Singularity theorems: Modern mathematical treatment

Leonardo García-Heveling*

November 27, 2023

Theorem 1 (Penrose singularity theorem 1965 [9]). *Let (M, g) be a smooth, 4-dimensional spacetime. Assume that*

(i) *(M, g) is globally hyperbolic and contains a non-compact Cauchy hypersurface,*

(ii) *the Einstein Equations hold for a matter model that satisfies the null energy condition,*

(iii) *(M, g) contains a (codimension 2) trapped surface $\mathcal{T}$.*

Then there is at least one future-incomplete null geodesic starting from $\mathcal{T}$.

Interpretation:

- Historical (see also Nobel prize text[1]): proves that under generic conditions, black holes are formed.
- Alternative/modern (see Senovilla [10]): proves that black holes have singularities inside. The argument of Senovilla is that assuming the existence of a trapped surface is almost like assuming from beforehand that there is a black hole (as in a region where even light cannot escape). Note that trapped surfaces usually (e.g. in Schwarzschild or Kerr) occur behind the event horizon, but in an arbitrary spacetime, it's not known if there will be an event horizon (according to Penrose's weak cosmic censorship conjecture, there should be, but it has not been proven).

The other very famous singularity theorem is due to Hawking.

Theorem 2 (Hawking singularity theorem [7]). *Let (M, g) be a smooth, 4-dimensional spacetime. Assume that*

(i) *(M, g) is globally hyperbolic,*

(ii) *the Einstein Equations hold for a matter model that satisfies the Strong Energy Condition (SEC), meaning $T_{ab} - \frac{1}{2} T g_{ab} X^a X^b \geq 0$ for every timelike vector X.*

*Mathematics Dept., Universität Hamburg. Email: `leonardo.garcia@uni-hamburg.de`

[1] Note that there it is not specified if the prize is for the theorem alone, or for all contributions (e.g. cosmic censorship conjectures) https://www.nobelprize.org/prizes/physics/2020/penrose/facts/

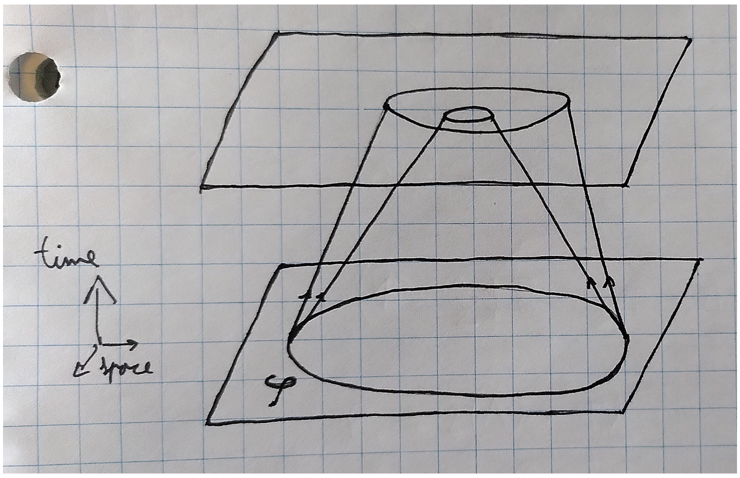


Figure 1: A trapped surface

(iii) *(M, g) contains a (codimension 1) Cauchy hypersurface Σ with expansion $H \geq \epsilon > 0$ for some constant ϵ.*

Then every timelike geodesic in M is past incomplete.

Here both the assumption and the conclusion are of a cosmological nature. Condition (iii) tells us that the Universe is expanding at some moment in time, with a uniform lower bound on the expansion rate. The conclusion is a Big Bang type singularity: it lies in the past and represents the beginning of the whole spacetime. Some more comments on Hawking's theorem:

- The SEC is stronger than the NEC (this can be seen by approximating null vectors by timelike ones). The SEC + Einstein Equations implies $R_{ab}X^aX^b \geq 0$, which is all that is used in the proof. Mathematicians often directly refer to this inequality for the Ricci as SEC (or NEC when X is null).
- A cosmological constant Λ (interpreted as part of the energy-momentum) violates SEC. The theorem can be adapted to allow for Λ, but then $\epsilon = \epsilon(\Lambda)$ is no longer arbitrary small.

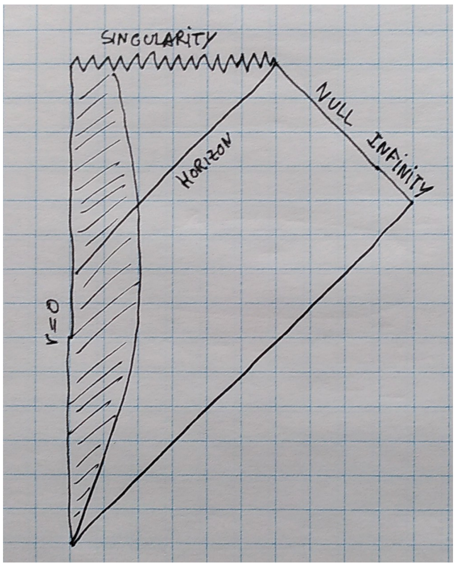
Hawking and Penrose also joined forces for a singularity theorem that does not assume global hyperbolicity:

Theorem 3 (Hawking–Penrose singularity theorem [7])**.** *Let (M, g) be a smooth, 4-dimensional spacetime. Assume that*

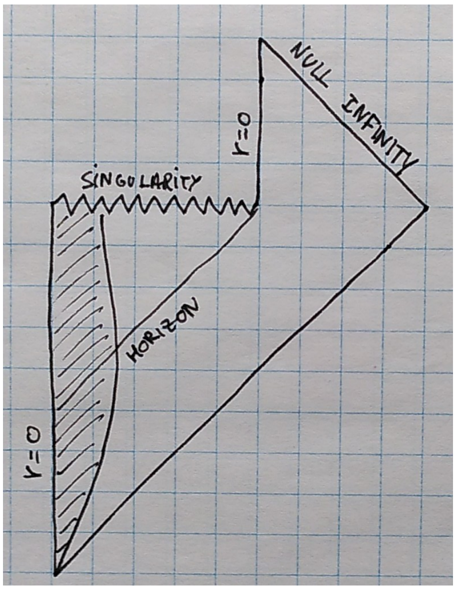
(i) *(M, g) contains no Closed Timelike Curves (CTCs) and is generic, meaning that for every causal geodesic γ, there exists at least one parameter value where $\dot{\gamma}_{[\alpha}R_{\beta]\pi\delta[\rho}\dot{\gamma}_{\sigma]}\dot{\gamma}^{\pi}\dot{\gamma}^{\delta} \neq 0$*

(ii) *the Einstein Equations hold for a matter model that satisfies the Strong Energy Condition (SEC),*

(iii) *(M, g) contains a (codimension 2) trapped surface $\mathcal{T}$.*



(a) Formation of a black hole in the Oppenheimer–Snyder model. The shaded region contains matter, the outside is vacuum.



(b) A black hole that forms and then evaporates

Then there is some past- or future-incomplete causal geodesic in M.

Here assumption (iii) can also be replaced by a cosmological assumption as in Hawking's theorem. The problem with Hawking–Penrose is that it does not tell us if the singularity is to the future or to the past, which makes it difficult to interpret. On the other hand, not assuming global hyperbolicity is a great advantage, especially if we want to take into account black hole evaporation due to Hawking ratiation. A spacetime containing evaporated black holes is believed to be non-globally hyperbolic (this is related to the informaton-loss paradox).

Recently, Minguzzi has shown a singularity theorem compatible with black hole evaporation which gives us more information about the structure of the singularity (at least as much as Penrose).

Theorem 4 (Minguzzi 2020 [8])**.** *Let (M,g) be a smooth, 4-dimensional spacetime. Assume that*

(i) (M,g) is past reflecting ($I^+(x) \subset I^+(y) \implies I^-(y) \subset I^-(x)$) and does not contain any compact spacelike hypersurfaces,

(ii) the Einstein Equations hold for a matter model that satisfies the null energy condition,

(iii) (M,g) contains a (codimension 2) trapped surface $\mathcal{T}$ and there are no CTCs crossing $\mathcal{T}$.

Then there is at least one future-incomplete null geodesic starting from $\mathcal{T}$.

Some comments:

- Here $I^+(x) := \{p \in M \mid \exists \text{ future-directed causal curve from } x \text{ to } p\}$, and I^- the same with past-directed.

- Globally hyperbolic spacetimes are past- and future reflecting ($I^+(x) \subset I^+(y) \iff I^-(y) \subset I^-(x)$), but for the evaporating black hole, the implication $\Longleftarrow$ is false.

- Asking that there are no compact spacelike hypersurfaces in M is weaker than sasking that there is one non-compact *Cauchy* hypersurface (this is not obvious; it turns out that global hyperbolicity puts topological restrictions on M).

- We even allow for CTCs (such as inside maximally extended Kerr) as long as they do not touch the trapped surface $\mathcal{T}$.

We have discussed how to relax the causality assumption in Penrose's theorem. There are also reasons to relax the other assumptions:

- The Null Energy Condition, because it is violated by e.g. quantum fields. This is meant in the following sense (cf. semi-classical Einstein Equation): If one takes a fixed background spacetime and a QFT on it, the expectation value of the energy-momentum tensor generally does not satisfy the NEC (it can even go to $-\infty$). However, some "averaged" or "smeared" versions of NEC hold, where some integral of the energy momentum is non-negative. Multiple works including [1, 2, 3, 4].

- The dimension (e.g. because of string theory extra dimensions). It is trivial to extend to higher spacetime dimension as long as the trapped surface is still of codimension 2. But if one wants different codimension (for example, if one wants *dimension* 2), then not trivial. The Hawking–Penrose theorem has been proven in this context [5], but there are some caveats about the energy condition.

- The smoothness of the spacetime metric g. We discuss this in a bit more detail now.

From a physical point of view, there is no reason to assume that the spacetime metric is smooth:

- Experimentally testing for (non-)smoothess is impossible, because it would require infinite resoultion.

- Some interesting spacetimes are non-smooth. For example Oppenheimer–Snyder, where it is assumed that the matter density is constant inside the star and 0 outside, causing a non-smooth transition at the surface.

This poses a problem: what if the geodesics predicted by Penrose's theorem are incomplete only because they encounter a point where the metric is non-smooth? Could it be that our incomplete smooth spacetime can be extended to a compete, non-smooth one? This would ruin the conclusion since non-smoothness is not per se a singularity (in the physical sense).

Theorem 5 (Graf 2020 [6])**.** *The Penrose singularity theorem holds for C^1 metric tensors.*

The metric tensor g is C^1 if it is continuous and $\frac{\partial}{x^k} g_{ab}$ exists and is continuous for any coordinate x^k. In general, the second partial derivatives could be discontinuous or even ill-defined. The callenges of considering C^1 metrics include:

- While the Christoffel symbols (which involve ∂g) are well-defined and continuous, they are not necessarily Lipschitz-continuous (as they would be in the smooth case). As a consequence, while the geodesic equation still has solutions for every initial condition, these solutions are no longer unique.
- The Riemann curvature tensor (which involves $\partial^2 g$) is not well-defined in the usual sense. However, it is still possible to define it in a distributional sense (think of Dirac δ). In this way, we can make sense of the NEC.
- The proof is based on approximating the metric by a family of smooth metrics.

References

[1] C. J. Fewster and G. J. Galloway. "Singularity theorems from weakened energy conditions". *Class. Quant. Grav.* 28 (2011), 125009. arXiv: `1012.6038 [gr-qc]`.

[2] C. J. Fewster and E.-A. Kontou. "A new derivation of singularity theorems with weakened energy hypotheses". *Class. Quant. Grav.* 37 (6) (2020), 065010. arXiv: `1907.13604 [gr-qc]`.

[3] C. J. Fewster and E.-A. Kontou. "A semiclassical singularity theorem". *Class. Quant. Grav.* 39 (7) (2022), 075028.

[4] B. Freivogel, E.-A. Kontou, and D. Krommydas. "The Return of the Singularities: Applications of the Smeared Null Energy Condition". *SciPost Phys.* 13 (1) (2022), 001. arXiv: `2012.11569 [gr-qc]`.

[5] G. J. Galloway and J. M. M. Senovilla. "Singularity theorems based on trapped submanifolds of arbitrary co-dimension". *Class. Quant. Grav.* 27 (2010), 152002.

[6] M. Graf. "Singularity theorems for C^1-Lorentzian metrics". *Comm. Math. Phys.* 378 (2) (2020), 1417–1450.

[7] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2023.

[8] E. Minguzzi. "A gravitational collapse singularity theorem consistent with black hole evaporation". *Lett. Math. Phys.* 110 (9) (2020), 2383–2396.

[9] R. Penrose. "Gravitational collapse and space-time singularities". *Phys. Rev. Lett.* 14 (1965), 57–59.

[10] J. M. M. Senovilla. "The influence of Penrose's singularity theorem in general relativity". *Gen. Rel. Grav.* 54 (11) (2022), 151. arXiv: `2206.13925 [gr-qc]`.