A Glimpse of

Quantum Field Theory in Curved Spacetime

Davide Lai

DESY Hamburg

i) QM RECAP

ii) QFT IN MINKOWSKI RECAP

iii) QFT IN FRW SPACETIMES

iv) UNRUH EFFECT

v) ALGEBRAIC ASPECTS

QUANTUM MECHANICS' RECAP

Before moving to quantum mechanics, let's state well-known results in classical mechanics that will serve as a basis for quantum systems.

<u>Def.</u> Let $\mathcal{M}$ be the phase space of a classical system with finite degrees of freedom. The dynamical evolution of $y \in \mathcal{M}$ is defined via $H : \mathcal{M} \to \mathbb{R}$ through:

$$(1) \qquad \frac{dy^\mu}{dt} = \tilde{\Omega}^{\mu\nu} \partial_\nu H \qquad \mu, \nu \in (1, \dots, 2n)$$

where $\tilde{\Omega}$ is an antisymmetric $2d \times 2d$ matrix such that $\tilde{\Omega}^{\mu\nu} = 1$ when $\mu = \nu - d$, $\tilde{\Omega}^{\mu\nu} = 0$ when $|\mu - \nu| \neq d$.

<u>Remark.</u> $\tilde{\Omega}$ defines a non-degenerate, closed 2-form on $\mathcal{M}^{(*)}$ via its inverse. Chosen as local frame on $\mathcal{M}$, $\{q^\mu, p_\mu\}$, we have

(*) symplectic

$$\Omega := dp_\mu \wedge dq_\mu$$

$$\tilde{\Omega}^{ab} \Omega_{bc} = \delta^a_c$$

<u>Remark.</u> Given $h^a = \Omega^{ab} \nabla_b H$, then any possible trajectory allowed by H is an integral curve of h^a.

Def. (OBSERVABLE) Given a phase space $\mathcal{M}$ together with a symplectic form Ω, the set

$$\mathcal{O} := \{ f : \mathcal{M} \to \mathbb{R} \mid f \text{ is } \mathcal{C}^\infty \}$$

enlarges to a Poisson algebra through

$$\{,\} : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$$

$$\{f, g\} := \Omega^{ab} \nabla_a f \nabla_b g$$

Remark. Consider now $q_\mu : \mathcal{M} \to \mathbb{R}$, $p_\nu : \mathcal{M} \to \mathbb{R}$.

we have:

$$(1) \qquad \{q_\mu, p_\nu\} = \delta_{\mu\nu}$$

$$(2) \qquad \{q_\mu, q_\nu\} = \{p_\mu, p_\nu\} = 0$$

Remark. (ii) Given $y_1, y_2 \in \mathcal{M}$, we have:

$$(3) \qquad \Omega(y_1, y_2) := p_{1\mu} q_{2\mu} - p_{2\mu} q_{1\mu}$$

Def. ("QUANTIZATION") By "quantization" of a classical system we mean a "correspondence map"

$$\hat{} : \mathcal{O} \to \hat{\mathcal{O}}$$

such that

$$[\hat{f}, \hat{g}] := i \widehat{\{f, g\}}$$

where the latter should be understood as an algebra $(\hat{\mathcal{O}}, [,])$ of self-adjoint operators on an Hilb.

space $\mathbb{F}$.

Remark. For linear dynamical systems, it's possible to associate to each coordinate p_μ, q_μ in $\mathcal{M}$ some operators $\hat{q}_\mu$, $\hat{p}_\mu$ s.t.

$$[\hat{q}_\mu, \hat{p}_\nu] = i \widehat{\{q_\mu, p_\nu\}} = i\delta_{\mu\nu} \, \mathrm{Id}_{\mathbb{F}}.$$
$$[\hat{q}_\mu, \hat{q}_\nu] = [\hat{p}_\mu, \hat{p}_\nu] = 0_{\mathbb{F}}.$$

Remark For observables at most linear in the momentum we can endow the set w/ an algebra structure consisting of bounded self-adjoint operators on the Hilbert space $\mathbb{F}$.

Remark. For other kind of observables, factor ordering ambiguities can arise. For theories of oscillators like the one we are interested in this lecture, only p, q and H are relevant, and these are not affected by factor ordering ambiguities.

Ex. Schrödinger QM is characterized by
$\mathbb{F} = L^2(\mathbb{R}^3)$, $\hat{q}_\mu := q_\mu$. and $\hat{p}_\mu := i\partial_\mu$

Ex. An important example that lead us to field theory is the classical harmonic oscillator (HO).

Given

$$H = \frac{p^2}{2} + \frac{w^2}{2} q^2$$

we can aim to a re-writing in terms of the classical analogue of the annihilation operator and its conjugate:

$$a := \sqrt{\frac{w}{2}}\, q + i \sqrt{\frac{1}{2w}}\, p$$

that leads upon quantization to:

$$[\hat{a}, \hat{a}^\dagger] = Id_{\mathcal{F}} = \mathbb{1}$$

$$\hat{H} = w\left(\hat{a}^\dagger \hat{a} + \frac{\mathbb{1}}{2}\right)$$

Claim: Given

$$\hat{O}_H := U_t^{-1}\, \hat{O}\, U_t$$

we have:

$$\hat{q}(t) = \sqrt{\frac{1}{2w}} \left(e^{-iwt} \hat{a} + e^{iwt} \hat{a}^\dagger \right)$$

We introduce now a very important concept that will return later:

Def. We define as vacuum of the theory $|0\rangle \in \mathcal{F}$ such that

$$(1) \quad a|0\rangle = 0_{\mathcal{F}}$$

Remark. The n-th excited state in the theory is given by:

$$|n\rangle := \frac{1}{\sqrt{n!}}(\hat{a}^{\dagger})^n |0\rangle$$

Trivially:

$$\hat{H}|n\rangle = \left(n + \frac{1}{2}\right)\omega |n\rangle$$

QFT IN MINKOWSKI RECAP

<u>Def</u>. (KLEIN GORDON FIELD IN M_2)

Given M_2 as the two-dim. Minkowski spacetime

$$ds^2 = dt^2 - dx^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

we define as <u>Klein-Gordon field</u> $\phi : M_2 \to \mathbb{R}$ a function on M_2 whose dynamics is described by the following action:

$$S = \frac{1}{2} \int d^2x \left(\partial_\mu \phi \, \partial^\mu \phi - m^2 \phi^2 \right)$$

<u>Def</u> (QUANTIZED KG FIELD). The quantization procedure of a KG field in M_2 is carried out using the prescription stated before. Considering the "fundam." variables $\phi(t,x)$, $\Pi(t,x)$ (where Π is the conjugate of ϕ) we construct a map

$$\hat{} : \theta \to \hat{\theta}$$

such that

$$[\hat{\phi}(t,x), \hat{\Pi}(t,y)] = i \widehat{\{\phi(t,x), \Pi(t,y)\}}$$
$$= i\,\delta(x-y)\, Id_{\mathcal{F}}$$

<u>Remark</u> In terms of modes' expansion: $(k^\mu := (\omega_k, k))$

$$\phi(t,x) = \int \frac{d^3\cancel{k}}{\sqrt{2\omega_k}} \left(\hat{a}_k e^{-ik^\mu x_\mu} + \hat{a}_k^+ e^{ik^\mu x_\mu} \right) \qquad (*)$$

where $[\hat{a}_k, \hat{a}^\dagger_{k'}] = \delta(k-k')\, Id_F$

$$[\hat{a}_k, \hat{a}_{k'}] = [\hat{a}^\dagger_k, \hat{a}^\dagger_{k'}] = 0_F.$$

Remark. Following the harmonic oscillator @ page 5, we see that a vacuum can be defined as:

$$\forall k, \quad \hat{a}_k |0\rangle = 0_F$$

Comment: Pay attention on the procedure that we did above: we used a time-translation symmetry to decompose ϕ (i.e. F, our Hilbert space) in positive and negative energy (i.e. the conserved charge corresp. to time translations). This makes the definition of a vacuum (and - as a consequence - the particle interpretation) self-consistent. What if the time-transl. symmetry / any other symmetries are not there?! We will come back later to this, but as a first naive comment we can certainly say that the notion of vacuum/particles cannot certainly be a universal concept.

Let's move to more interesting situations.

QFT IN FRW SPACETIMES

A first example different from QFT in Minkowski is the following

Def. (FRW METRIC)

$$ds^2 := dt^2 - a^2(t)\, d\vec{x}^2 \qquad (1)$$

Def. (CONFORMAL TIME)

Let $t_\emptyset$ be an arbitrary constant. The conformal time η is defined as:

$$\eta(t) := \int_{t_\emptyset}^{t} dt/a(t)$$

Claim. Let $d=2$. Then (1) in the new coordinates is expressed as:

$$ds^2 := a^2(\eta)\left(d\eta^2 - dx^2\right)$$

Claim. Consider the action of a KG field in two-dimensions in a FRW background:

$$S_{FRW} := \frac{1}{2}\int d\,\mathrm{Vol}_{FRW}\left(\partial^\alpha\phi\,\partial_\alpha\phi - m^2\phi^2\right)$$

$$= \frac{1}{2}\int d\eta\, dx\; a^2(\eta)\left(\partial_\eta\phi\,\partial_\eta\phi - \partial_x\phi\,\partial_x\phi - m^2\phi^2\right)$$

Using

$$\chi(\eta,x) := a(\eta)\,\phi(\eta,x) \Rightarrow$$

(we omit functional dependencies and we denote $\partial_\eta f := f'$

$$\phi' = \partial_\eta\left(\frac{\chi}{a}\right) = -\frac{a'}{a^2}\chi + \frac{\chi'}{a}$$

$$(\phi')^2 = \frac{(a')^2}{a^4}\chi^2 + \frac{(\chi')^2}{a^2} - 2\frac{a'\chi'\chi}{a^3}$$

$$a^2(\phi')^2 = \left(\frac{a'}{a}\right)^2\chi^2 + (\chi')^2 - 2\frac{a'}{a}\chi'\chi$$

But...

$$-2\frac{a'}{a}\chi'\chi = \frac{a'}{a}\left((\chi')^2\right)' = \underbrace{-\left(\frac{a'}{a}(\chi')^2\right)'}_{\text{total derivative}} + \left(\frac{a'}{a}\right)'\chi^2$$

i.e. $a^2(\phi')^2 = (\chi')^2 + \left(\frac{a''}{a}\right)\chi^2$ + total derivative

We obtain:

$$S_{FRW} := \frac{1}{2}\int d\eta\, dx\left(\partial_\eta\chi\,\partial_\eta\chi - \partial_x\chi\,\partial_x\chi - M^2(\eta)\chi^2\right)$$

where

$$M(\eta) := m^2 a^2(\eta) - \frac{\partial_\eta^2 a(\eta)}{a(\eta)}$$ "effective mass"

<u>Comment</u>. We reduced the dynamics of a KG field in a FRW metric to a problem of a KG field w/ a η-dependent mass in Minkowski. Thus we can

admit a mode expansion as $\left(d\!\!\!/k = \frac{dk}{(2\pi)^3} \right)$

$$\hat{\chi}(\eta, x) := \int \frac{d\!\!\!/k}{\sqrt{2}} \left(e^{ik\cdot x}\, v_k^*(\eta)\, \hat{a}_k + e^{-ik\cdot x}\, v_k(\eta)\, \hat{a}_k^\dagger \right)$$

where now $v_k(\eta)$ are mode functions satisfying the analogue of the ODE describing an harmonic oscillator w/ time-dependent frequency:

Def. (MODE FUNCTION OF A FRW FIELD)

We def. as mode function of a KG field in the FRW metric in (η, x) coordinates a solution of:

$$v_k'' + \omega_k^2(\eta)\, v_k = 0\,, \quad \omega_k^2 := k^2 + M^2(\eta)$$

Comment: Notice that the choice of $v_k(\eta)$ will lead to different definitions of $\hat{a}_k, \hat{a}_k^\dagger$, leading to different vacua / particles!

Remark. Let choose two different sets of mode functions, $\{u(k), v(k)\}$. It holds that (with the suitable normalisations):

$$(1) \qquad v_k^*(\eta) := \alpha_k u_k^*(\eta) + \beta_k u_k(\eta)$$

$$(2) \qquad |\alpha_k|^2 - |\beta_k|^2 = 1$$

Claim: Since

$$\hat{\chi}(\eta, x) := \int \frac{d\kappa}{\sqrt{2}} \left(e^{i\kappa \cdot x} v_\kappa^*(\eta)\, \hat{a}_\kappa + e^{-i\kappa \cdot x} v_\kappa(\eta)\, \hat{a}_\kappa^\dagger \right)$$

$$= \int \frac{d\kappa}{\sqrt{2}} \left(e^{i\kappa \cdot x} u_\kappa^*(\eta)\, \hat{b}_\kappa + e^{-i\kappa \cdot x} u_\kappa(\eta)\, \hat{b}_\kappa^\dagger \right)$$

$$\Rightarrow \quad (1) \begin{cases} \hat{b}_\kappa = \alpha_\kappa \hat{a}_\kappa + \beta_\kappa^* \hat{a}^\dagger_{-\kappa} \\ \hat{b}_\kappa^\dagger = \alpha_\kappa^* \hat{a}_\kappa^\dagger + \beta_\kappa \hat{a}_{-\kappa} \end{cases}$$

(notice the $-\kappa$, we want to factor a $e^{i\kappa x}$)

Def. (BOGOLIUBOV TRANSFORMATIONS)

Given two solutions of:

$$u_\kappa'' + \omega_\kappa^2(\eta)\, u_\kappa = 0 \,, \quad \omega_\kappa^2 := \kappa^2 + M^2(\eta)$$

namely $v_\kappa(\eta)$ corresponding to the operators $\hat{a}_\kappa, \hat{a}_\kappa^\dagger$ and $u_\kappa(\eta)$ corresponding to $\hat{b}_\kappa, \hat{b}_\kappa^\dagger$, these two related by

$$(3) \qquad v_\kappa^*(\eta) := \alpha_\kappa u_\kappa^*(\eta) + \beta_\kappa u_\kappa(\eta)$$

$$(4) \qquad |\alpha_\kappa|^2 - |\beta_\kappa|^2 = 1$$

we define as Bogoliubov transformations the eqs. in (1).

Def. (a-VACUUM / b-VACUUM). It follows from (1) that we can associate a vacuum for each Bogoliubov-transformed set of modes:

i.e.

$$|0\rangle_a \;\Big|\; a_k |0\rangle_a = 0_{\mathcal{F}} \quad \forall k$$

or

$$|0\rangle_b \;\Big|\; b_{k'} |0\rangle_b = 0_{\mathcal{F}} \quad \forall k'$$

Claim: Let $n_k^{(b)}$ be the density of b particles (k^{th} mode) seen in the a-vacuum. We have:

$$n_k^{(b)} = |\beta_k|^2$$

Proof:

$$_a\langle 0| \underbrace{b_k^\dagger b_k}_{N_b^{(k)}} |0\rangle_a =$$

$$_a\langle 0| (\alpha_k^* a_k^\dagger + \beta_k a_{-k})(\alpha_k a_k + \beta_k^* a_{-k}^\dagger) |0\rangle_a$$

$$= {}_a\langle 0| (\beta_k a_{-k})(\beta_k^* a_{-k}^\dagger) |0\rangle_a = |\beta_k|^2 \delta(k-k)$$

$$= |\beta_k|^2 \delta(0)$$

Considering the quantization in a 1d box, we would replace $\delta(0)$ by L. Computing the density would have given:

$$n_k^{(b)} = |\beta_k|^2$$

Remark If $\int |\beta_k|^2 dk$ is divergent, it means that the back reaction of this particle production cannot be neglected!

FULLING-DAVIES-UNRUH EFFECT

[FULLING, 1973]
[DAVIES, 1975]
[UNRUH, 1976]

Comment. By the equivalence principle:

GRAVITY ~ NON-INERTIAL FRAME

we should be able to detect the a/b vacua effect of FRW metric also in M_2, modulo the fact that we adopt a non-inertial frame. The simplest frame that we can adopt is a constant proper acceleration frame. Considering M_2 in the usual patch (t,x), we define as usual the proper time as

$$\tau \ \Big| \ ds^2 := d\tau^2$$

and:

$$u^\mu := \frac{dx^\mu}{d\tau} \ , \quad a^\mu := \frac{d^2 x^\mu}{d\tau^2} \ \Big| \ a^\mu a_\mu = -|a|^2 = -a^2$$

that leads to:

$$(1) \quad \begin{aligned} x(\tau) &= x_0 - \frac{1}{a} + \frac{1}{a}\cosh(a\tau) \\ t(\tau) &= t_0 + \frac{1}{a}\sinh(a\tau) \end{aligned}$$

and choosing $x_0 = \frac{1}{a}$, $t_0 = 0$ we obtain a simpler form of (1).

Obs. We can obtain a frame where the accel. observer sit @ $\xi=0$, with proper time τ:

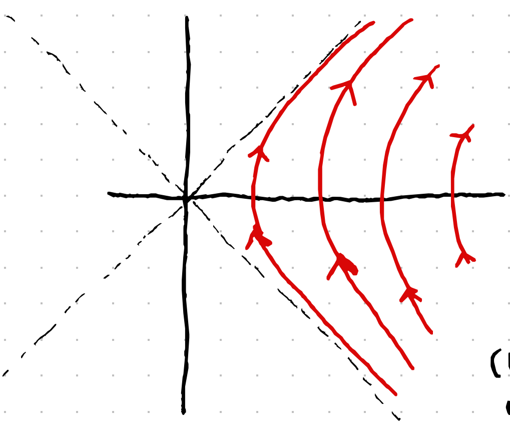
$$t(\tau,\xi) := \frac{1}{a} e^{a\xi} \sinh(a\tau)$$
$$x(\tau,\xi) := \frac{1}{a} e^{a\xi} \cosh(a\tau) \qquad (1)$$

w/ metric element:

$$(2) \quad ds^2 := e^{2a\xi}(d\tau^2 - d\xi^2)$$

conformally flat!

Def. (RINDLER SPACETIME) We define as Rindler spacetime the metric obtained by (2) defining a new coord. $\hat{\xi} \mid \xi := \frac{1}{a} \ln(1 + a\hat{\xi})$



(1) means that we cover the right Rindler wedge.

Def. (RINDLER MODE EXPANSION) We consider a massless KG field.

$$\hat{\phi}(\tau,\xi) := \int d\cancel{k} \frac{1}{\sqrt{2|k|}} \left(e^{-i|k|\tau + ik\xi} b_k + e^{i|k|\tau - ik\xi} b_k^\dagger \right)$$

Def. (RINDLER VACUUM)

$$|0\rangle_R \;\Big|\; b_k |0\rangle_R = 0 \quad \forall k$$

Question: how $|0\rangle_R$ is seen by $|0\rangle_{M_2}$?

Def. (LIGHTCONE COORDINATES)

We define the M_2 half-cone coordinates as:

$$x^- := t - x$$

$$x^+ := t + x$$

that reduce the metric to:

$$ds^2 := dx^+ dx^-$$

We define the (conformal) Rindler half-cone coordinates as:

$$u := \tau - \xi$$

$$v := \tau + \xi$$

such that:

$$ds^2 := dx^+ dx^- = e^{a(v-u)} du\, dv$$

Notice:

$$x^+ = -\frac{1}{a} e^{-au} \quad ; \quad x^- = \frac{1}{a} e^{av} \quad v$$

Obs. The field equations become:

M_2 lightcone: $\partial_{x^+}\partial_{x^-}\phi(x^+,x^-)=0$

$$\Rightarrow \quad \phi(x^+,x^-) = \phi^+(x^+) + \phi^-(x^-)$$

Rindler lightcone: $\partial_u\partial_v\,\phi(u,v)=0$

$$\Rightarrow \quad \phi(u,v) = \phi^\uparrow(u) + \phi^\downarrow(v)$$

Claim. Considering $\omega := |k|$, we accomplish to separate the ~~mode~~ expansion in u, v components / x^+, x^- comp. components. It focus on x^+, u parts, respectively. This is consistent with the above observation.

$$\phi^+(x^+) := \int_0^{+\infty} d\omega \frac{1}{\sqrt{2\omega}}\left(e^{-i\omega x^+}a_\omega + e^{i\omega x^+}a_\omega^\dagger\right)$$

$$\phi^\uparrow(u) := \int_0^{+\infty} d\Omega/\sqrt{2\Omega}\left(e^{-i\Omega u}b_\Omega + e^{i\Omega u}b_\Omega^\dagger\right)$$

Obs. Since

$$x^+ \to u \quad \text{and} \quad x^- \to v$$

we obtain:

$$\int_0^{+\infty} d\omega/\sqrt{2\omega}\left(e^{-i\omega x^+}a_\omega + e^{i\omega x^+}a_\omega^\dagger\right)$$

$$= \int_0^{+\infty} d\Omega/\sqrt{2\Omega}\left(e^{-i\Omega u}b_\Omega + e^{i\Omega u}b_\Omega^\dagger\right) \qquad (\bullet)$$

We have now to take the Fourier transform on both sides.

$$\int_{-\infty}^{+\infty} du\, e^{i\Omega u} \phi^{+}(x^{+}) = \int_{0}^{+\infty} \frac{d\omega}{\sqrt{2\omega}} \Big(F(\omega,\Omega)\, a_\omega + F(-\omega,\Omega)\, a_\omega^{+} \Big)$$

where

$$F(\omega,\Omega) := \int_{-\infty}^{+\infty} du\, e^{i\Omega u - i\omega x^{+}}$$

$$= \int_{-\infty}^{+\infty} du\, \exp\left(i\Omega u + i\frac{\omega}{a} e^{-au} \right)$$

$$\int_{-\infty}^{+\infty} du\, e^{i\Omega u} \phi^{\uparrow}(u) = \frac{1}{\sqrt{2|\Omega|}} \cdot \begin{cases} b_\Omega & \text{if } \Omega > 0 \\ b_\Omega^{+} & \text{if } \Omega < 0 \end{cases}$$

Comparing the above eqs. for $\Omega > 0$:

$$b_\Omega = \int_0^{+\infty} d\omega \;\; \alpha(\omega,\Omega)\, a_\omega + \beta(\omega,\Omega)\, a_\omega^{+}$$

where $\alpha(\omega,\Omega)$, $\beta(\omega,\Omega)$ are functions of the frequencies*. It can be shown that:

$$\int d\omega \left(\alpha(\omega,\Omega)\, \alpha^{*}(\omega,\Omega') - \beta(\omega,\Omega)\beta^{*}(\omega,\Omega') \right) = \delta(\Omega - \Omega')$$

* See the next page

Obs. For completeness:

$$\alpha(\omega,\Omega) := \sqrt{\frac{\Omega}{\omega}}\, F(\omega,\Omega) = \sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{+\infty} du\, \exp\left(i\Omega u + i\frac{\omega}{a} e^{-au}\right)$$

$$\beta(\omega,\Omega) \cdot \sqrt{\frac{\Omega}{\omega}}\, F(-\omega,\Omega) = \sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{+\infty} du\, \exp\left(+i\Omega u - i\frac{\omega}{a} e^{-au}\right)$$

Property It can be shown

$$|\beta(\omega,\Omega)|^2 = \frac{\Omega}{\omega}\,|F(-\omega,\Omega)|^2$$

$$= e^{-2\pi\Omega/a}\,|\alpha(\omega,\Omega)|^2$$

This can be proven by:

$$F(\omega,\Omega) := \frac{1}{2\pi a} \exp\left(i\frac{\Omega}{a} \ln\left(\frac{\omega}{a}\right) + \frac{\pi\Omega}{2a}\right) \cdot \Gamma\left(-\frac{i\Omega}{a}\right) \quad \begin{matrix}\omega>0\\ a>0\end{matrix}$$

Claim (UNRUH EFFECT)

$$\langle N_\Omega\rangle = {}_{M_2}\langle 0|\, b^\dagger_\Omega b_\Omega\,|0\rangle_{M_2} = \int d\omega\, d\omega'\, \beta^*(\Omega,\omega')\beta(\Omega,\omega) \cdot \underbrace{\langle \hat{a}_{\omega'}\, \hat{a}^\dagger_\omega\rangle}_{\sim\,\delta(\omega-\omega')}$$

$$= \int d\omega\, |\beta(\Omega,\omega)|^2 \sim \frac{V}{e^{\Omega/T}-1}$$

w/ $T = \frac{a}{2\pi}$.

Proof.

$$\int \left(\alpha(\omega, \Omega) \alpha^*(\omega, \Omega') - \beta(\omega, \Omega) \beta^*(\omega, \Omega') \right) d\omega$$

$$= \delta(\Omega - \Omega')$$

Considering $\Omega \sim \Omega'$ and a q.tization in a box, we have:

$$\int d\omega \left(|\alpha(\omega, \Omega)|^2 - |\beta(\omega, \Omega)|^2 \right) \sim V$$

But we invoke the property stated below:

$$\int d\omega \left(e^{2\pi/a\,\Omega} - 1 \right) |\beta(\omega, \Omega)|^2 \sim V$$

$$\langle N_\Omega \rangle \sim \frac{V}{e^{2\pi/a\,\Omega} - 1} \qquad \square$$

i.e. an accelerated observer sees the Minkowski vacuum as a thermal bath w/ a Bose-Einstein type of radiation w/ $T = a/2\pi$.