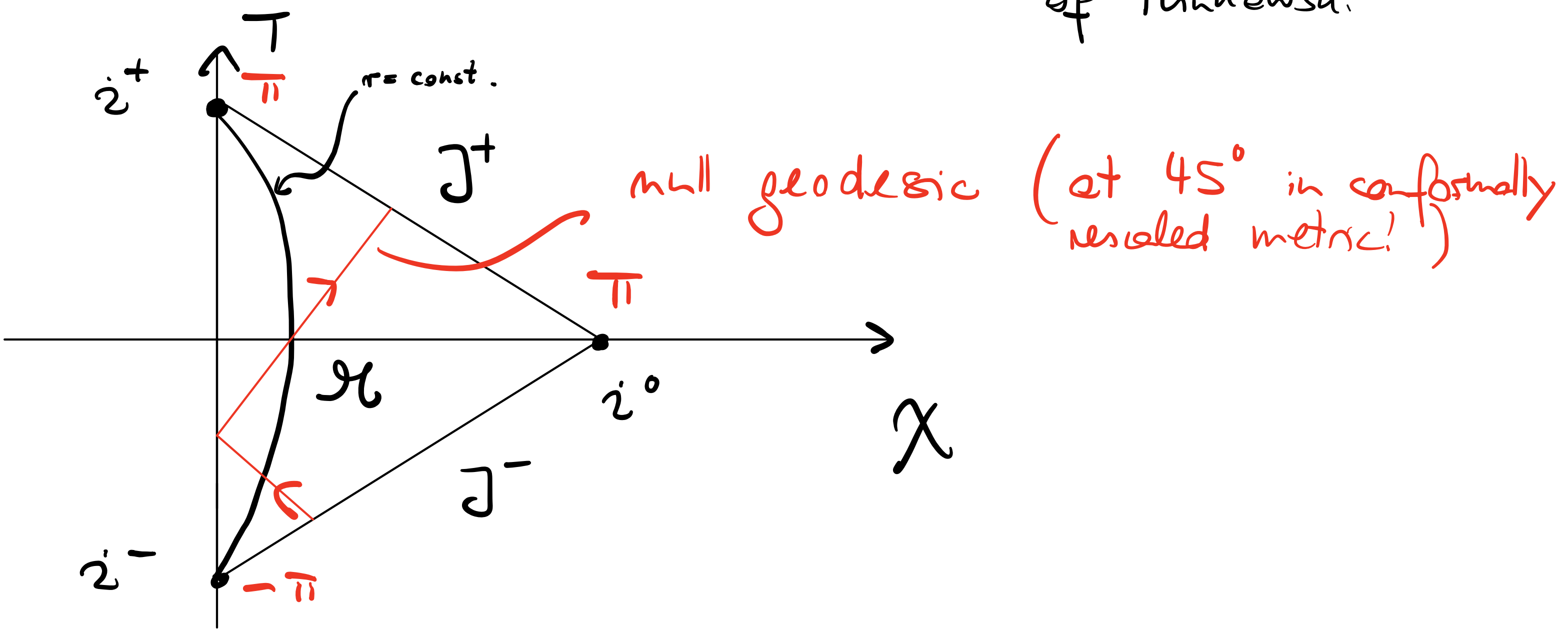
$\mathcal{J}^-$: past null infinity: $t - r \to -\infty$, $t + r = \text{const}$

$\Longleftrightarrow u \to -\infty$, v const

$\Longleftrightarrow p = -\frac{\pi}{2}, \quad -\frac{\pi}{2} < q < \pi/2$

$$\boxed{\mathcal{J}^- : T - \chi = -\pi}$$

Projection onto (T, χ) plane : Penrose diagram of Minkowski



* Each point represents an S^2 $((\theta, \varphi))$!
* Boundary of Penrose diagram = infinity of original spacetime

- Can extend Minkowski w/ rescaled metric

$$(M, \bar{g}) \qquad T \in (-\pi, \pi), \quad \chi \in [0, \pi)$$

to

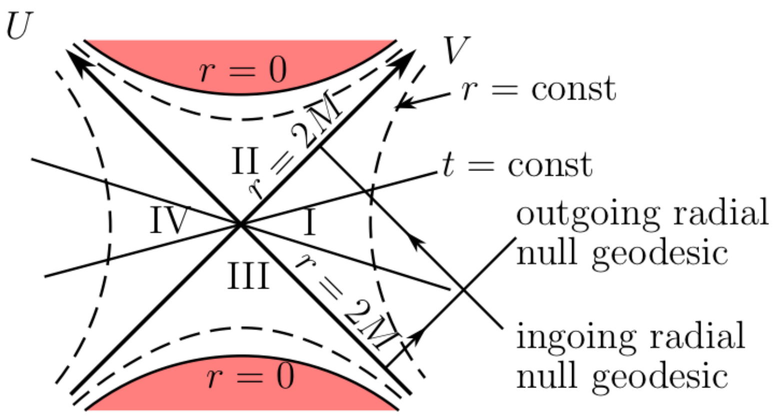
$$(\bar{M}, \bar{g}) \qquad T \in (-\infty, \infty), \quad \chi \in [0, \pi]$$ (Einstein static universe)

with

$$\bar{M} \supset \partial M = i^+ \cup i^- \cup i^0 \cup \mathcal{J}^+ \cup \mathcal{J}^-$$

2) Penrose Diagram for Kruskal Spacetime

Reminder: Kruskal extension of BH:



E.g.: Region I: $U = -e^{-u/4M}, \quad V = +e^{v/(4M)}$

$\left(dr^* = \dfrac{dr}{1 - 2M/r}\right) \qquad u = t - r_*, \qquad v = t + r_*$

$$\Rightarrow \quad UV = -e^{r_*/2M} = -e^{r/2M}\left(\frac{r}{2M} - 1\right)$$

$$V/U = -e^{t/2M}$$

From Kruskal to Penrose diagram:

* Define suitable coordinates ψ, χ of finite range
* Find conf. factor Ω^2 s.t. $\Omega = 0$ at infinity

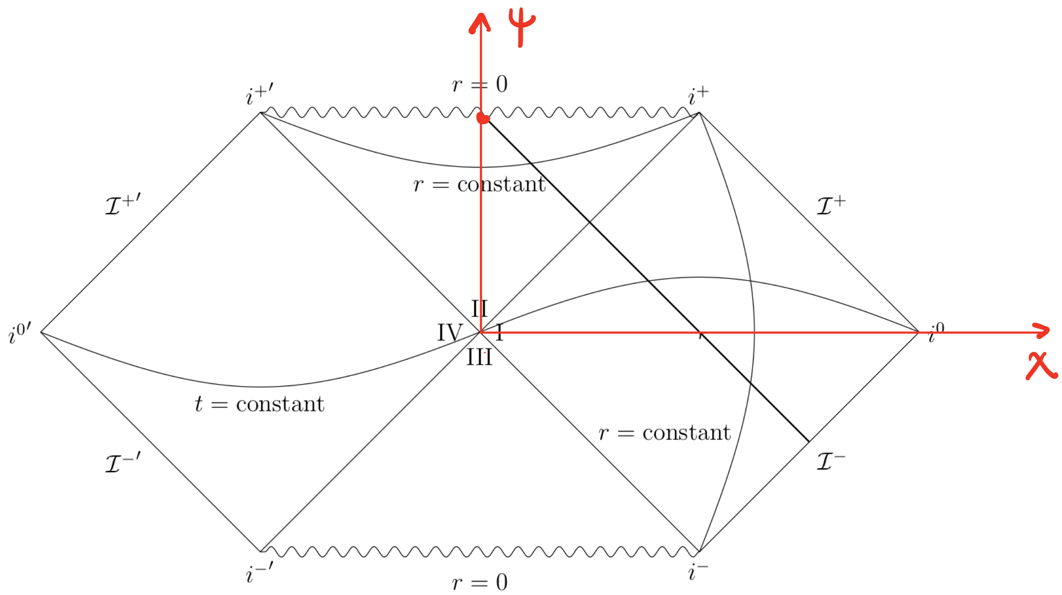
Explicitly: $V = \tan \frac{1}{2}(\psi + \chi)$

$U = \tan \frac{1}{2}(\psi - \chi)$

$ds^2_g = f(r)\left(-d\psi^2 + d\xi^2\right) + r^2 d\omega^2$

* Expect: Spacetime at ∞ : 2 copies of ∞ of Minkowski (I, IV)
* Pick Ω^2 s.t. $r = 0$ is straight line

*

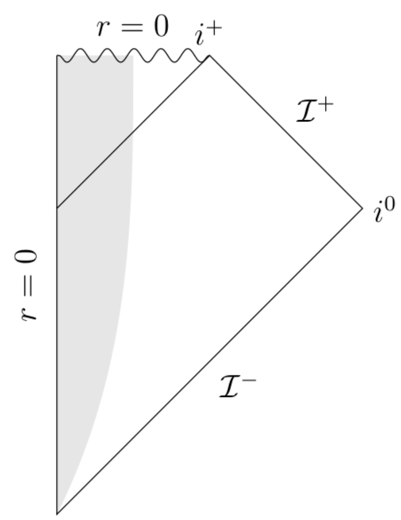


Novelty compared to Minkowski:

$\bar{g}$ (unphysical conformally rescaled metric) singular at $i^{\pm}$, $i'^{\pm}$ (due to intersection with $r = 0$)

& also at i^0, i'^0

Penrose diagram of gravitational collapse:



IV) Reissner – Nordström BHs

Charged BHs of conceptual importance in string theory, quantum gravity/Swampland

Einstein – Maxwell Theory

$$S = \frac{1}{16\pi G}\int \sqrt{-g}\left(R + F_{\mu\nu}F^{\mu\nu}\right) \qquad G \equiv 1$$

Generalisation of Birkhoff's Theorem:

$\exists$ unique solution to Einstein–Maxwell e.o.m.

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi T_{\mu\nu}(F), \quad \nabla^\mu F_{\mu\nu} = 0, \quad dF = 0$$

with non-constant area-radius function & spherical symmetry:

$$ds^2 = -\frac{\Delta}{r^2}dt^2 + \frac{r^2}{\Delta}dr^2 + r^2 d\Omega^2 \qquad \Delta = r^2 - 2Mr + e^2$$

$$A = -\frac{Q}{r}dt - P\cos\theta d\phi \qquad e = \sqrt{Q^2 + P^2}$$

Q: electric charge, P: magnetic charge

$\longrightarrow$ static (outside horizon r_+) & asymptotically flat.

At $r=0$: $\exists$ curvature singularity

From

$$\Delta = (r-r_+)(r-r_-) \quad , \quad r_\pm = M \pm \sqrt{M^2 - e^2}$$

(when it exists)

see :

i) If $M < e$, then $\Delta > 0 \quad \forall r > 0$

$\Longrightarrow$ $r=0$ is a naked singularity, which is forbidden by Weak Cosmic Censorship conjecture

Consistently : Spherically symmetric ball of matter w/ $M < e$ is self-repulsive

$\Longrightarrow$ no gravitational collapse

ii) $M \geq e$ Extremality bound for RN solutions

$M = e$: $r_+ = r_-$ extremal

$M > e$: $r_+ > r_- > 0$ subextremal

($M < e$: No RN solution super-extremal)

Subextremal RN BH

Assume $M > e \implies 0 < r_- < r_+$

In region $r > r_+$, define ingoing Eddington-Finkelstein coordinates:

- $v := t + r_*$ $\qquad dr_* = \frac{r^2}{\Delta} dr$

$$r_* = r + \frac{1}{2\kappa_+}\log|\frac{r-r_+}{r_+}| + \frac{1}{2\kappa_-}\log|\frac{r-r_-}{r_-}| + \text{const.} \qquad \kappa_\pm = \frac{r_\pm - r_\mp}{2r_\pm^2}$$

Significance: $v = \text{const}$ is ingoing null geodesic!

In coordinates (v, r, θ, φ):

$$ds^2 = -\frac{\Delta}{r^2} dv^2 + 2\, dv\, dr + r^2 d\Omega^2$$

smooth $\forall\, r \neq 0 \implies$ Singularity at r_+ ($\&$ r_-) is only coordinate sing.)

$\implies$ Can analytically continue to $0 < r < r_+$

Note: Killing vector $k = \frac{\partial}{\partial t} = \frac{\partial}{\partial v}$ w/ $k^2 = g_{vv}$

$$\left.\begin{array}{ll} k^2 < 0 & r > r_+ \\ k^2 > 0 & r_- < r < r_+ \end{array}\right\}$$ analogous to Schwarzschild for $r > r_s$ vs $r < r_s$

but: $k^2 < 0 \qquad 0 < r < r_-$!

By same reasoning: $r = r_+$ is (outer) horizon

$\Longrightarrow$ Black hole

Global structure of RN spacetime

<u>Goal:</u> Find maximal analytic, spherically symmetric extension of RN solution for $r > r_+$

$\Longrightarrow$ Kruskal coordinates

<u>1) $r > r_+$</u>

$$u = t - r_* \quad , \quad v = t + r_*$$

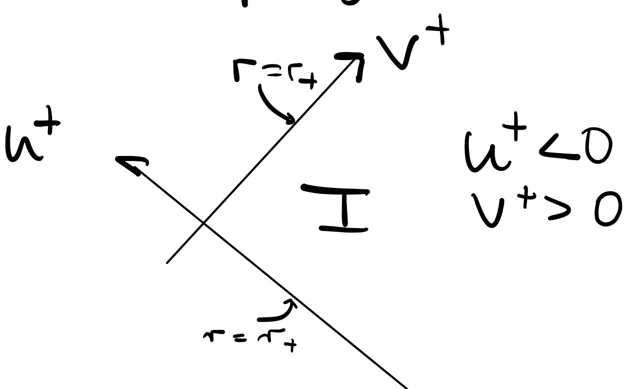
$$\kappa_\pm = \frac{r_\pm - r_\mp}{2r_\pm^2}$$

<u>Def:</u> $$U^+ = -e^{-\kappa_+ u} \quad , \quad V^+ = +e^{\kappa_+ v}$$

$$\Longrightarrow \quad U^+ < 0 \quad , \quad V^+ > 0$$

$$\& \quad -U^+ V^+ = e^{2\kappa_+ r} \left(\frac{r - r_+}{r_+}\right)\left(\frac{r_-}{r - r_-}\right)^{\kappa_+/|\kappa_-|}$$

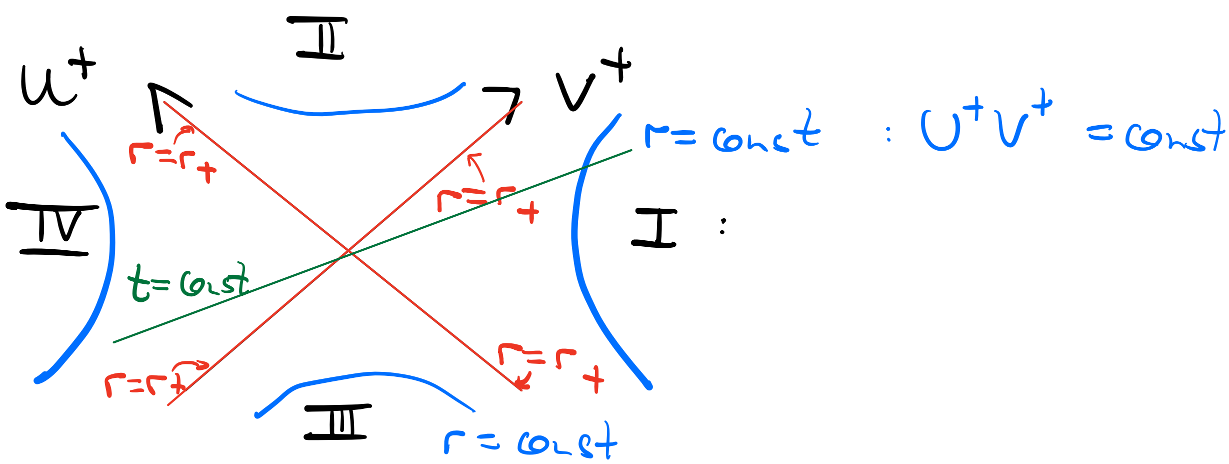
$$ds^2 = f(r) \quad dU^+ dV^+ + r^2 d\Omega^2$$



2) $r_+ > r > r_-$

Since ds^2 smooth at $u^+=0$ or $v^+=0$, can analytically continue to include also

$$u^+ \geq 0 \quad \& \quad v^+ \leq 0$$

$\Longrightarrow$ 

Note: $r=r_-$ lies at $u^+v^+=+\infty$

I, IV : $r > r_+$

II, III $r_+ > r > r_-$

3) $r < r_-$

EF coordinates guarantee extendibility to $r < r_-$ as well

Idea: Ingoing EF well defined on region II

$$v = t + r_* \implies t = v - r_*$$

Def: $u = t - r_* = v - 2r_*$ on region II

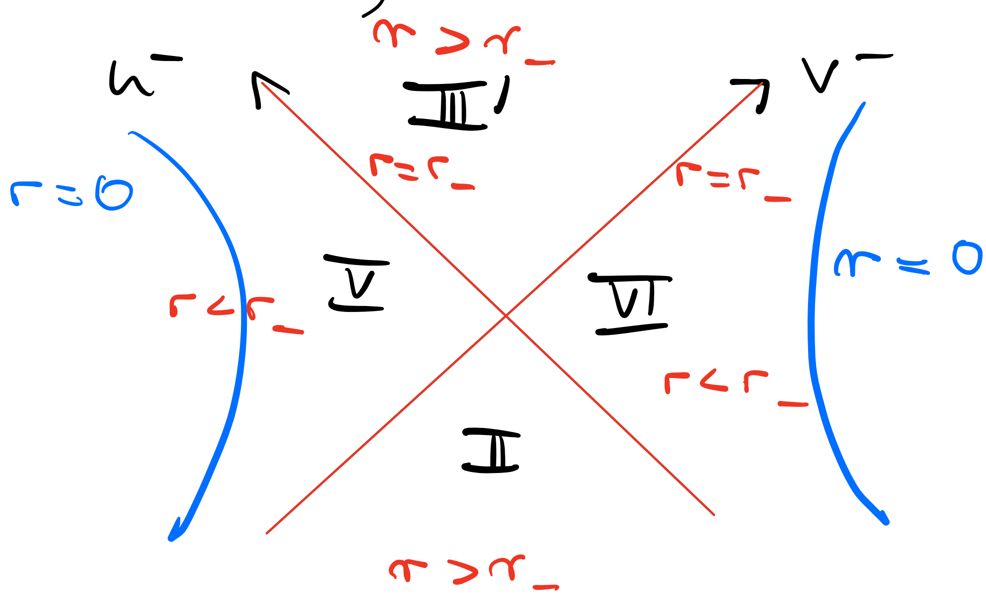
Def:
$$\left.\begin{aligned} U^- &:= -e^{\kappa_- u} \\ V^- &:= -e^{\kappa_- v} \end{aligned}\right\} \text{ in region II}$$

$$U^- V^- = e^{-2|\kappa_-| r}\left(\frac{r-r_-}{r_-}\right)\left(\frac{r_+}{r_+-r}\right)^{|\kappa_-|/\kappa_+}$$

$U^- V^- > 0$ in region II

Analytically continue solution to include also

$U^- \geq 0, \quad V^- \geq 0$

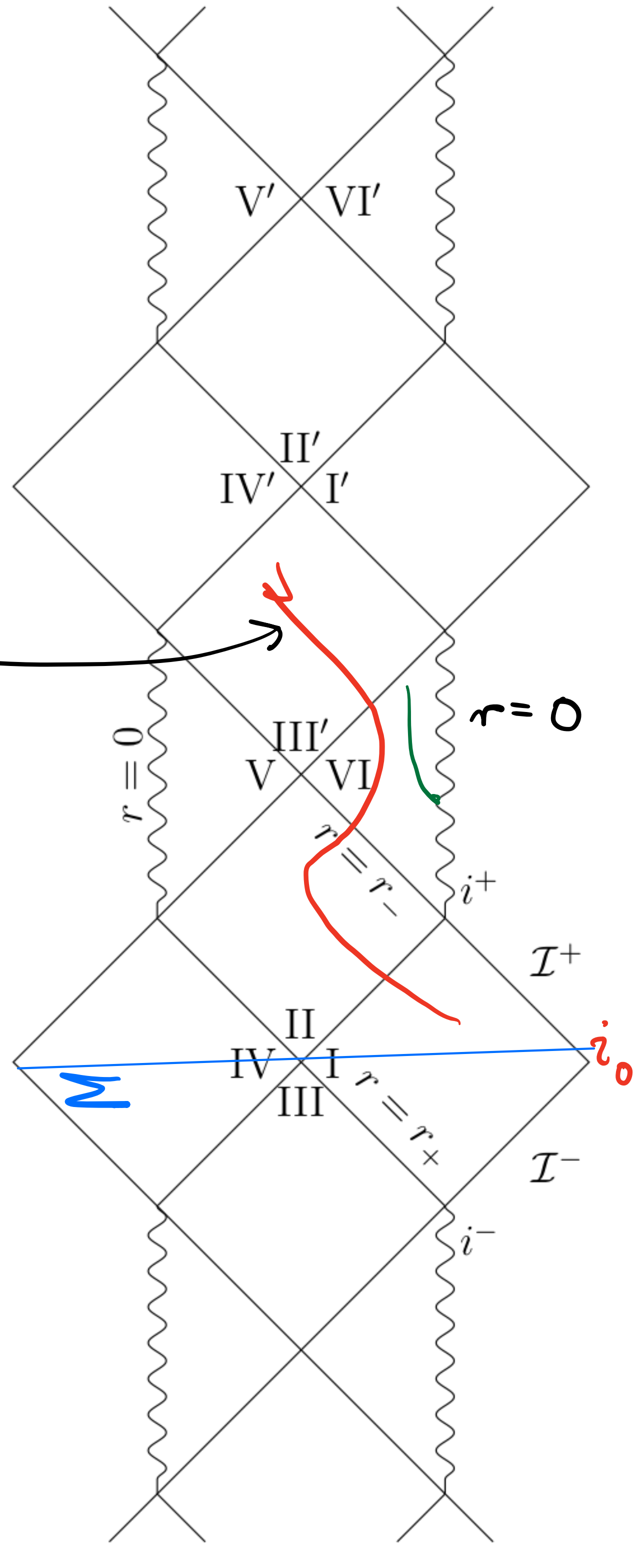


→ singularity! is timelike (different from Schwarzschild!)

Region III' ($r_+ > r > r_-$) turns out to be isometric to III

$\Rightarrow$ Analytically continue back to $r > r_+$ by defining $(U^+)'$, $(V^+)'$ & continue

Penrose diagram of maximally analytically exten-ded RN Spacetime



Singularity at $r = 0$ timelike ("locally naked")

$\Rightarrow$ Timelike & null geodesics can avoid singularity !

Reason:

Δ changes sign twice from $r > r_+ \to r_+ > r > r_- \to r_- > r$

Problem w/ RN spacetime

Physics in regions VI, V not uniquely determined by boundary conditions along spacelike hypersurface in I, IV

Reason: $\exists$ past-directed causal geodesics in VI ending at $r = 0$ not traversing Σ in diagram

$\Rightarrow$ Σ is not Cauchy surface for all of RN

i.e. $r = r_-$ is Cauchy horizon for Σ

This violates the Strong Cosmic Censorship conjecture

if this holds for any small perturbation of initial data on Σ that violate spherical symmetry or add massive sources

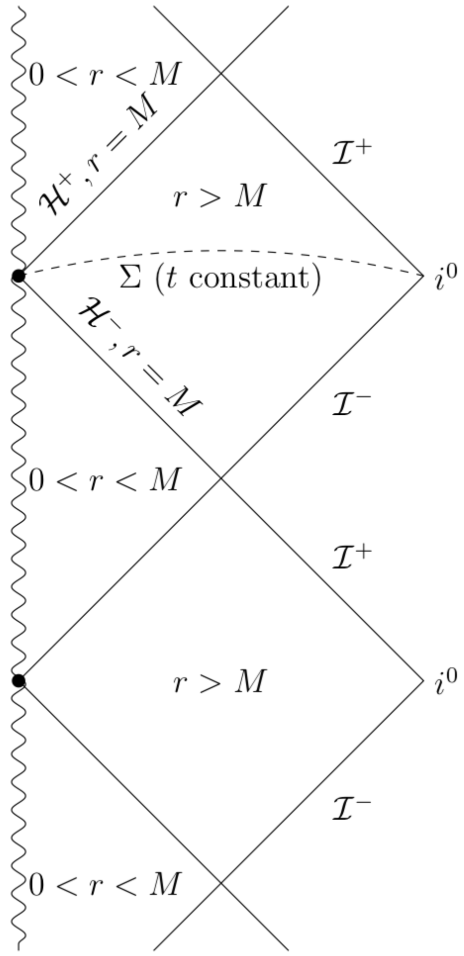
Conjecture: Such small perturbations lead to singularity at $r = r_-$ or in region II more generally

$\Rightarrow$ active field of Mathematical Relativity

Extremal RN

$e = M \Longrightarrow r_+ = r_- = M$

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega^2$$



$\mathcal{H}^\pm$: Cauchy horizons for Σ !

Near horizon geometry :

$r = M(1 + \lambda)$

$$ds^2 \cong \underbrace{-\lambda^2 dt^2 + M^2 \frac{d\lambda^2}{\lambda^2}}_{AdS_2} \times \underbrace{+ M^2 d\Omega^2}_{S^2}$$