Notes on Hawking radiation

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Abstract: In these notes we explore the basics of Hawking radiation in black holes' physics. First we review the original Hawking derivation and we then present a modern derivation, closer to the usual intuition, and also the rigorous deirvation from aximatic approches to quantum field theories (in curved background). Notes prepared for the Desy theory Workshop (13.06.2023) and Desy theory workshop (09.01.2024).

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1 Recap of Black holes and Penrose diagrams

In this section we firstly recall the basics of black holes; in particular we proceed by using as explicit case of study the Schwarzschild metric, i.e. the metric obtained by considering a spherical (uncharged and non-rotating) black hole. We will describe the main feature of this metric in the Kruskal coordinates and we will describe its Penrose diagram. The Schwarzschild metric is derive in appendix B and it is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right) \ , \tag{1.1}$$

where M is the black hole mass and G is the Newton constant. At $r = 0$ the metric is singular; this divergence can be understood in a coordinate independent way by considering the divergence of the fully contracted Riemann tensor $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$.
The radius $r_s = 2GM$ is called the Schwarzschild radius; the metric in equation (1.1) seems to be divergent also for $r = r_s$, however this last singularity is just due to a bad choice of coordinates and indeed the fully contracted Riemann tensor is not singular in that point. Nevertheless for $r = r_s$ one can easily observe that the sign of the coefficients in front of $\mathrm{d}t^2$ and $\mathrm{d}r^2$ switch. The coordinate r becomes timelike and any particle that falls in the region $r < r_s$ proceed necessarily toward the singularity. For this reason the surface defined by

$r = r_s$ is called the black hole horizon.
In order to study the near horizon physics it is convenient to choose different coordinates; a null geodesic in Schawrzschild is given by $t = \pm r_\star + C$, where C is a constant and $r_\star$ is the tortoise coordiante

$$r_\star = r + \log(r-1) \ , \tag{1.2}$$

where we choose from here on $r_s = 1$. Let's define the Kruskal coordinate

$$U = -e^{(r_\star - t)/2} \ , \qquad V = e^{(r_\star + t)/2} \ . \tag{1.3}$$

By construction lines defined by U or V constant are null geodesic and the singularity is when $UV = 1$, while the horizon is defined by either $U = 0$ or $V = 0$. The metric is now

$$\mathrm{d}s^2 = -\frac{4}{r}e^{-r}\mathrm{d}U\mathrm{d}V + r^2\mathrm{d}\Omega^2 \ , \tag{1.4}$$

where r can be implicitly given by $UV = (1-r)e^r$. The off-diagonality of such a coordinates can be removed by defining

$$U = T - X \ , \qquad V = T + X \ . \tag{1.5}$$

Indeed in these coordinates

$$\mathrm{d}s^2 = \frac{4}{r}e^{-r}(-\mathrm{d}T^2 + \mathrm{d}X^2) + r^2\mathrm{d}\Omega^2 \ . \tag{1.6}$$

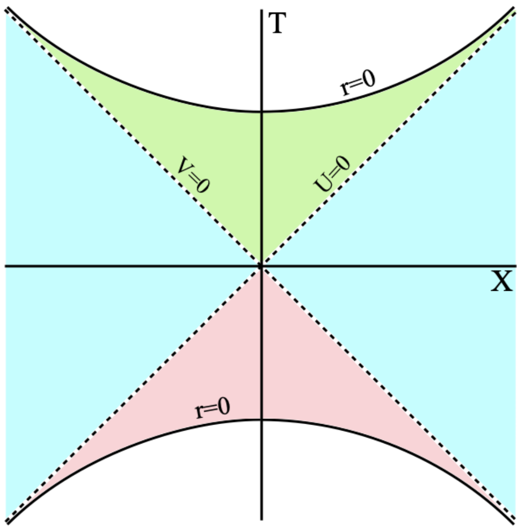

Figure 1. The Schawzschild solution in the (T, X); the light blue region is the region defined by $r > 1$, the $r < 1$ spacetime is the given by the green and the red $(X^2 - T^2 < 0)$; finally the singularity is given by $X^2 - T^2 = -1$. Figure taken by [1].

It is usually convenient to draw simplest diagrams to understand the causality proprieties of the black hole geometry. A way to do that is to observe that conformally equivalent metric, i.e. metrics related by

$$\tilde{g}_{\mu\nu}(x) = e^{2\omega(x)} g_{\mu\nu}(x) \ , \tag{1.7}$$

have the same null geodesics and timelike/spacelike curves in one metric will be time-like/spacelike curves in the other. A skatch of the proof is given in appendix A but, with this simple observation, it is possible to define the Penrose diagrams in the following way:

1. Choose a set of the coordinates of the sapcetime defined in a finite range such that

$$\mathrm{d}s^2 = \frac{1}{\omega^2(x)} \mathrm{d}\hat{s}^2 \ , \tag{1.8}$$

   where $\mathrm{d}\hat{s}^2$ is regular on the boundary, i.e. at the infinity of the previous spacetime coordinates.

2. The spacetime defined by $\mathrm{d}\hat{s}^2$ has the same causality proprieties of the spacetime defined by $\mathrm{d}s^2$ and therefore we can study the causality proprieties of $\mathrm{d}s^2$ by studying $\mathrm{d}\hat{s}^2$. This spacetime is defined in a finite range.

Let's apply this procedure to the Schwarzschild case; starting from the first region T and X one can define

$$T' + X' = \arctan(T + X) \ , \qquad T' - X' = \arctan(T - X) \ , \tag{1.9}$$

and these parameters are defined in a finite domain $-\frac{\pi}{2} < \tilde{U}, \tilde{V} < \frac{\pi}{2}$; the metric can be written as

$$\mathrm{d}s^2 = \overbrace{\frac{e^{-r}}{4r\cos^2(T'+X')\cos^2(T'-R')}}^{\omega^{-2}} \overbrace{\left(-(\mathrm{d}T')^2 + (\mathrm{d}X')^2 + r^2 \mathrm{d}\Omega^2\right)}^{\mathrm{d}\hat{s}^2} \tag{1.10}$$

The new coordinates are such that $|X' \pm T'| \leq \pi/2$ (and throw out the region $|T| > \pi/4$). Therefore we obtain the Penrose diagram in figure 2.

2 The Hawking radiation

The Hawking radiation is one of the most important phenomena in black hole physics [2]. It explains how a black hole can collapse even if it seems to violate the fact that the area of the black hole cannot decrease. This violation must be caused by a flux of negative energy across the event horizon which balances the positive energy flux emitted to infinity. Just outside the horizon there would be pairs of particles, one with positive and one with negative energy. The negative particle can tunnel inside the black hole where the Killing vector which represents time translations is spacelike. In this region the particle can exists as a real particle with a timelike momentum vector even if its energy relatie to infinity as measured by the time translation Killing vector is negative. The other particle can escape to infinity where it consitutes a part of the thermal emission we are going to describe. The

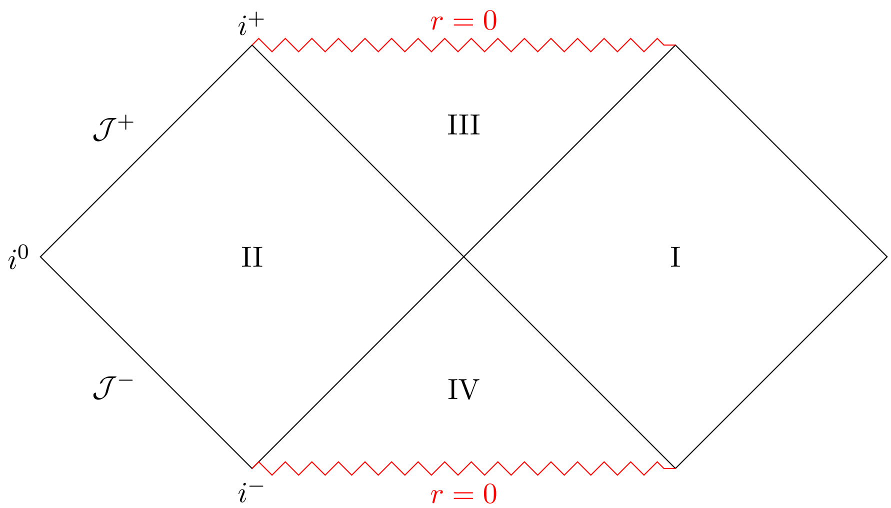

Figure 2. Penrose diagram of a Schwarschild spacetime. $i^{\mp}$ is the past/future timelike infinity, $\mathcal{J}^{\mp}$ is the past/future null infinity and i^0 is the spatial infinity. $i^{\pm}$ are where timelike geodesics come from. The red zigzag lines are where the singularity is located.

probability of the negative energy particle tunnelling the horizon is governed by the surface gravity κ[1], which indeed measure how fast the Killing vector becomes spacelike.
In the following we will study a free massless scalar theory so that computations can be really be done explicitly, however let us just comment that a fully non-perturbative proof of the Unruh effect (and similarly Hawking radiation) is possible for every field theory satisfying the Wightmann axioms. For a case of a generic scalar theory (with arbitrary potential) see [3].
Let's be concrete and consider a free massless field satisfying the equation of motion [2]

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = 0 \ . \tag{2.1}$$

The field operator ϕ can be expanding as

$$\phi = \sum_i f_i \boldsymbol{a}_i + \overline{f}_i \boldsymbol{a}_i^\dagger \ , \tag{2.2}$$

where f_i satisfy the wave equations $g^{\mu\nu}\nabla_\mu\nabla_\nu f_i = 0$ and, by choosing the f_i so that they satisfy the orthonormality condition

$$\frac{i}{2}\int_{\mathcal{J}^-}\left(f_i\nabla_\nu\overline{f}_j - \overline{f}_j\nabla_\nu f_i\right) \mathrm{d}\Sigma^\nu = \delta_{ij} \ , \tag{2.3}$$

$\boldsymbol{a}_i/\boldsymbol{a}_i^\dagger$ are the annihilation and construction operators for particle at past null infinity ($\mathcal{J}^-$); those are ingoing particles. Observe that there is an ambiguity in the definition of the ingoing particles because of the presence of the white hole solution from which particles

[1] Remember that, if k^a is the properly normalized Killing vector, the surface gravity is defined as $k^a\nabla_a k^b = \kappa k^b$. In the Schwarzschild case $\kappa = 1/(4GM) = 1/(2r_s)$.
[2] Same results can be obtained by using conformally invariant wave equation $g^{\mu\nu}\nabla_\mu\nabla_\nu\phi + R/6\phi = 0$.

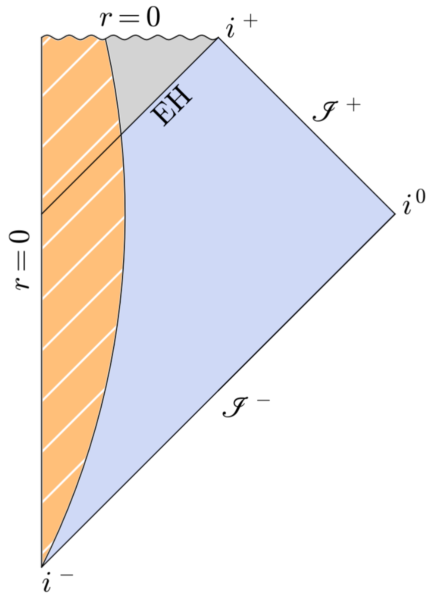

Figure 3. Penrose diagram of a collapsing spherical body. $i^{\mp}$ is the past/future timelike infinity, $\mathcal{J}^{\mp}$ is the past/future null infinity and i^0 is the spatial infinity. $i^{\pm}$ are where timelike geodesics come from. The red zigzag lines are where the singularity is located.

can be consider[4]. However this ambiguity is eliminated by the fact that we will consider a collapsing spherical body instead of the Schwarschild solution (see Penrose diagram in Figure 3): in this case the white hole region simply does not exists are we conclude that ingoing particles are only considering in $\mathcal{J}^-$. The field ϕ can be expressed everywhere as in (2.2), however in the region outside the event horizion one can express it in terms of future null infinity ($\mathcal{J}^+$) data:

$$\phi = \sum_i p_i \boldsymbol{b}_i + \overline{p}_i \boldsymbol{b}_i^\dagger + q_i \boldsymbol{c}_i + \overline{q}_i \boldsymbol{c}_i^\dagger \ , \tag{2.4}$$

where $\{p_i\}$ are solutions purely outgoing, i.e. with zero Cauchy data on the event horizon, and $\{q_i\}$ are solution with zero Cauchy data on $\mathcal{J}^+$. This splitting is due to the fact that the outgoing Hilbert space is given by

$$\mathcal{H}_{\text{out}} = \mathcal{H}_{\text{out},\mathcal{J}^+} \oplus \mathcal{H}_{\text{out,BH}} \ . \tag{2.5}$$

The p_i and q_i satisfy

$$\frac{i}{2}\int_{\mathcal{J}^+} \left(p_i \nabla_\nu \overline{p}_j - \overline{p}_j \nabla_\nu p_i\right) \mathrm{d}\Sigma^\nu = \delta_{ij} \ , \tag{2.6}$$

and

$$\frac{i}{2}\int_{\text{Event Hor.}} \left(q_i \nabla_\nu \overline{q}_j - \overline{q}_j \nabla_\nu q_i\right) \mathrm{d}\Sigma^\nu = \delta_{ij} \ , \tag{2.7}$$

respectively. To pass from the decomposition in equation (2.2) to the decomposition in (2.4) we can apply the Bogolibov transformation

$$p_i = \sum_j \alpha_{ij} f_j + \beta_{ij} \overline{f}_j \ , \qquad q_i = \sum_j \gamma_{ij} f_j + \eta_{ij} \overline{f}_j \ , \tag{2.8}$$

$$\boldsymbol{b}_i = \sum_j \overline{\alpha}_{ij}\boldsymbol{a}_j - \overline{\beta}_{ij}\boldsymbol{a}_j^\dagger \ , \qquad \boldsymbol{c}_i = \sum_j \overline{\gamma}_{ij}\boldsymbol{a}_j - \overline{\eta}_{ij}\boldsymbol{a}_j^\dagger \ . \tag{2.9}$$

It is known that the Bogolibov's transformations "do not preserve the vacuum", in the sense that the vacuum of $\mathcal{J}^-$, i.e. the vacuum defined by $\boldsymbol{a}_i\,|0,in\rangle$, is not the vacuum in $\mathcal{J}^+$ since

$$\langle 0,in|\,\boldsymbol{b}_i^\dagger\boldsymbol{b}_i\,|0,in\rangle = \sum_j |\beta_{ij}|^2 \ . \tag{2.10}$$

The proof of the above equation is provided in appendix C. β_{ij} contain the information about the particles created by the gravitation field and emitted to infinity and we then want to estimate these quantities. Let us expand the solution in spherical harmonics

$$f_{\omega' lm} \sim r^{-1}(\omega')^{-1/2}F_{\omega'}(r)e^{i\omega' v}Y_{lm}(\theta,\varphi) \ , \tag{2.11}$$

$$p_{\omega lm} \sim r^{-1}\omega^{-1/2}P_{\omega}(r)e^{i\omega u}Y_{lm}(\theta,\varphi) \ , \tag{2.12}$$

where v and u are andvaned and retarted coordinates

$$v/u = t \pm r \pm 2M\log\left|\frac{r}{2M}-1\right| \ . \tag{2.13}$$

In general

$$p_\omega = \int_0^\infty \left(\alpha_{\omega\omega'}f_{\omega'} + \beta_{\omega\omega'}\overline{f}_{\omega'}\right)\mathrm{d}\omega' \ . \tag{2.14}$$

There are two-component of p_ω: one will be scattered outside the collapsing body and will end up on $\mathcal{J}^-$; the other will enter in the collapsing body, call it $\tilde{p}_\omega$. In order to estimate the last contribution near the latest time that a null geodesic could leave $\mathcal{J}^-$, v_0, we can define a null vector tangent to the horizon l^μ and n^μ the future directed null vector at x which is directly radially and such that $l^\mu n_\mu = -1$. A vector $-\epsilon n^\mu$ connects a point x on the event horizon with a nearby null surface of constant retarded time u. We have to transport (l^μ, n^μ) back along the point in which future and past event horizon intersected. Define the affine parameter λ [3] which is related to the retarded time u on the part horizon by

$$\lambda = -Ce^{-\kappa u} = (v_1 - v_0) \ , \tag{2.15}$$

where C is a constant and κ is the surface gravity. Therefore

$$u = -\frac{1}{\kappa}\log(v-v_0) \ , \tag{2.16}$$

which means that

$$e^{-i\omega u} \sim (v-v_0)^{i\omega/\kappa} \ , \tag{2.17}$$

therefore

$$\tilde{p}_\omega \sim \omega^{-1/2}r^{-1}P_\omega(2M)(v-v_0)^{i\omega/\kappa} \ . \tag{2.18}$$

[3]The affine parameter is that it satisfies the geodesic equation. Another way is to say that if the parametrization is affine, parallel transport preserves the tangent vector.

The Frourier transform of $\tilde{p}_\omega$ will give us $\tilde{\alpha}_{\omega\omega'}$ and $\tilde{\beta}_{\omega\omega'}$; in particular for large ω'

$$\tilde{\alpha}^{(2)}_{\omega\omega'} \sim e^{i(\omega-\omega')v_0}\left(\frac{\omega'}{\omega}\right)^{1/2}\Gamma\left(1-\frac{i\omega}{\kappa}\right)(-i\omega')^{-1+i\omega/\kappa} \ , \tag{2.19}$$

$$\tilde{\beta}_{\omega\omega'} \sim -i\tilde{\alpha}_{\omega(-\omega')} \ . \tag{2.20}$$

The logarithmic singularity of $(-i\omega')^{-1+i\omega/\kappa}$ in $\omega' = 0$ can be cured analytically continue $\tilde{\alpha}$ anticlockwise round the singularity. Therefore

$$|\tilde{\alpha}_{\omega\omega'}| \sim e^{\pi\omega/\kappa}|\tilde{\beta}_{\omega\omega'}| \ . \tag{2.21}$$

The total number of created particle in $\mathcal{J}^+$ in the frequency range $\omega + \mathrm{d}\omega$ is given by $\mathrm{d}\omega\int_0^\infty|\beta_{\omega\omega'}|\mathrm{d}\omega'$ which is divergent. However this divergence is due to the fact that there is a finite steady rate of emission continuing for an infinite time. Construct then the wave-packets

$$p_{jn} = \frac{1}{\sqrt{\epsilon}}\int_{j\epsilon}^{(j+1)\epsilon}\mathrm{d}\omega\ e^{2\pi in\omega/\epsilon}p_\omega \ , \tag{2.22}$$

where j and n are integers. For ϵ small these wavepackets have frequency $j\epsilon$ and are peacked around $u = 2\pi n/\epsilon$ (with width ϵ^{-1}). We can express p_{jn} in terms of the f_ω as

$$p_{jn} = \int_0^\infty(\alpha_{jn\omega'}f_{\omega'} + \beta_{jn\omega'}\overline{f}_{\omega'})\mathrm{d}\omega' \ , \tag{2.23}$$

where

$$|\alpha_{jn\omega'}| = \left|\frac{1}{\sqrt{\epsilon}}\int_{j\epsilon}^{(j+1)\epsilon}\mathrm{d}\omega\ e^{2\pi in\omega/\epsilon}\alpha_{\omega\omega'}\right| \sim \omega^{-1/2}\Gamma\left(1-\frac{i\omega}{\kappa}\right)(\epsilon\kappa)^{-1}(\omega')^{-1/2}\sin(\epsilon/(2\kappa)) \ . \tag{2.24}$$

The expectation value of the number of particles created and emitted to infinity ($\mathcal{J}^+$) in the wavepacket mode p_{jn} is given by

$$\int_0^\infty\mathrm{d}\omega|\beta_{jn\omega}|^2 \ , \tag{2.25}$$

and this quantity can be computed by considering the fraction Γ_{jn} of wavepacket that will enter in the collapsing body

$$\Gamma_{jn} = \int_0^\infty\mathrm{d}\omega\ \left(|\tilde{\alpha}_{jn\omega}|^2 - |\tilde{\beta}_{jn\omega}|^2\right) \ , \tag{2.26}$$

where the minus sign is because negative frequency contribution make negative contribution to the flux. From (2.21) it's clear that

$$|\tilde{\alpha}_{jn\omega'}| = e^{\pi\omega/\kappa}|\tilde{\beta}_{jn\omega'}| \ . \tag{2.27}$$

Therefore the total number of particle created in the mode p_{jn} is given by

$$\frac{\Gamma_{jn}}{e^{2\pi\omega/\kappa}-1} \ . \tag{2.28}$$

Observe that this is the emission cross-section is exactly that for a body with a temperature of $T = \kappa/2\pi$. This temperature is called Hawking temperature and observe that it is related to the mass of the black hole, unintuitively, as $T \sim 1/M$[4]. Therefore the temperature of the black hole increase when the black hole is collapsing; or in another way the smaller is the black hole mass the more it emits particles. Further observe that all this calculation can be repeted for a massless free fermions (e.g. an approximation for neutrinos) and it gives the very same result up to equation (2.26) where the sign in front of $|\beta_{jn\omega}|$ is the opposite (i.e. a plus); this is because negative frequency components gives, due to the spin-statistics, a positive contribution to the flux. The number of particles emitted is therefore

$$\frac{\Gamma_{jn}}{e^{2\pi\omega/\kappa}+1} , \tag{2.29}$$

which is the Fermi-Dirac statistics. To better interpret this phenomena simply recall that the ingoing Hilbert space is just $\mathcal{H}_{\text{in},\mathcal{J}^-}$, whereas the outgoing Hilbert space is given in equation (2.5); the outgoing Fock-space is then given by

$$\mathcal{F}(\mathcal{H}_{\text{out}}) = \mathcal{F}(\mathcal{H}_{\text{out},\mathcal{J}^+}) \otimes \mathcal{F}(\mathcal{H}_{\text{out,BH}}) . \tag{2.30}$$

This means that the out-states are "joint" (entangled) states of particles that reach $\mathcal{J}^+$ and particles that will fall down in the black holes.

3 Back reaction on the metric

The back reaction on the metric due to the particle creation implies the slow decreasing of the black hole mass. Unfortunately it is not meaninful to talk about local energy-momentum of creating particles; this is similar to the problem of defining gravitational energy in classical general relativity. Nevertheless we can define an total flux, by integrating over a suitable surface. The stress energy tensor of a free scalar is given by[5]

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi . \tag{3.1}$$

let k^μ be a time parameter defined along the generators of the horizon in the final quasi-stationary state. We are interesting in the quantity

$$\frac{1}{u_1-u_2}\int_{u_1}^{u_2} d^dx \, \langle 0, in| : T_{\mu\nu} : |0, in\rangle \, k^\mu \mathrm{d}\Sigma^\nu , \tag{3.2}$$

where the integration is performed by fixing r, considering two retarded times u_1 and u_2 and $|0, in\rangle$ is the vacuum of $\mathcal{J}^-$. For convenience let's define the wave-packets $x_{jn} = p^{(2)}_{jn} + q^{(2)}_{jn}$ and $y_{jn} = p^{(1)}_{jn} + q^{(1)}_{jn}$; the first quantity represents the part of p_{jn} and q_{jn} passes through the collapsing body, the second quantities do not contain any negative frequencies and therefore do not contribute to the flux of the stress tensor. On the other hand

$$x_{jn} = \int_0^\infty \mathrm{d}\omega \left(\chi_{jn\omega} f_\omega + \xi_{jn\omega}\overline{f}_\omega\right) , \tag{3.3}$$

[4]Reintroducing also $\hbar$, c and G we have $T = \hbar c^3/(8\pi GM)$

[5]In general we have to renormalize this quantity; however every tensor which is stationary, satisfies $\nabla_\mu T^{\mu\nu}$ and agree near $\mathcal{J}^+$ gives the same result for the flux.

but close enough to $\mathcal{J}^+$ we have that $x_{jn} \simeq \sqrt{\Gamma_{jn}} p_{jn}$. This implies that

$$\int_{u_1}^{u_2} d^d x \ \langle 0, in| : T_{\mu\nu} : |0, in\rangle \ k^\mu \mathrm{d}\Sigma^\nu =$$
$$= \mathrm{Re}\left[\sum_{jn}\sum_{pl}\int_0^\infty \mathrm{d}\omega'' \int_{u_1}^{u_2} \mathrm{d}u \ \omega\omega' \sqrt{\Gamma_{jn}} p_{jn} \bar{\xi}_{jn\omega'} \left(\sqrt{\bar{\Gamma}_{pl}} \bar{p}_{pl} \chi_{pl\omega'} - (\sqrt{\Gamma_{pl}} p_{pl} \xi_{pl\omega'}\right)\right] , \tag{3.4}$$

where ω and ω'' are the frequencies of the wave-packets p_{jn} and p_{pl} respectively. By considering $u_2 - u_1 \gg 1$ we only the first term in the integrand contribues; furthermore by repeating argument similar to the discussion in the above section we conclude that

$$\int_0^\infty |\xi_{jn\omega'}| \ \mathrm{d}\omega' = \frac{1}{e^{2\pi\omega/\kappa} - 1} . \tag{3.5}$$

Therefore

$$\frac{1}{u_1 - u_2} \int_{u_1}^{u_2} d^d x \ \langle 0, in| : T_{\mu\nu} : |0, in\rangle \ k^\mu \mathrm{d}\Sigma^\nu = \int_0^\infty \Gamma_\omega \frac{\omega}{e^{2\pi\omega/\kappa} - 1} , \tag{3.6}$$

where $\Gamma_\omega = \lim_{n\to\infty} \Gamma_{jn}$ is the fraction of wave-packet of frequency that would be absorbed by the black hole.

3.1 Thermodynamic interpretation

The energy flux computed in (3.6) exactly compensate the thermal emission computed above (see equation (2.29) for the number of emitted particles). This energy flux will cause the area of the event horizon to decrease and so the black hole will not, in fact, be in a stationary state. This can also be interpreted from the thermodynamic point of view considering the standard entropy definition

$$\frac{\mathrm{d}S}{\mathrm{d}E} = \frac{1}{T} ; \tag{3.7}$$

in the case of the black hole we discussed above that

$$T = \frac{\kappa}{2\pi} = \frac{1}{8\pi G M} . \tag{3.8}$$

Therefore by using this temperature in the above definition and by identifying M as the energy we have that

$$S = \frac{A}{4G} , \tag{3.9}$$

where A is the area of the black hole. This entropy seems to suggests that the area of a black hole can never decrease; nevertheless, due to the Hawking radiation discussed above, $\langle T_{\mu\nu}\rangle$ does not satisfy the energy condition assumed in the proof of the area law (i.e. the law $\delta A \geq 0$). Nevertheless it is clear that also the second law of thermodynamics for the matter outside the black hole is violated; indeed the entropy S_m of the matter outside the black hole can decrease by dropping matter inside the black hole. Notice however that the violation of the two laws compensate each other, in the sense that when $\delta S_m < 0$ by dropping matter in the black hole then $\delta A > 0$, whereas when $\delta A < 0$ the black hole emits particles that increase the entropy of the matter outside the black hole $\delta S_m > 0$. In order to solve this problem Bekenstein proposed the *generalized entropy* [5][6]

$$S' = S_m + \frac{A}{4G} \ , \tag{3.10}$$

and the second law, given by

$$\delta S' \geq 0 \ , \tag{3.11}$$

is then valid. As shown above the black hole is not in a stationary states, because of particle emission. This implies that the black hole is evaporating. The Penrose diagram is given in figure 3; the generalized second law makes this process possible since the fact that the area of the black hole is decreasing is compensate by the particle production.
The last comment concern the fact that the Hawking temperature is very small when the mass of the black hole is smaller then the mass Plank. This implies the number of particle produced by a massive (mass grater then the Planck mass) (per unit of time) black hole is small. Therefore, even if the black hole is not in a stationary state it is a reasonable assumption to consider the black hole in a sequence of stationary states (quasi-stationary states).

4 Hawking radiation and tunneling

The derivation of the Hawking radiation presented above is the original one. It is quite simple but it does not include non-perturbative effects and it does not really fit with the usual pictorial picture. In fact the derivation presented above make explicitly use of the equations of motion of the free fields and it is not clear the connection between that derivation and the usual picture which account the fact that one of the particle of the particle anti-particle pair created just inside of just outside the horizon can tunnel the horizon changing sign of its energy.
This picture of tunneling of particles or anti-particles was making concrete and precise by Parikh and Wilczek [6]. We will follow the original derivation of [6]. The idea is to use the WKB approximation to compute Hawking radtion as tunneling
We first have to introduce coordinates which are not singular at the horizon. One possibility is the metric [7]

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + 2\sqrt{\frac{2M}{r}}\mathrm{d}t\mathrm{d}r + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \ , \tag{4.1}$$

where we send the Schwarzschild time

$$t \to t + 2\sqrt{2Mr} + 2M\ln\left(\frac{\sqrt{r} - \sqrt{2M}}{\sqrt{r} + \sqrt{2M}}\right) \ . \tag{4.2}$$

[6]Originally Bekenstein proposed the generalized entropy before the Hawking paper on black hole radiation; the law was based on merely observations, but the Hawking radiation put solid ground on the theoretical meaning of such an entropy.

[7]This coordinates are firstly used by Painlevé to criticize General Relativity for allowing singularities to come and go.

In this way the metric is explicitly non-singular in $r = 2M$. Observe that the coordinates describe a stationary spacetime. We can define a vacuum state such that it annihilates modes with negative frequency with respect to t. This is not the Unruh vacuum, but this choice do not affect the late-time radiation. The radial null geodesic is given by

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm 1 - \sqrt{\frac{2M}{r}} \ . \tag{4.3}$$

In [7] it was shown that when the mass of the balck hole is free to fluctuate, and self-gravitating shells are considered we have that the shall of energy ω travels along a line element which is

$$\mathrm{d}s^2 = -\left(1 - \frac{2(M-\omega)}{r}\right)\mathrm{d}t^2 + 2\sqrt{\frac{2(M-\omega)}{r}}\mathrm{d}t\mathrm{d}r + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \ , \tag{4.4}$$

In fact effectively $M \to M - \omega$. What follow is a WKB approximation. This approximation can be easily justified by remember that, when an outgoing wave is traced back towards the horizon, its wavelength is blue-shifted; the radial wave-number approaches infinity and WKB is justified. We have now to compute the imaginary part of the action for an s-wave outgoing particle with positive energy which crosses the horizon outwards from r_{in} to r_{out}. It is

$$\operatorname{Im} S = \operatorname{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} p_r \ \mathrm{d}r = \operatorname{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} \int_0^{p_r} \mathrm{d}p \ \mathrm{d}r \ . \tag{4.5}$$

Without enerting the detail of the solution we have

$$\operatorname{Im} S = \operatorname{Im} \int_M^{M-\omega} \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{\mathrm{d}r}{\dot{r}}\mathrm{d}H = \operatorname{Im} \int_0^{\omega} \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{\mathrm{d}r}{1 - \sqrt{\frac{2(M-\omega')}{r}}}(-\mathrm{d}\omega') \ , \tag{4.6}$$

where we used the Hamilton equation

$$\dot{r} = \frac{\mathrm{d}H}{\mathrm{d}p_r} \ , \tag{4.7}$$

and we change variables in the integrated. The integral can be computed and we have

$$\operatorname{Im} S = 4\pi\omega\left(M - \frac{\omega}{2}\right) \ . \tag{4.8}$$

In order to do that we have to deform the contour, such that $\omega \to \omega - i\epsilon$, and set $r_{\text{in}} > r_{\text{out}}$. Hawking radiation can be also regarded as pair creation outside the horizon, with the negative energy particle tunneling into the black hole. The latter propagates back in time. Observe that time-reversal corresponds to $\sqrt{\frac{2M}{r}} \to -\sqrt{\frac{2M}{r}}$ and since the anti-particle sees a geometry of fixed black hole, then $m \to M + \omega$. Therefore, deforming the countour such that $\omega \to \omega + i\epsilon$, we have

$$\operatorname{Im} S = \operatorname{Im} \int_0^{-\omega} \int_{r_{\text{out}}}^{r_{\text{in}}} \frac{\mathrm{d}r}{\sqrt{\frac{2(M+\omega')}{r}} - 1} \ \mathrm{d}\omega' = 4\pi\omega\left(M - \frac{\omega}{2}\right) \ . \tag{4.9}$$

Finally the expondential part of the semi-classical emission rate is

$$\Gamma \sim e^{-2\,\mathrm{Im}\,S} = e^{-8\pi\omega\left(M-\frac{\omega}{2}\right)} = e^{\Delta S_{\mathrm{B.H.}}} \ . \tag{4.10}$$

The quadratic terms comes with conservation of energy, which raises the effective temperature of the hole as it radiates. Following the standard argument, the linear contribution in the exponential implies that the density is given by the the Planck spectral flux

$$\rho(\omega) = \frac{\mathrm{d}\omega}{2\pi}\frac{|T(\omega)|^2}{e^{8\pi M\omega}-1} \ , \qquad \frac{1}{T} = 8\pi M \ , \tag{4.11}$$

where $T(\omega)$ is the greybody trasmission coefficient. A fast way to see this is to observe that the linear term is the Boltzmann weight when the temperature is $T = 1/(8\pi M)$. In [6] the same procedure is also proposed for charged balck holes.

5 The algebraic approach

Another possible approach to study quantum field theories is to use to made use of axioms, inspired by free field theory, to construct directly the correlation functions in a mathematically rigorous way. The most important results are contained in the very famous book [8]. In this notes we will mainly follow [9]. Before going on we have to spend some time to review those axioms.

5.1 The Wightman axioms and ”standard” results

For this short review I will use [10] and the Online Lecture by Slava Rychkov. It is important to define the fields in quadruple $(\mathcal{H}, \varphi, \Psi, U)$, where

- ⋆ $\mathcal{H}$ is an Hilbert space;
- ⋆ $\varphi : \mathcal{S}(X) \to$ *operators in* $\mathcal{H}$, is a liner map and X is the spacetime. To connect with the usual fields we first have to define the operators $\varphi(x)$, which are more technically called *operator valued distributions*, i.e. are distributions such that

  $$\varphi(f) = \int \mathrm{d}^d x \ f(x)\varphi(x) \ , \tag{5.1}$$

  are operators acting on an Hilbert space $\mathcal{H}$. This also ensure that $\varphi(f)\Psi$ is a state of finite norm.
- ⋆ Ψ is the (unique!!) vaccum, i.e. a unit vector of $\mathcal{H}$, which is cylic with respect to the algebra $\mathcal{A}$ of polynomials of $\{\varphi(f), f \in \mathcal{S}(X)\}$, i.e. the set of vectors generated by applying polynomials of the field operator to the vacuum is dense in the Hilbert space $\mathcal{H}$;
- ⋆ U a continuous representation of the symmetry group G in $\mathcal{H}$.

The quadrule defined above has to satisfy the following axioms

(W1) $\varphi(f_1)\dots\varphi(f_n)\Psi$ is continuous w.r.t. each state of the elements $f_1,\dots f_n \in \mathcal{S}(X)$;

(W2) Hermicity: $\varphi(f)^\star = \varphi(\overline{f})$ on $\mathcal{A}\Psi$;

(W3) Covariace:

$$U(g)\Psi = \Psi \ , \qquad U(g)\varphi(f)U(g^{-1}) = \varphi(f_g) \ , \tag{5.2}$$

where $f_g = f(g^{-1}x)$, for every $f \in \mathcal{S}(X)$, $g \in G$;

(W4) Locality: $[\varphi(f), \varphi(h)] = 0$ if the support of f and h are space-like separated;

(W5) Spectrum condition:Ψ is the ground state w.r.t. the time traslational group; i.e. its infinitesimal generator is a positive operator.

From the assumption above many interesting theorems were proved. Among them the PCT theorem, the spin-statics,the Haag, the Reeh-Schlieder theorems and many others. For a review of some of them with proofs see e.g. my online notes. Here I will just state the relevant ones for the following.

C1 (PCT) There is an unique conjugation J_0 of $\mathcal{H}$ such that

$$J_0\varphi(f_1)\dots\varphi(f_n)\Psi = \varphi(f_1^\dagger)\dots\varphi(f_n^\dagger)\Psi \ , \tag{5.3}$$

where $f^\dagger(x) = \overline{f}(-x)$.

C2 (Reeh-Schlieder) Let $\Lambda \subset X$, and Λ open.Let $\mathcal{A}(\Lambda)$ be the subalgebra of $\mathcal{A}$ of polynomials of the field $\varphi(f)$ where supp $f \subset \Lambda$. Then $\mathcal{A}(\Lambda)\Psi$ is dense in $\mathcal{H}$.

C3 (Bisognano-Wichmann)Let $L(\tau)$ be the one-parameter subgroup of G corresponding to the Lorentz boosts [8] and let $i\boldsymbol{K}$ its infinitesimal generator. Then

$$J_0\boldsymbol{\rho}e^{-\pi\boldsymbol{K}}A\psi = A^\star\Psi \ , \qquad \forall A \in \mathcal{A}(X^+) \ , \tag{5.4}$$

where J_0 is the PCT conjugation, $\boldsymbol{\rho} = U(\rho)$ where ρ is the partial inversion $\left(x^0, x^1, x^2, x^3\right) \to \left(x^0, x^1, -x^2, -x^3\right)$ and $X^\pm$ are the submanifold of X such that $x^1 > |x^0|$ and $x^1 < -|x^0|$ respectively.

The proof for Hawking radiation will concretely generalizes the Bisognano Wichmann theorem for the case in which the spacetime is not flat. The equation (5.4) will be interpreted as the KMS condition for the thermal states.

Before going on we have to review the basics of statistical mechanics and thermal states. In order to do that it is important to re-define the vacuum. Assuming that the dynamics of the system is taken to be given by a continuous unitary representation V of the additive reals, in $\mathcal{H}$ and such that $\mathcal{A}$ is closed under transformations $A \to V(t)AV(-t)$. Let $i\boldsymbol{K}$ be the generator of $V(\mathbb{R})$. Take a vector Ψ such that $\mathcal{A}\Psi$ is dense in $\mathcal{H}$. If $\boldsymbol{K}$ is positive and it satisfy the Kubo-Martin-Schwinger(KMS) condition

$$\left(\exp\left(-\frac{1}{2}\beta\boldsymbol{K}\right)A\Psi, \exp\left(-\frac{1}{2}\beta\boldsymbol{K}\right)B\Psi\right) = (B^\star\Psi, A^\star\Psi) \ , \tag{5.5}$$

[8] More precisely $L(\tau)\left(x^0, x^1, x^2, x^3\right) = \left(x^0\cosh(\tau) + x^1\sinh(\tau), x^1\cosh(\tau) + x^0\sinh(\tau), x^2, x^3\right)$.

for every $A, B \in \mathcal{A}$. Equivalently

$$J \exp\left(-\frac{1}{2}\beta \boldsymbol{K}\right) A\Psi = A^{\star}\Psi \ , \tag{5.6}$$

for every $A, B \in \mathcal{A}$, where J is the unique conjugation of $\mathcal{H}$, i.e. the antilinear transformation such that $J^2 = 1$ and $(Jf, Jg) = (g, f)$. In order to justify this property we are requiring to the ground state in order to be thermal we show here a standard derivation. The density matrix of the theory is $\rho = e^{-\beta \boldsymbol{K}}$. Then at the level of the correlation functions

$$\begin{aligned}
\langle A(t)B(t')\rangle_\beta &= \mathcal{Z}^{-1}\,\mathrm{Tr}\left[e^{-\beta \boldsymbol{K}} A(t)B(t)\right] = \\
&= \mathcal{Z}^{-1}\,\mathrm{Tr}\left[e^{-\beta \boldsymbol{K}} e^{i\boldsymbol{K}t} A e^{-i\boldsymbol{K}t} e^{i\boldsymbol{K}t'} B e^{-i\boldsymbol{K}t'}\right] = \\
&= \mathcal{Z}^{-1}\,\mathrm{Tr}\left[e^{-\beta \boldsymbol{K}} e^{i\boldsymbol{K}t} A e^{-i\boldsymbol{K}t} e^{\beta \boldsymbol{K}} e^{-\beta \boldsymbol{K}} e^{i\boldsymbol{K}t'} B e^{-i\boldsymbol{K}t'}\right] = \\
&= \mathcal{Z}^{-1}\,\mathrm{Tr}\left[e^{-\beta \boldsymbol{K}} e^{i\boldsymbol{K}t'} B e^{-i\boldsymbol{K}t'} e^{i\boldsymbol{K}(t+i\beta)} A e^{-i\boldsymbol{K}(t+i\beta)}\right] = \\
&= \langle B(t')A(t+i\beta)\rangle_\beta
\end{aligned} \tag{5.7}$$

It is already clear at this point that the Bisognano-Wichmann conclusion, in equation (5.4) it is very reminiscent of a thermal state. Before going to the curved background case let us first study the case of an uniformly accelerated observer (Rindler spacetime).

5.2 Uniformly accelerated observer

The best description for the uniformly accelerated observer is the Rindler spacetime. We have to introduce the coordinates (ξ, τ, x^2, x^3), where $\xi \in \mathbb{R}_+$ and $\tau \in \mathbb{R}$, such that

$$x^0 = \xi \sinh \tau \ , \qquad x^1 = \xi \cosh \tau \ . \tag{5.8}$$

The metric for X^+ is therefore

$$\mathrm{d}s^2 = \xi^2 \mathrm{d}\tau^2 - \mathrm{d}\xi^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2 \ . \tag{5.9}$$

As usual the acceleration of the observer is $\alpha = 1/\xi$. If $L^+(\mathbb{R})$ is the one-parameter isometries corresponding to time translations for the accelerated observer, i.e.

$$L^+(\tau)\left(\xi, \tau', x^2, x^3\right) = \left(\xi, \tau' + \tau, x^2, x^3\right) \ , \tag{5.10}$$

then, by invoking the Reeh-Schlieder and the Bisognano-Wichmann theorem we conclude that the restriction of the state Ψ to $\mathcal{A}(X^+)$ is a thermal state satisfying the KMS condition w.r.t. the time-translation group $L^+(\mathbb{R})$, i.e.

$$J \exp\left(-\pi K^+\right) A\Psi = A^{\star}\Psi \ , \tag{5.11}$$

for every $A \in \mathcal{A}(X^+)$, where $i\boldsymbol{K}^+$ is the generator of $\boldsymbol{L}^+(\mathbb{R})$ and $J = J_0\boldsymbol{\rho}$. Observe that this result implies that Ψ, which is the ground state for the inertial observer, corresponds to a thermal state for the accelerated observer. Observe that the temperature is given by $T = 1/2\pi$, as expected. However this is the temperature associated with τ instead of the proper time τ_α. By rescaling the time from τ to τ_α we have the observed temperature which is

$$T_\alpha = \frac{\alpha}{2\pi} \ . \tag{5.12}$$

This is the rigorous proof of the Unruh's result.

5.3 Quantum fields on Manifolds

We have now to formulate the axiomatic approach fro the curved background case. We consider space-time manifolds of the form $\mathbb{R}^2 \times Y$, with the metric of the form

$$\mathrm{d}s^2 = A(t^2 - \omega^2, y)(\mathrm{d}t^2 - \mathrm{d}\omega^2) - B(t^2 - \omega^2, y)\mathrm{d}\sigma^2(y) \ , \tag{5.13}$$

where A, B are positive valued smooth functions and $\mathrm{d}\sigma^2$ is a positive metric on Y. If we consider $X^{\pm}$ defined by $\omega > |t|$ and $\omega < -|t|$ we have that the $X^{\pm}$ are isometric with $\mathbb{R}_{\pm} \times \mathbb{R} \times Y$ under

$$\omega = \xi \cosh \tau \ , \qquad t = \xi \sinh \tau \ , \tag{5.14}$$

so that

$$\mathrm{d}s^2 = A(-\xi^2, y)(\xi^2 \mathrm{d}\tau^2 - \mathrm{d}y^2) - B(-\xi^2, y)\mathrm{d}\sigma^2(y) \ . \tag{5.15}$$

Observe that from those metric it is clear that, in general, $t \to t + c$ is not an isometry for this space-time metric. However it does become an isometry when restricted to E, E' defined by $\omega \pm t = 0$ respectively. This is equivalent to say that $\partial/\partial t$ is a Killing vector on E and E'. Those surfaces corresponds to past and future event horizons respectively. Also time translations $\tau \to \tau + c$ are isometries on those surfaces and in fact these are Lorentz transformations $t \to t \cosh b + \omega \sinh b$, $\omega \to \omega \cosh b + t \sinh b$. In formula

$$L^{\pm}(\tau)(\xi, \tau', y) = (\xi, \tau + \tau', y) \ . \tag{5.16}$$

$E \sim \mathbb{R} \times Y$ and we can use coordinates (t, y). The restriction of $L(\tau)$ on E is

$$L_E(\tau)(t, y) = \left(te^{-\tau}, y\right) \ . \tag{5.17}$$

Furthermore the time translations are

$$T_E(t)(t', y) = (t' + t, y) \ . \tag{5.18}$$

Let $E^{\pm}$ be defined as subspace such that $t \geq 0$ and $t < 0$ respectively. Those are both stable under L_E but not under T_E. A quantum field on X is defined as a quintuple $(\mathcal{F}, \mathcal{H}, \varphi, \Psi, \boldsymbol{L})$, where $\mathcal{F} = \mathcal{S}(\mathbb{R}^2) \times \mathcal{S}(Y)$ and $\boldsymbol{L}$ is an unitary representation of $\mathbb{R}$ in $\mathcal{H}$, such that it satisfy the generalization of the Wightmann axioms

A1) $\varphi(f_1 \otimes g_1) \ldots \varphi(f_n \otimes g_n)\Psi$ is continuous in $f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R}^2)$, $g_1, \ldots g_n \in \mathcal{S}(Y)$. This point is necessary to define Withmann functions, but it will not be necessary here;

A2) Hermiticity: $\varphi(f)^{\star} = \varphi(\overline{f})$;

A3) Covariance (w.r.t. $L(\mathbb{R})$):

$$\boldsymbol{L}(\tau)\Psi = \Psi \ , \qquad \boldsymbol{L}(\tau)\varphi(f)\boldsymbol{L}(-\tau) = \varphi(f_{\tau}) \ , \tag{5.19}$$

where $f_{\tau}(x) = f(L(\tau)x)$.

A4) Locality: $[\varphi(f), \varphi(h)] = 0$ if the support of f and h are space-like separated;

A5) This is much more complicated since the Hamiltonian is not defined everywhere. However we substitute it with an axiom designed to yield a field on the sub-manifold E, which does have time translations. Let $\{h_{1n}\}\dots\{h_{k,n}\}$ be a k arbitrary sequences of positive $\mathcal{S}(\mathbb{R})$ functions such that $\int \mathrm{d}t h_{jn}(t) = 1$ and $\operatorname{supp} h_{j,n} \to 0$, as $n \to \infty$. For $F_j \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(Y)$ let

$$\tilde{F}_{j,n}(t,\omega,y) = \frac{\partial}{\partial t} F_j\left(\frac{t-\omega}{2}, y\right) h_{j,n}(t+\omega) \ . \tag{5.20}$$

Then $\varphi(\tilde{F}_{1,n})\dots\varphi_E(\tilde{F}_{k,n})\Psi$ converges to a multilinear vector function $\Phi(F_1,\dots F_k)$ such that

a. $\Phi_E(e_1 \otimes g_1,\dots,e_k \otimes g_k)$ is continous w.r.t. $e_1,\dots e_k \in \mathcal{S}(\mathbb{R})$, $g_1,\dots,g_k \in \mathcal{S}(Y)$.

b. Exist $F_1,\dots F_k$ such that Φ_E is not constant and

$$F_{l,\tau}(x_E) = F_l(L_E(-\tau)x_E) \ . \tag{5.21}$$

The latter axiom is desined such that the field φ induced a field φ_E on the event horizon E. This is done by

$$\varphi_E(t,y) = -\frac{\partial}{\partial t}\varphi(t,-t,y) \ . \tag{5.22}$$

In fact it is possible to prove (see [9] for the proof) that there is a field $(\mathcal{F}_E, \mathcal{H}_E, \varphi_E, \Psi, \boldsymbol{L}_E)$ on E such that $\Phi_E(F_1,\dots F_k) = \varphi(F_1)\dots\varphi(F_k)\Psi$ and

E1. $\varphi_E(e_1 \otimes g_1)\dots\varphi_E(e_n \otimes e_n)\Psi$ is continuous w.r.t. $e_1,\dots e_k \in \mathcal{S}(\mathbb{R})$, $g_1,\dots,g_k \in \mathcal{S}(Y)$;

E2. Hermiticity: $\varphi_E(F)^\star = \varphi_E(\overline{F})$;

E3. Covariance (w.r.t. $L(\mathbb{R})$): $\boldsymbol{L}_E(\tau)$ is the restriction of the $\boldsymbol{L}(\tau)$ on $\mathcal{H}_E$ and

$$\boldsymbol{L}_E(\tau)\Psi = \Psi \ , \qquad \boldsymbol{L}_E(\tau)\varphi(F)\boldsymbol{L}_E(-\tau) = \varphi(F_\tau) \ , \tag{5.23}$$

where $F_\tau(x) = F(L_E(\tau)x_E)$.

E4. Locality: $[\varphi(F), \varphi(h)] = 0$ if the support of F and F are space-like separated;

E4'. Further Locality: $[\varphi(e_1 \otimes g_1), \varphi(e_2 \otimes g_2)] = 0$ if the support of e_1 and e_2 are disjoint;

E5. Covariance: There is a continous unitary representation $\boldsymbol{T}_E$ of $\mathbb{R}$ on $\mathcal{H}_E$ such that

$$\boldsymbol{T}_E(t)\Psi = \Psi \ , \qquad \boldsymbol{T}_E(t)\varphi_E(F)\boldsymbol{T}_E(-t) = \varphi_E(F_t) \ , \tag{5.24}$$

where $F_t(x_E) = F\left(T_E(-t)x_E\right)$.

E6. Spectrum condition:The generator of $\boldsymbol{T}_E$, $\boldsymbol{K}$ is a selfadjoint, positive operator.

Now E1-E6 are very similar to the Wighmann axioms which are enough to show that gravitation induces thermal states as we shown in the case of the Rindler space-time. Indeed the following results can be proved

R1. Let $\mathcal{I}$ be in interval in $\mathbb{R}$ and $\mathcal{A}_E(\mathcal{I})$ the algebra of polynomials in $\{\varphi_E(e\otimes g)|e\in \mathcal{S}(\mathbb{R}), \operatorname{supp} e\subset\mathcal{I}, g\in\mathcal{S}(Y)\}$. Then $\mathcal{A}_E(\mathcal{I})$ is dense in $\mathcal{H}_E$.

R2. There is an unique conjugation J_E of $\mathcal{H}_E$ such that

$$J_E\varphi_E(F_1)\dots\varphi_E(F_k)\Psi = \varphi_E(F_k^\dagger)\dots\varphi_E(F_1^\dagger)\Psi\ , \tag{5.25}$$

where $F^\dagger(t,y)=\overline{F}(-t,y)$.

R3. Let $\mathcal{A}_E^\pm$ be the algebra of polynomials in $\{\varphi_E(F)|F\in\mathcal{S}(\mathbb{R})\times\mathcal{S}(Y), \operatorname{supp} F\subset E^\pm\}$. Then Ψ is cyclic w.r.t. $\mathcal{A}^\pm$ in $\mathcal{H}_E$ and

$$J_E\exp\left(\mp\pi\boldsymbol{K}_E\right)A_E\Psi = A_E^\star\Psi\ , \tag{5.26}$$

for every $A_E\in\mathcal{A}_E^\pm$. Since E^+ is stable under $L_E(\mathbb{R})$ it follows that Ψ satisfy the KMS condition w.r.t. $\boldsymbol{L}(\mathbb{R})$ for the restriction of φ_E on E^+. The temperature (the non-oberved one) is $T=1/2\pi$.

The proofs of those two theorems are really similar to those of the Reeh-Schlider, PCT and Bisognano-Wichmann theorems. The differences are technical: for references see [9].
Finally we have to discuss the results for fields in $X^\pm$. Those are fields $\varphi^\pm$ whose test functions have support in $X^\pm$. Denoted by $\boldsymbol{L}^\pm$ the unitary representation of $\mathbb{R}$ in $\mathcal{H}^\pm$ defined by

$$\boldsymbol{L}^\pm(\tau)\Psi=\Psi\ ,\qquad \boldsymbol{L}^\pm(\tau)\varphi^\pm(f)\boldsymbol{L}^\pm(-\tau)=\varphi^\pm(f_\tau)\ , \tag{5.27}$$

it is clear that $\boldsymbol{L}^\pm$ is the restriction of $\boldsymbol{L}$ on $X^\pm$. Similarly, because of E3, $\boldsymbol{L}_E$ is the restriction of $\boldsymbol{L}$ on E. To conclude what we need we need one more assumption. Observe that $\mathcal{H}_E\subset\mathcal{H}^+$ since $\mathcal{A}_E^+\Psi$ is dense in $\mathcal{H}_E$ and by A5 vectors in $\mathcal{A}_E^+\Psi$ can be approximated arbitrarily closely by vector in $\mathcal{A}^+\Psi$. What we actually need is either

C1. $\mathcal{H}_E=\mathcal{H}^+$; This is related to the assumption that the fields on X^+ can be determined by fields in E in such a way $\mathcal{A}'_E\subset\mathcal{A}^{+'}$ where $\mathcal{A}'_E$ and $\mathcal{A}^{+'}$ are the weak commutants. In fact if $\boldsymbol{P}_E$ is the projection from $\mathcal{A}^+$ to $\mathcal{A}_E$, it has to lie in $\mathcal{A}'_E$ and therefore

$$||(\mathbf{1}-\boldsymbol{P}_E)A\Psi||^2=((\mathbf{1}-\boldsymbol{P}_E)A\Psi,A\Psi)=(A^\star A\Psi,(\mathbf{1}-\boldsymbol{P}_E)\Psi)=0\ , \tag{5.28}$$

since $P_E\in\mathcal{A}'_E\subset\mathcal{A}^{+'}$.

C2. The restriction of Ψ on $\mathcal{A}^+$ corresponds to an equilibrium state of the field on X^+ at some temperature (possibly zero). This is an assumption on the stablity of the state of the field on X^+.

Under all the assumption A and either C1 or C2 it is possible to prove that the restriction of Ψ to the field on X^+ corresponds to a thermal state of temperature $T=1/2\pi$, w.r.t. the dynamical group $\boldsymbol{L}^+(\mathbb{R})$, i.e. there is a conjugation J of $\mathcal{H}$ such that

$$J\exp\left(-\pi\boldsymbol{K}^+\right)A\Psi=A^\star\Psi\ . \tag{5.29}$$

Observe that from the metric the proper time for a local observer in X^+ is given by

$$\tau_p = \sqrt{A(-\xi^2, y)}\tau \ , \tag{5.30}$$

and therefore the local temperature is

$$T = \frac{1}{2\pi\sqrt{A(-\xi^2, y)}} \ . \tag{5.31}$$

This is exactly the generalization of the Hawking temperature. The proof of the statement above is given in all the details in [9].

A Conformally equivalent metrics

We prove here that a null geodesic for for a metric is a null geodesic also for any conformally equivalent metric. The trajectory of a null geodesic is

$$g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu = 0 \tag{A.1}$$

and this trajectory is the same for $\hat{g}_{\mu\nu}$ in fact

$$\hat{g}_{\mu\nu}\dot{X}^\mu \dot{X}^\nu = e^{2\omega} g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu = 0 \ . \tag{A.2}$$

The same is true if we consider a space/time-like curve, since $e^{2\omega} > 0$. However observe that the geodesic equations in one metric is different from the geodesic equation in the other one and therefore time/space-like geodesics in one metric are not necessarily time/space-like geodesics in the other one.

B Schwarzschild solution

Let's solve the Einstein equations for a particular case: let's assume that the system is static, which means that is invariant under spacetime translations and temporal inversion, and spherically symmetric. A particular solution for the Minkowski case is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right)$$

Let's use it as a starting point and let us deformed it; a general ansaz satisfying the above conditions is

$$\mathrm{d}s^2 = -e^{2A(r)}\mathrm{d}t^2 + e^{2B(r)}\mathrm{d}r^2 + e^{2C(r)}\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right)$$

Assuming that $\partial_r C(r)$ is not vanishing, introducing the coordinate $\tilde{r} = e^{C(r)}$ and redefining $\tilde{r} = r$ one can obtain

$$\mathrm{d}s^2 = -e^{2A(r)}\mathrm{d}t^2 + e^{2B(r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right)$$

We have only two function to be found: $A(r)$ and $B(r)$. Let's use the Einstein equation in vacuum when $T^{\mu\nu} = 0$. Let's use the Vielbein formalism [9] and consider

$$e^0 = e^{A(r)}\mathrm{d}t \quad e^1 = e^{B(r)}\mathrm{d}r \quad e^2 = r\mathrm{d}\theta \quad e^3 = r\sin\theta\mathrm{d}\varphi$$

[9] The same calculation without using the Vielbein formalism is provided in [11].

We want to find the spin connection. In order to do that we have to impose the null torsion condition and the condition $\nabla_\mu \eta_{ab} = 0$ (which defines the Levi-Civita connection); these two conditions can be written as

$$\mathrm{d}e^a + \omega^a_{\ b} \wedge e^b = 0 \qquad \omega_{ab} = -\omega_{ba}$$

The first equation is

$$\omega^0_{\ b} \wedge e^b = \omega^0_{\ 1} \wedge e^1 + \omega^0_{\ 2} \wedge e^2 + \omega^0_{\ 3} \wedge e^3 = -\mathrm{d}e^0 = -\dot{A}e^A \mathrm{d}r \wedge \mathrm{d}t$$

Where we read $\omega^0_{\ 2} \propto e^2$, $\omega^0_{\ 3} \propto e^3$ and $\omega^0_{\ 1} \wedge e^1 = \dot{A}e^{A-B}\mathrm{d}t \wedge e^1$. Then one can write

$$\omega^1_{\ b} \wedge e^b = \omega^1_{\ 1} \wedge e^1 + \omega^1_{\ 2} \wedge e^2 + \omega^1_{\ 3} \wedge e^3 = -\mathrm{d}e^1 = 0$$

from which $\omega^1_{\ 2} \propto e^2$, $\omega^1_{\ 3} \propto e^3$ and $\omega^1_{\ 0} \propto e^0$ and so the first solution is $\omega^0_{\ 1} = \dot{A}e^{A-B}\mathrm{d}t$. From the equation

$$\omega^2_{\ b} \wedge e^b = \omega^2_{\ 1} \wedge e^1 + \omega^2_{\ 2} \wedge e^2 + \omega^2_{\ 3} \wedge e^3 = -\mathrm{d}e^2 = -\mathrm{d}r \wedge \mathrm{d}\theta$$

one can conclude that $\omega^2_{\ 3} \propto e^3$ and so $\omega^0_{\ 2} = 0$ e $\omega^1_{\ 2} = -e^{-B}\mathrm{d}\theta$. The last equation is

$$\omega^3_{\ b} \wedge e^b = \omega^3_{\ 1} \wedge e^1 + \omega^3_{\ 2} \wedge e^2 + \omega^3_{\ 3} \wedge e^3 = -\mathrm{d}e^3 = -\sin\theta \mathrm{d}r \wedge \mathrm{d}\varphi - r\cos\theta \mathrm{d}\theta \wedge \mathrm{d}\varphi$$

form which $\omega^0_{\ 3} = 0$ and so $\omega^1_{\ 3} = -\sin\theta e^{-B}\mathrm{d}\varphi$ and $\omega^2_{\ 3} = -\cos\theta\mathrm{d}\varphi$.
Summarizing

$$\begin{array}{cc} \omega^0_{\ 1} = \dot{A}e^{A-B}\mathrm{d}t & \omega^0_{\ 2} = 0 \\ \omega^0_{\ 3} = 0 & \omega^1_{\ 3} = -\sin\theta e^{-B}\mathrm{d}\varphi \\ \omega^2_{\ 3} = -\cos\theta\mathrm{d}\varphi & \omega^1_{\ 2} = -e^{-B}\mathrm{d}\theta \end{array}$$

Recalling that the curvature tensor is

$$R^a_{\ b} = \mathrm{d}\omega^a_{\ b} + \omega^a_{\ c} \wedge \omega^c_{\ b}$$

with simple steps one can find

$$\begin{array}{cc} R^0_{\ 1} = -\left(\ddot{A} + \dot{A}^2 - \dot{A}\dot{B}\right)e^{-2B}e^0 \wedge e^1 & R^0_{\ 2} = -\frac{\dot{A}}{r}e^{-2B}e^0 \wedge e^2 \\ R^0_{\ 3} = -\frac{\dot{A}}{r}e^{-2B}e^0 \wedge e^2 & R^1_{\ 2} = \frac{\dot{B}e^{-2B}}{r}e^1 \wedge e^2 \\ R^1_{\ 3} = \frac{\dot{B}e^{-2B}}{r}e^1 \wedge e^3 & R^2_{\ 3} = \frac{1-e^{2B}}{r}e^2 \wedge e^3 \end{array}$$

From which the Ricci tensor is[10]

$$R_{00} = \left(\ddot{A} + \dot{A}^2 - \dot{A}\dot{B} + \frac{2\dot{A}}{r}\right)e^{-2B} \qquad R_{11} = \left(-\left(\ddot{A} + \dot{A}^2 - \dot{A}\dot{B}\right) + \frac{2\dot{B}}{r}\right)e^{-2B}$$

$$R_{22} = R_{33} = \left(-\dot{A} + \dot{B} + \frac{e^{2B} - 1}{r}\right)\frac{e^{-2B}}{r}$$

[10]Observe that $R_{22} = R_{33}$, as we expected from the symmetry of the problem.

The fact that $G_{ab} = 0$ means that $R_{ab} = 0$ and so $R_{00} + R_{11} = 0$ which means $\dot{A} + \dot{B} = 0$. From this one can conclude $B = -A + c$ but rescaling properly the time and setting $c = 0$ the result is $A = -B$. Moreover $R_{22} = 0$ means that

$$e^{2A} = 1 + \frac{c}{r}$$

and the equation $R_{00} - R_{11}$ is automatically solved.
In order to find the value of c one can recall that for $r \to \infty$ the metric has to be flat. So using the weak field approximation

$$h_{00} = -\frac{c}{r} = -2\Phi = 2\frac{GM}{r} \Rightarrow c = -2GM$$

In conclusion for an spherically symmetric object [11] the metric is

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)$$

This is the *Schwarzschild solution*.

C Proof of equation (2.10)

In order to prove equation (2.10) simply substistute $\boldsymbol{b}_i$ and $\boldsymbol{b}_i^\dagger$ with their decomposition in $\boldsymbol{a}_i$ and $\boldsymbol{a}_i^\dagger$. In particular

$$\begin{aligned}
\langle 0, in|\, \boldsymbol{b}_i^\dagger \boldsymbol{b}_i \,|0, in\rangle &= \sum_j \sum_l \langle 0, in|\,(\alpha_{il}\boldsymbol{a}_l^\dagger - \beta_{il}\boldsymbol{a}_l)(\overline{\alpha}_{ij}\boldsymbol{a}_j - \overline{\beta}_{ij}\boldsymbol{a}_j^\dagger)\,|0, in\rangle = \\
&= \sum_j \sum_l \langle 0, in|\,\beta_{il}\overline{\beta}_{ij}\boldsymbol{a}_l\boldsymbol{a}_j^\dagger\,|0, in\rangle = \\
&= \sum_j \sum_l \delta_{jl}\beta_{il}\overline{\beta}_{ij} = \sum_j |\beta_{ij}|^2 \ .
\end{aligned} \tag{C.1}$$

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