Black Hole Mechanics

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BASED ON:

- "The four laws of black hole mechanics" (1972) by J. M. BARDEEN, B. CARTER, S. W. HAWKING
- "Black holes" by H. REALL
- "Student book on black holes" by S. VANDOREN

GOAL:

Introduce the four laws that govern the mechanics of black holes from a CLASSICAL PERSPECTIVE. Formally, the laws are similar to the four laws of thermodynamics.

ZEROTH LAW OF BLACK HOLE MECHANICS

Recall the zeroth law of thermodynamics:

Bodies in thermal equilibrium among each other are characterized by the SAME TEMPERATURE.

The zeroth law of black hole mechanics states that some classes of black holes are also characterized by a constant quantity - the SURFACE GRAVITY, which we now define.

Consider a spacetime $\mathcal{M}$, with local metric g — we will denote this couple as $(\mathcal{M}, g)$.

Within this space we consider HYPERSURFACES Σ defined via

$$S(x) = \text{constant}$$

NORMAL VECTORS ℓ to the hypersurfaces are

$$\ell = \underbrace{N(x)}_{\text{normaliz.}} \underbrace{\partial^\mu S}_{\equiv \ell^\mu} \partial_\mu$$

NULL HYPERSURFACES Σ satisfy

$$\ell^2 \big|_\Sigma = \ell^\mu \ell_\mu \big|_\Sigma = 0$$

$\Rightarrow$ on Σ, ℓ is orthogonal to itself

We choose $N(x)$ such that

$$\ell^\sigma \nabla_\sigma \ell^\mu \big|_\Sigma = 0$$

With null hypersurfaces we can define:

KILLING HORIZON: a null hypersurface Σ for which there exists a KILLING VECTOR ξ that is orthogonal to Σ.

We fix the normalization of ξ in such a way that $\lim_{r\to\infty} \xi^2 = -1$.

Thus, in general, $\xi = f(x)\, \ell$, and ξ obeys:

$$\xi^\sigma \nabla_\sigma \xi^\mu \big|_\Sigma = \xi^\sigma \partial_\sigma \log f = \underset{\substack{\uparrow \\ \text{SURFACE GRAVITY}}}{\kappa}\, \xi^\mu$$

But how are Killing horizons related to black hole horizons?

THEOREM (Hawking, 1974):
Given a black hole spacetime that is:
- analytic
- stationary
- asymptotically flat

the black hole FUTURE HORIZON $\mathcal{H}^+$ is a Killing horizon.

How can we formally define the future horizon?

First, we identify the:

- BLACK HOLE REGION:

$$\mathcal{B} = \mathcal{M} - \left[\mathcal{M} \cap J^-(\mathcal{I}^+)\right]$$

causal past

future null infinity

needed because J^- and $\mathcal{I}^+$ are defined in the compactified $\mathcal{M}$

Let us recall how $\mathcal{I}^+$ and J^- are defined.

- Future null infinity: the locus where null geodesics end
- Causal past: the causal past of a set U is the union of U with the set of points of M which can be reached by future directed causal curves starting on U.

Intuitively: $J^-(\mathcal{I}^+)$ is the causal past of all null geodesic that are able to "escape" towards infinity

$\Rightarrow$ $M \wedge J^-(\mathcal{I}^+)$ is the portion of M where null paths (light rays) cannot escape and are "trapped".

Given the black hole region, we define:

- FUTURE EVENT HORIZON: $\mathcal{H}^+ = \dot{\mathcal{B}}$

We can then formalize the zeroth law as:

ZEROTH LAW OF BLACK HOLE MECHANICS

The surface gravity k is constant on the future event horizon $\mathcal{H}^+$ of a stationary black hole obeying the dominant energy condition, namely $T_{\mu\nu}K^\mu K^\nu$ $\forall$ timelike K^μ (weak energy condition) and $-T^\mu{}_\nu \tilde{K}^\nu$ must be future-pointing for every future pointing $\tilde{K}^\nu$.

EXAMPLES

1) Surface gravity of a Schwarzschild black hole

Use Eddington - Finkelstein incoming coordinates:

$$v = t + r_*, \quad r_* = r + 2M \log\left(\frac{r-2M}{2M}\right)$$

so that the metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dv^2 + 2\,dv\,dr + r^2 d\Omega^2$$

The metric is STATIC, and $\xi = \partial_t$ is a Killing vector.

Computing:

$$\xi^\alpha \nabla_\alpha \xi^\nu \Big|_\Sigma = \underbrace{\frac{1}{4M}}_{\text{constant } \kappa} \xi^\nu$$

2) Surface gravity for RN black hole

As above, one finds: $r_\pm = M \pm \sqrt{M^2 - Q^2}$

$$\xi^\alpha \nabla_\alpha \xi^\nu \Big|_\Sigma = \left(\frac{r_\pm - r_\mp}{2 r_\pm^2}\right) \xi^\nu$$

3) Surface gravity for a Kerr black hole

The metric is stationary, not just static

$$dv = dt + \frac{(r^2 - a^2)}{\triangle}dr \quad , \quad d\chi = d\phi + \frac{a}{\triangle}dr \ .$$

$$ds^2 = -\frac{(\triangle - a^2\sin^2\theta)}{\rho^2}dv^2 + 2dvdr - \frac{2a\sin^2\theta(r^2 + a^2 - \triangle)}{\rho^2}dvd\chi$$
$$- 2a\sin^2\theta d\chi dr + \frac{[(r^2+a^2)^2 - \triangle a^2 \sin^2\theta]}{\rho^2}\sin^2\theta d\chi^2 + \rho^2 d\theta^2 \ ,$$

with $\Delta = (r - r_+)(r - r_-)$

Killing vector: $\xi = \partial_v + \Omega_H \partial_\chi$

and

$$\kappa_\pm = \frac{r_\pm - r_\mp}{2(r_\pm^2 + a^2)}$$

FIRST LAW OF BLACK HOLE MECHANICS

The first law of thermodynamics states:

For a closed system, the internal energy U changes as

$$dU = T\,dS - p\,dV$$

The first law of black hole mechanics is similar, in that it says how the black hole MASS changes with respect to other black hole quantities, namely the AREA, ANGULAR MOMENTUM and CHARGE.

But how are these quantities defined?

Consider, for instance, the MASS.
In General Relativity there is no unique way to define masses.
However, for stationary spaces that are asymptotically flat black hole masses can be defined passing via the KOMAR QUANTITY.

Consider an hypersurface Σ. We assume Σ to be spacelike. We introduce:

$$a^\mu \quad \text{timelike, normal to } \Sigma$$

with induced metric $\gamma_{\mu\nu} = g_{\mu\nu} + a_\mu a_\nu$

And we consider $\partial\Sigma$, with:

$$b^\mu \quad \text{spacelike, normal to } \partial\Sigma, \text{ s.t. } a^\mu b_\mu = 0$$

with induced metric $q_{\mu\nu} = \gamma_{\mu\nu} - b_\mu b_\nu$

We introduce:

$$dS_{\mu\nu} = (b_\mu a_\nu - b_\nu a_\mu)\sqrt{q}\, d^2x$$

$$dS_\mu = a_\mu \sqrt{\gamma}\, d^3x$$

KOMAR QUANTITY: Given a Killing vector ξ, there exists the following conserved quantity

$$Q_K = -\frac{1}{4\pi}\int_{\partial\Sigma_\infty} dS_{\mu\nu}\, \nabla^\mu \xi^\nu$$

We want to apply this definition to STATIONARY, AXISYMMETRIC black holes; we choose:

$$\Sigma = \mathcal{M} - \mathcal{B} \quad (\text{exterior of the black hole region})$$

$$\partial\Sigma = \mathcal{H}^+ \cup \partial\Sigma_\infty$$

$\hookrightarrow$ component at $r=\infty$

Recalling that we have two Killing vectors:

$$\xi_t = \partial_t = k \quad , \quad \xi_\phi = \partial_\phi = \tilde{k}$$

we can define two conserved quantities

MASS: $$M = -\frac{1}{4\pi}\int_{\partial\Sigma_\infty} dS_{\mu\nu}\,\nabla^\mu k^\nu$$

ANGULAR MOMENTUM: $$J = -\frac{1}{8\pi}\int_{\partial\Sigma_\infty} dS_{\mu\nu}\,\nabla^\mu \tilde{k}^\nu$$

Recall that $\mathcal{H}^+$ is a Killing horizon, with Killing vector:

$$\xi = \partial_t + \Omega \partial_\phi$$

$$\Rightarrow \quad k^\mu = \xi^\mu - \Omega \tilde{k}^\mu$$

Then, the mass can be recast as:

$$M = -\frac{1}{4\pi} \int_{\partial\Sigma_\infty} dS_{\mu\nu} \nabla^\mu k^\nu$$

$$= \frac{1}{4\pi} \int_\Sigma dS_\mu R^\mu_\nu k^\nu + \frac{1}{4\pi} \int_{\mathcal{H}^+} dS_{\mu\nu} \nabla^\mu k^\nu$$

Gauss theorem

$$= \frac{1}{4\pi} \int_\Sigma dS_\mu \left(2T^\mu_\nu - T\delta^\mu_\nu\right) k^\nu$$

$$+ 2\Omega J + \frac{1}{4\pi} \int_{\mathcal{H}^+} dS_{\mu\nu} \nabla^\mu \xi^\nu$$

but we can relate the last term to the surface gravity:

$$\kappa = -n_\mu \xi_\nu \nabla^\nu \xi^\mu$$

with n_μ null-vector orthogonal to $\mathcal{H}^+$ such that $\xi^\mu n_\mu = -1$

$$= \frac{1}{4\pi} \int_\Sigma dS_\mu \left(2T^\mu_\nu - T\delta^\mu_\nu\right) k^\nu$$

$$+ 2\Omega J + \frac{\kappa}{4\pi} A$$

We can take small variations of the parameters; we use the following hypotheses:

$$\delta k^\mu = \delta \tilde{k}^\mu = 0$$

$$\delta \xi^\mu = \delta \Omega \, \tilde{k}^\mu$$

and assume

$$\delta T^\mu_{\ \nu} = 0 \quad \text{in} \quad \Sigma$$

and we arrive at:

FIRST LAW OF BLACK HOLE MECHANICS:

$$\delta M = \Omega \, \delta J + \frac{\kappa}{8\pi} \delta A$$

Note that:

- for charged black holes, we add

$$\Phi \, \delta Q$$

  with Φ the electrostatic potential difference between radial infinity and the horizon

- we have two interpretations:

  1) "PASSIVE" one: how the background changes if we change the parameters that define the solution;

  II) "ACTIVE" one: how the black hole parameters change if we throw a particle with quantum number $(\delta M, \delta J, \delta Q)$ inside the black hole.

SECOND LAW OF BLACK HOLE MECHANICS

The first law does not constrain the variations of the black hole quantities. Indeed, it may seem that "any" change of the black hole parameters is allowed. The second law aims at (partially) constraining the allowed variations, and – in spirit – is similar to the second law of thermodynamics:

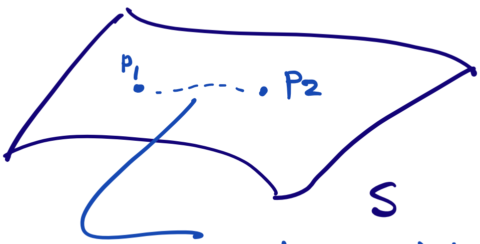
In an irreversible or spontaneous change from one equilibrium state to another, the entropy always increases:

$$\delta S \geq 0$$

For black holes, the quantity of interest that does not decrease in time is the HORIZON AREA.

However, to arrive at a formal definition of the second law, we need to introduce some definitions:

ACHRONAL SURFACE: Consider a surface $S \subset \mathcal{M}$; S is ACHRONAL if $\forall\, p \in S$ $\nexists\, q \in S$ s.t. $q \in I^+(p)$.

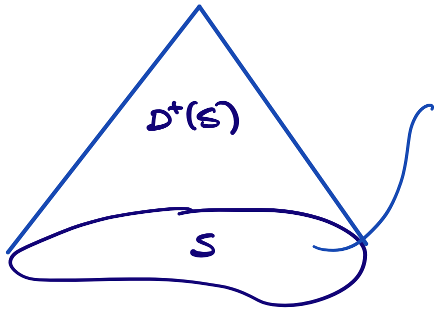


no two points can be connected by a timelike curve.

Theorem: S is a three-dimensional C^0 submanifold of $\mathcal{M}$.

FUTURE DOMAIN OF DEPENDENCE: Consider an achronal surface S; the future domain of dependence is defined as:

$$D^+(S) = \left\{ p \in \mathcal{M} \;\middle|\; \begin{array}{l}\text{every past causal curve} \\ \text{through } p \text{ intersects } S\end{array} \right\}$$



the knowledge of the conditions on S are enough to determine what happens in $D^+(S)$

Analogously, one defines the **PAST DOMAIN OF DEPENDENCE** $D^-(S)$, and the **DOMAIN OF DEPENDENCE** $D(S) = D^-(S) \cup D^+(S)$

CAUCHY SURFACE: an achronal surface Σ is a **CAUCHY SURFACE** if

$$D(\Sigma) = \mathcal{M}$$

GLOBAL HYPERBOLICITY: A spacetime $(\mathcal{M}, g)$ is **GLOBALLY HYPERBOLIC** if it admits a Cauchy surface.

STRONGLY ASYMPTOTICALLY PREDICTIVITY: An asymptotic flat spacetime $(\mathcal{M}, g)$ is **STRONGLY ASYMPTOTICALLY PREDICTABLE** if $\exists$ $\bar{V} \subset \bar{\mathcal{M}}$ open such that

- $\overline{\mathcal{M} \cap J^-(\mathcal{J}^+)} \subset \bar{V}$
- $(\bar{V}, \bar{g})$ is globally hyperbolic

With these definitions, we can formulate the

SECOND LAW OF BLACK HOLE MECHANICS (AREA THEOREM)

Consider a spacetime $(\mathcal{M}, g)$ with the following properties:

- strongly asymptotically predictable
- Einstein equations satisfied with the null energy condition, namely

$$R_{\mu\nu} k^\mu k^\nu \geq 0 \qquad \forall \text{ null } k^\mu$$

Take the globally hyperbolic region $\bar{V} \subset \mathcal{M}$.
Recall that $\overline{J^-(\mathcal{I}^+)} \subset \bar{V}$.

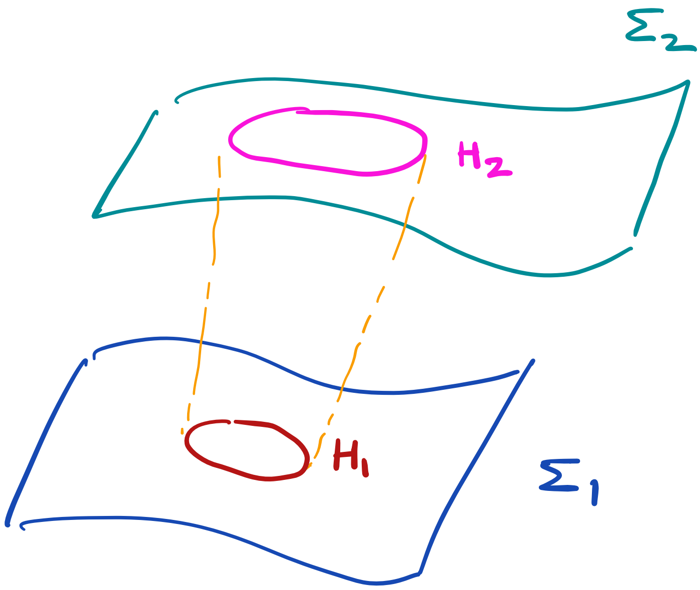
Within $\bar{V}$, take two spacelike Cauchy surfaces:

$$\Sigma_1 \, , \qquad \Sigma_2 \subset J^+(\Sigma_1)$$

Within them, define:

$$H_1 = \mathcal{H}^+ \cap \Sigma_1$$

$$H_2 = \mathcal{H}^+ \cap \Sigma_2$$



Then:

$$A(H_2) \geq A(H_1)$$

Loosely speaking: the area of the black hole (future) horizon cannot decrease in time.

This is also true for <u>multiple</u> black holes: the sum of the areas of the horizons cannot decrease in time.

EXAMPLES AND APPLICATIONS

• MERGING OF TWO SCHWARZSCHILD B.H.'s :

Consider two Schwarzschild black holes, with masses M_1, M_2 and horizon areas A_1, A_2.

Assume that they collide and merge into a new black hole with mass M_3 and area A_3.

The Area theorem tells that:

$$A_3 \geq A_1 + A_1$$

but $A = 16\pi M^2$

$$\Rightarrow \quad M_3 \geq \sqrt{M_1^2 + M_2^2}$$

One can define the energy radiated as:

$$E_{rad} = M_1 + M_2 - M_3$$

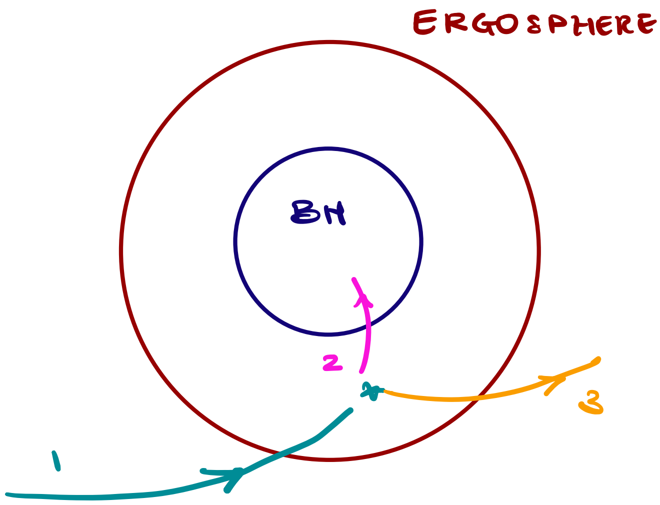
or the efficiency as:

$$\text{efficiency} = \frac{M_1 + M_2 - M_3}{M_1 + M_2} \leq 1 - \frac{1}{\sqrt{2}}$$

- Penrose process

It is a process that extracts rotational energy from black holes.

Consider a particle (1) that enters the black hole ergosphere. Therein, it decays into two particles (2) and (3); (2) falls into the black hole, (3) manages to escape.



$p_1^\mu = (E, 0, 0, L)^T$

It happens that

$$E^{(3)} \geq E^{(1)}$$

For the second particle, we have (in order to move forward in time)

$$p_2^\mu \xi_\mu < 0 \implies L^{(2)} \leq \frac{E^{(2)}}{\Omega_H}$$

$$\hookrightarrow \xi = \partial_t + \Omega_H \partial_\phi$$

From the first law, we know that

$$\frac{\kappa}{8\pi G} \delta A = \delta M - \Omega_H \delta J$$

We have: $\delta M = E^{(2)}$, $\delta J = L^{(2)}$; thus, the Area theorem tells that

$$E^{(2)} - \Omega_H L^{(2)} \geq 0$$

which is indeed verified.

THIRD LAW OF BLACK HOLE MECHANICS

The third law forbids certain "extreme" processes to take place.

It is inspired by the third law of thermodynamics:

It is impossible by any procedure, no matter how idealized, to reduce the temperature of any closed system to zero temperature ie a FINITE NUMBER of FINITE OPERATION.

Analogously, for black holes, the third law states:

THIRD LAW OF BLACK HOLE MECHANICS:

It is impossible by any procedure, no matter how idealized, to reduce the surface gravity κ to zero in a FINITE SEQUENCE of operations.

For Reissner-Nordström black holes, the surface gravity is given by:

$$\kappa = \pm \frac{\sqrt{M^2 - Q^2}}{(M \pm \sqrt{M^2 - Q^2})^2} = \pm \frac{r_+ - r_-}{2 r_\pm^2}$$

Thus:

$$\kappa \to 0 \iff \text{Black hole becomes extremal}$$

Thus, the third law is often formulated as:

THIRD LAW OF BLACK HOLE MECHANICS (2)

It is impossible by any procedure, no matter how idealized, to render a subextremal black hole extremal with FINITE SEQUENCE of operations.

This law is definitely more qualitative and less formal than the other three, but:

- Werner Israel tried to make the law more formal in 1986
- there are possible COUNTEREXAMPLES; for instance, see 2211.15742

TOWARDS BLACK HOLE "QUANTUM" MECHANICS

The black hole laws of mechanics resemble the laws of thermodynamics: this analogy gives key hints about the black hole QUANTUM DATA.

Consider the first law of thermodynamics:

$$dE = T\,dS + \mu\,dJ$$

against the first law of black hole mechanics:

$$\delta M = \frac{\kappa}{8\pi}\,\delta A + \Omega\,\delta J$$

We can then identify:

$$E = M\,, \quad T = \underbrace{\lambda}_{\text{constant}}\kappa$$

$$S = \frac{A}{8\pi\lambda}\,, \quad \mu = \Omega$$

These relations are the starting point to study black hole THERMODYNAMICS.