Basics of Hawking radiation

DESY theory seminar – Black holes
Albert Bekov, 19.12.2023

References:

Original – Hawking '75 „Particle creation by BH"

Same computation also nicely explained in
- Reall 2017 „Black Holes" (lecture notes)
- Wald 1984 „General Relativity" (book)

Outlook:

1) Wave equation in Schwarzschild spacetime
2) Spacetime of a gravitational collapse
3) Back-propagation of a wave packet
4) Back-propagation of a single mode
5) Particle creation via negative frequency mode

1. Wave equation in Schwarzschild spacetime

Goal: Construct solutions for a massless, scalar field in the presence of a BH

Consider the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\Omega^2$$

Change coordinates to <u>Regge-Wheeler coordinate</u>

$$r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right), \quad dr_* = \left(1 - \frac{2M}{r}\right)^{-1} dr$$

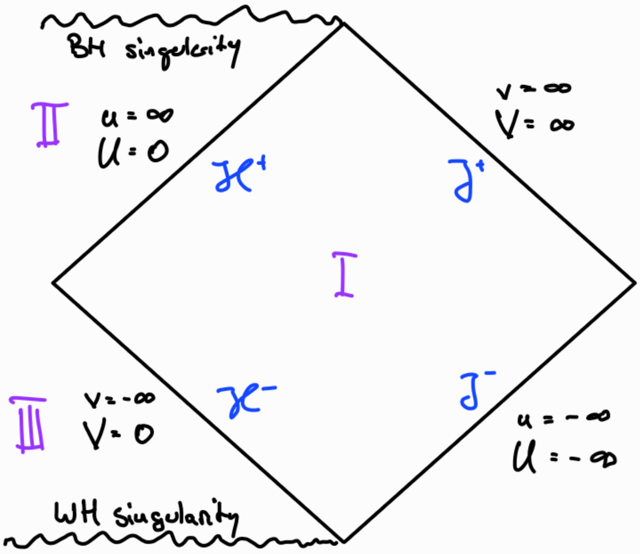
such that $ds^2 = -\left(1 - \frac{2M}{r}\right)(dt^2 - dr_*^2) + r^2 d\Omega^2$.

Further we define:

Retarded time $v = t + r_*$ and $V = e^{\kappa v}$

Advanced time $u = t - r_*$ and $U = -e^{-\kappa u}$

with $\kappa = 1/4M$ being the surface gravity of the BH, and (U, V) being the Kruskal coordinates



Massless scalar field obey Klein-Gordon eq.

$$0 = \nabla_a \nabla^b \Phi = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \Phi \right)$$

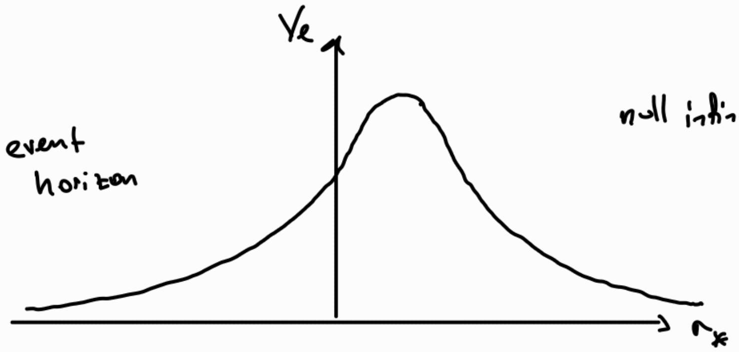
Use spherical symmetry to make an ansatz:

$$\Phi = \sum_{\ell m} \frac{1}{r} \phi_{\ell m}(t,r)\, Y_{\ell m}(\vartheta, \varphi)$$

This leads to a 2d wave equation with eff. potential

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell(r) \right) \phi_{\ell m}(t,r) = 0$$

with $\quad V_\ell(r(r_*)) = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3} \right)$



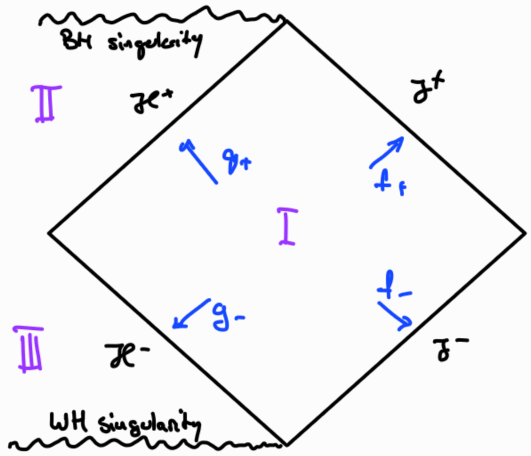
Note: The potential vanishes for $r_* \longrightarrow \pm\infty$.

Consider a localized wave packet. At late times it will be in a superposition of wave packet propagating from „left" ($r_* \to -\infty$) and „right" ($r_* \to \infty$)

$$\phi_{\ell m}(t, r_*) = f_\pm(t - r_*) + g_\pm(t + r_*)$$
$$= f_\pm(u) + g_\pm(v)$$

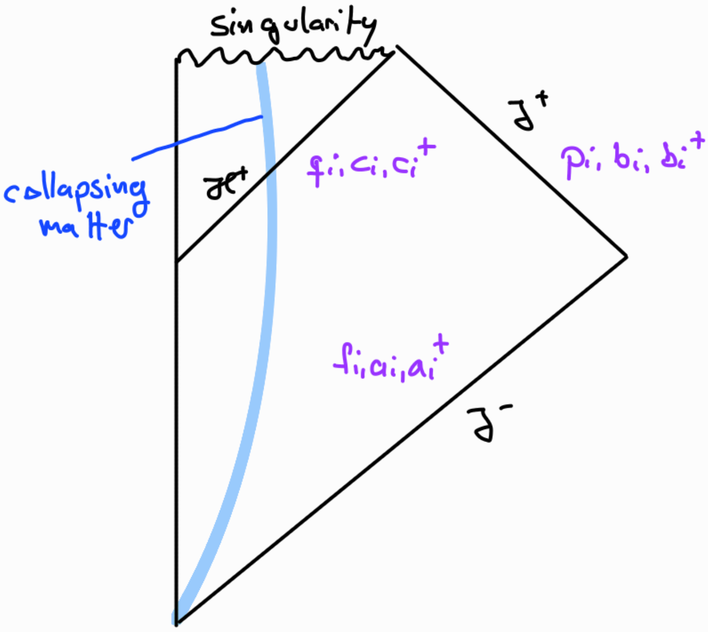
Note that in this limit $V(r_*) \longrightarrow 0$, so solutions are plane waves with freq. ω

$$\{ e^{\pm i\omega u}, e^{\pm i\omega v} \}$$



2. Spacetime of a gravitational collapse

Consider a spacetime describing a gravitational collapse.



- non-stationary spacetime, i.e. particle creation is expected
- steady flux of particles will also occur at late times

Early times: - No past event horizon $\mathcal{H}^-$ (no white hole)

Late times: - Future event horizon $\mathcal{H}^+$
- Future null infinity $\mathcal{J}^+$

Outside the collapsing matter we have a <u>stationary</u> configuration $\rightsquigarrow$ spacetime is described by Schwarzschild sol. Inside collapsing matter, spacetime is <u>not stationary</u>, especially not on the event horizon.

Wave function admits two expansion:

Early times: $\phi = \sum f_i a_i + \bar{f}_i a_i^+$ on $\mathcal{J}^-$

Late times: $\phi = \underbrace{\sum p_i b_i + \bar{p}_i b_i^+}_{\text{on } \mathcal{J}^+} + \underbrace{\sum q_i c_i + \bar{q}_i c_i^+}_{\text{on } \mathcal{H}^+}$

In general p_i, f_i and $\bar{p}_i, \bar{f}_i$ can be chose to describe pos. & neg. frequencies. While $q_i, \bar{q}_i$ is an arbitrary basis.

Change from one basis to the other,
Bogolivbov-transformation

$$p_i = \sum_j A_{ij} f_j + B_{ij} \bar{f}_j$$

$$b_i = \sum_j \bar{A}_{ij} a_j - \bar{B}_{ij} a_j^+$$

Assum that there are no particles at early times
$\longrightarrow$ Vacuum $|0\rangle$ defined by

$$a_i |0\rangle = 0 \qquad \forall i$$

Possibly non-zero particles at late times:

$$N_i = \langle 0| b_i b_i^+ |0\rangle = \sum_j B_{ij} \bar{B}_{ji} = (BB^+)_{ii}$$

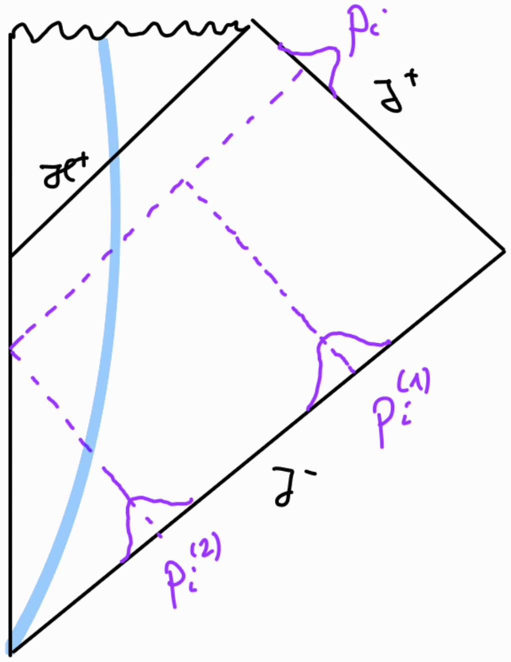
Remaing part: Determine Bogolivbov coeff. B_{ij}

3. Back-propagation of a wave packet

Consider a wave packet p_i (i labels freq. distr. & (l,m)) localized around some referded time u_i containing only pos. freq. on $\mathcal{J}^+$

$$p_i \sim \sum f_i$$

Evolving p_i backwards has two outcomes:



- p_i is scattered outside collapsing matter $\longrightarrow p_i^{(1)}$
- p_i enters the collapsing matter (would end up in $\mathcal{H}^-$, now also $\mathcal{J}^-$) $\longrightarrow p_i^{(2)}$

Two observations:

① Defining a suitable norm we can set

$$\langle p_i | p_i \rangle = 1$$

With $p_i = p_i^{(1)} + p_i^{(2)}$, we have

$$\begin{aligned} 1 &= \langle p_i^{(1)} + p_i^{(2)} | p_i^{(1)} + p_i^{(2)} \rangle \\ &= \langle p_i^{(1)} | p_i^{(1)} \rangle + \langle p_i^{(2)} | p_i^{(2)} \rangle \\ &= R_i^2 + T_i^2 \end{aligned}$$

Wave packet splits in two parts
$\longrightarrow$ Reflection & transmission coeff.

② $p_i^{(1)}$ stays in stationary regime
$\longrightarrow$ no mixing of pos. & neg. frequencies

$$p_i^{(1)} = \sum_{ij} A_{ij} f_j$$

$p_i^{(2)}$ evolves into collapsing matter,
i.e. non stationary regime

$$p_i^{(2)} = \sum_j A_{ij}^{(2)} f_j + B_{ij}^{(2)} \bar{f}_j$$

In total, we obtain

$$A_{ij} = A_{ij}^{(1)} + B_{ij}^{(2)} \quad \text{and} \quad B_{ij} = B_{ij}^{(2)}$$

Only $p_i^{(2)}$ contributes to the mixing of pos. & neg. frequencies, which leads to particle creation.

4. Back-propagation of a single mode

Goal: Find an expression for $p_i^{(2)}$ on $\mathcal{J}^-$ for a single mode at late times on $\mathcal{J}^+$

Consider a single mode with frequency ω

$$p = p_{\omega\ell m} = A\, e^{-i\omega u} \qquad \text{(pos. frequency)}$$

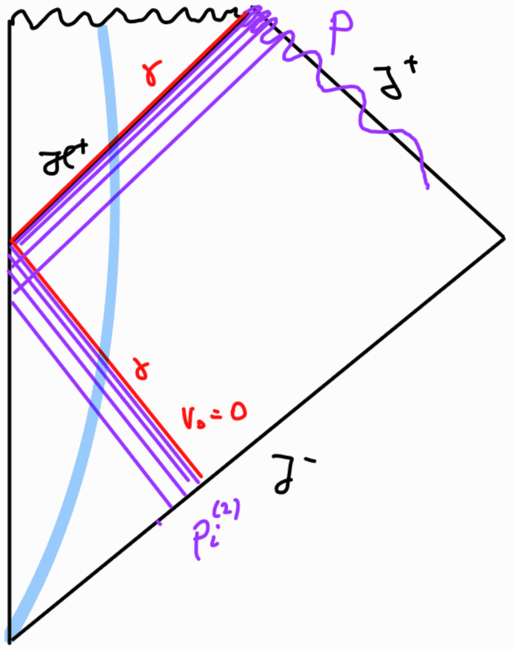
Comment: This is not normalizable and will lead to divergencies. However, computations will be simpler as to compared dealing with wave-packets and result will be the same.

In terms of Kruskal coordinates this reads

$$p = A \exp\left(+ i \frac{\omega}{\kappa} \log(-U) \right)$$

(with $U = -e^{-\kappa u}$ and $\kappa = 1/4M$).

Since $u \to 0$ at the event horizon there are infintely many oscillations close to $\mathcal{H}^+$

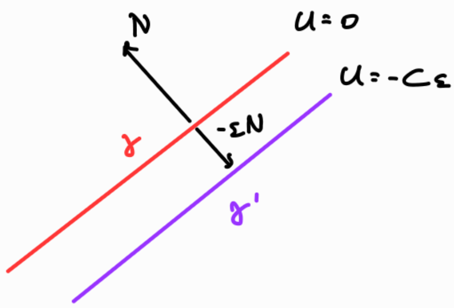


Infintely many oscillations
~ geometric optics
→ Surfaces of constant phase are null surfaes
(because wave equation)

Let γ be the null geodesic generator of the event horizon. Tracing it back to early times, it intersects $\mathcal{J}^-$ at some point $v_0 = 0$, we set to zero.
Let N be a normal null vector to γ, so

$$N = c \frac{\partial}{\partial u} \quad \text{for some } c.$$

and let γ' be a null geodesic parallel to γ (surface of constant phase), separated by $-\varepsilon N$.
So γ' is located at $u = -c\varepsilon$.



Parallel transporting N on γ, implies that $N = D \frac{\partial}{\partial v}$ for some D.

Implying that γ' intersects $\mathcal{J}^-$ at $-D\varepsilon$.
So in total, the position of γ' transforms from

$$u = \frac{c}{D} v \equiv \alpha v$$

such that on $\mathcal{J}^-$:

$$\boxed{p(v) = \tilde{A} \exp\left(\frac{i\omega}{\kappa} \log(-\alpha v)\right)}, \quad v < v_0 = 0$$

and zero for $v > 0$.

5. Particle creation via negative frequency modes

As expected, $p(v)$ also contains neg. freq. modes.

Specifically, we expand in terms of solutions

$$f_\theta(v) = e^{-i\theta v} \quad \text{and} \quad \bar{f}_\theta = e^{+i\theta v}$$

with freq. v on $\mathcal{J}^-$.

$$p(v) = \int_0^\infty d\theta \, N_\theta \tilde{p}(\theta) e^{-i\theta v} + \int_0^\infty d\theta \, \bar{N}_\theta \tilde{p}(-\theta) e^{i\theta v}$$

Up to normalisation, this corresponds to the Bogoliubov coefficients

$$A^{(2)}_{\omega\theta} = N_\theta \tilde{p}(\theta) \quad \text{and} \quad B^{(2)}_{\omega\theta} = \bar{N}_\theta \tilde{p}(-\theta)$$

It holds that: $\tilde{p}(-\theta) = -e^{-\omega\pi/\kappa} \tilde{p}(\theta)$ (*)

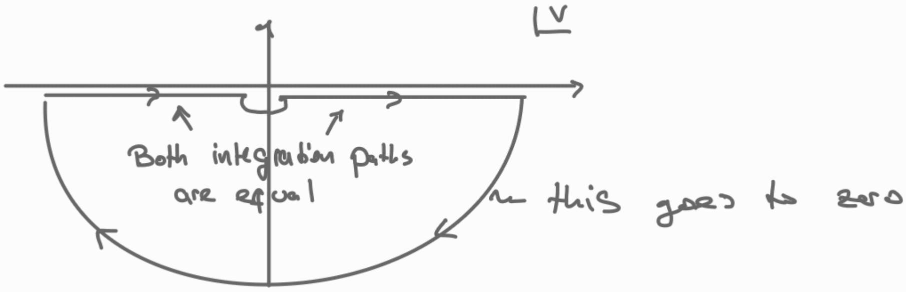
such that:

$$\boxed{|B^{(2)}_{\omega\theta}|^2 = e^{-2\omega\pi/\kappa} |A^{(2)}_{\omega\theta}|^2}$$

Explanation for (*):

Back-transformation : $\tilde{p}(-\theta) = \tilde{A} \int_{-\infty}^{0} dv\, e^{-i\theta v} \exp\left(i \frac{\omega}{\kappa} \log(-\alpha v)\right)$

complex plane



Cauchy's theorem:

$$\tilde{p}(-\theta) = -\tilde{A} \int_{0}^{\infty} dv\, e^{-i\theta v} \exp\left(i \frac{\omega}{\kappa} \log(-\alpha v)\right)$$

Use $\log(-\alpha v) = \log(\alpha v) + i\pi$

$$\tilde{p}(-\theta) = -\tilde{A}\, e^{-\omega\pi/\kappa} \int_{0}^{\infty} dv\, e^{-i\theta v} \exp\left(i \frac{\omega}{\kappa} \log(\alpha v)\right)$$

$$\overset{v \to -v}{=} -\tilde{A}\, e^{-\omega\pi/\kappa} \int_{-\infty}^{0} dv\, e^{+i\theta v} \exp\left(i \frac{\omega}{\kappa} \log(-\alpha v)\right)$$

$$= -e^{-\omega\pi/\kappa}\, \tilde{g}(\theta)$$

We this result we can express the transmission coeff. as follows:

$$T_\omega^2 = \langle p_\omega | p_\omega \rangle = \int d\theta\, |A_{\omega\theta}^{(2)}|^2 \underset{\uparrow}{-} |B_{\omega\theta}^{(2)}|^2$$

Because neg. freq. modes

$$= \left(1 - e^{-2\pi\omega/\kappa}\right) \int d\theta\, |B_{\omega\theta}^{(2)}|^2$$

$$\sim \left(1 - e^{-2\pi\omega/\kappa}\right) (BB^{+})_{ii}$$

Finally, we can state the number of particles on $\mathcal{J}^+$

$$\boxed{N_\omega = \int d\theta\, |B_{\omega\theta}^{(2)}|^2 = \frac{T_\omega^2}{1 - e^{-2\pi\omega/\kappa}}}$$

Hawking radiation

Two comments :

- Bosonic thermal distribution of particles.
  (If fermions were considered → Fermionic distr.)
  Temperature is given by:

$$\boxed{T = \frac{\kappa}{2\pi} = \frac{1}{8\pi M}}$$ Hawking temperature

- $\Gamma_\omega = T_\omega^2$ is the grey-body factor, and can be derived from the scattering process.
  It depends on the considere dimension d, spin ℓ and mass m of the corresponding field.
  For example: • In $d=2$ no factor at all
  • Higher spins are more and more opressed
  • Masses only occur after $T \gtrsim m$.