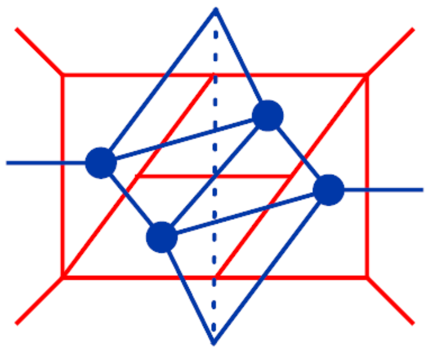
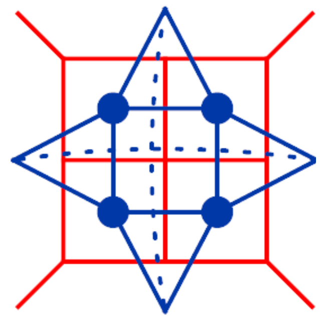
Higher Loop Amplitudes in N=4 Super-Yang-Mills Theory





Lance Dixon (SLAC)

DESY Theory Workshop

28 September 2007

Dedicated to Pief Panofsky 1919-2007

N=4 SYM and AdS/CFT

- N=4 SYM for gauge group $SU(N_c)$:
  scale-invariant (conformal) field theory
  for all g: $\beta(g) = 0$
- AdS/CFT duality

Maldacena; Gubser, Klebanov, Polyakov; Witten; Barbón, this workshop

suggests that weak-coupling perturbation series in $\lambda = g^2 N_c$ for planar limit (large N_c) should have special properties, because

strong-coupling limit ←→ weakly-coupled gravity/string theory

Gluon scattering in N=4 SYM

• Some quantities are protected, unrenormalized, so the series in λ is trivial (e.g. energies of BPS states)

• 2 → 2 gluon scattering amplitudes are not protected

How does series organize itself into simple result, from gravity/string point of view? Anastasiou, Bern, LD, Kosower

• Cusp anomalous dimension $\gamma_K(\lambda)$ is a new, nontrivial example, solved to all orders in λ using integrability

Beisert, Eden, Staudacher; Aryutunov, Beisert, this workshop

• Proposal: $\gamma_K(\lambda)$ is one of just four functions of λ alone, which fully specify gluon scattering to all orders in λ, for any scattering angle θ (value of t/s). And they specify n-gluon MHV amplitudes. Bern, LD, Smirnov

• Recent confirmation for 2 → 2 at strong coupling.

• n-point? Alday, Maldacena, 0705.0303[th]; Maldacena, this workshop

Some questions you might have

- What are gluons? They're certainly not the gauge-invariant local operators found in the usual AdS/CFT dictionary.
  Alday, Maldacena
- What does scattering mean in a conformal field theory, in which the interactions never shut off?
- What are the other functions of λ?
- What is the evidence for this proposal at weak coupliing?

“String Theory Meets Collider Physics”

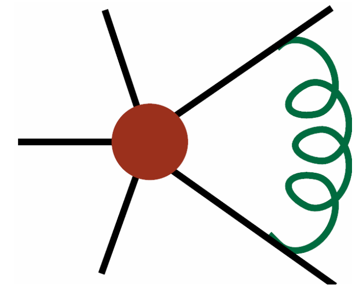
- Gluons (in QCD, not N=4 SYM) are the objects colliding at the LHC (most of the time).
- Interactions between gluons never turn off in QCD either. In fact, it is worse, due to asymptotic freedom – the coupling grows at large distances.
- We use dimensional regularization, with $D=4-2\varepsilon$, to regulate these long-distance, infrared (IR) divergences. (Actually, dimensional reduction/expansion to preserve all the supersymmetry.) At the LHC, IR divergences in loop diagrams cancel against real emission of gluons.
- In string theory, gluons can be “discovered” by tying open string ends to a D-brane in the IR, and using the kinematics (large s and t) to force the string to stretch deep into the UV. Alday, Maldacena

Dimensional Regulation in the IR

One-loop IR divergences are of two types:

Soft

$$\int_0 \frac{d\omega}{\omega} \rightarrow \int_0 \frac{d\omega}{\omega^{1+\epsilon}} \propto \frac{1}{\epsilon}$$



$D = 4 - 2\epsilon$

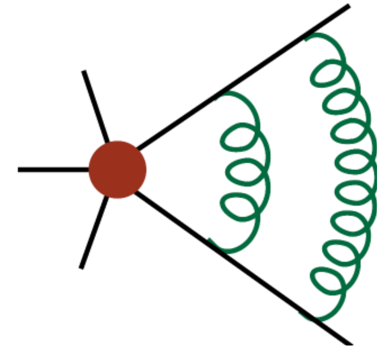
$\epsilon < 0$

Collinear (with respect to massless emitting line)

$$\int_0 \frac{dk_T}{k_T} \rightarrow \int_0 \frac{dk_T}{k_T^{1+\epsilon}} \propto \frac{1}{\epsilon}$$

Overlapping soft + collinear divergences imply leading pole is $\frac{1}{\epsilon^2}$ at 1 loop

$\frac{1}{\epsilon^{2L}}$ at L loops



IR Structure in QCD and N=4 SYM

• Pole terms in ε are predictable due to soft/collinear factorization and exponentiation – long-studied in QCD, straightforwardly applicable to N=4 SYM

Akhoury (1979); Mueller (1979); Collins (1980); Sen (1981); Sterman (1987);
Botts, Sterman (1989); Catani, Trentadue (1989); Korchemsky (1989)
Magnea, Sterman (1990) ; Korchemsky, Marchesini, hep-ph/9210281
Catani, hep-ph/9802439 ; Sterman, Tejeda-Yeomans, hep-ph/0210130

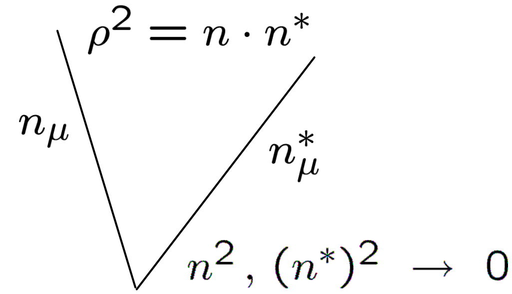
In the planar limit, for both QCD and N=4 SYM, pole terms are given in terms of:

- the beta function $\beta(\lambda)$ [= 0 in N=4 SYM]
- the cusp (or soft) anomalous dimension $\gamma_K(\lambda)$
- a "collinear" anomalous dimension $\mathcal{G}_0(\lambda)$

Cusp anomalous dimension

VEV of Wilson line with kink or cusp in it obeys renormalization group equation:

$$\left(\rho\frac{\partial}{\partial\rho}+\beta(g)\frac{\partial}{\partial g}\right)\ln W(\rho,g) = -2\,\gamma_K(g)\,\ln\rho^2+\mathcal{O}(\rho^0)$$



Polyakov (1980); Ivanov, Korchemsky, Radyushkin (1986); Korchemsky, Radyushkin (1987)

Cusp (soft) anomalous dimension $\gamma_K(g)$ also controls

large-spin limit of anomalous dimensions γ_j of leading-twist operators with spin *j*: $\bar{q}(\gamma^+D_+)^j q$

$$\gamma_j = \frac{1}{2}\gamma_K(g)\,\ln j\ +\ \mathcal{O}(j^0)$$

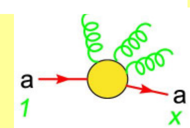
Korchemsky (1989);
Korchemsky, Marchesini (1993)

Related by Mellin transform to $x\longrightarrow 1$ limit of DGLAP kernel for evolving parton distribution functions $f(x,\mu_F)$:

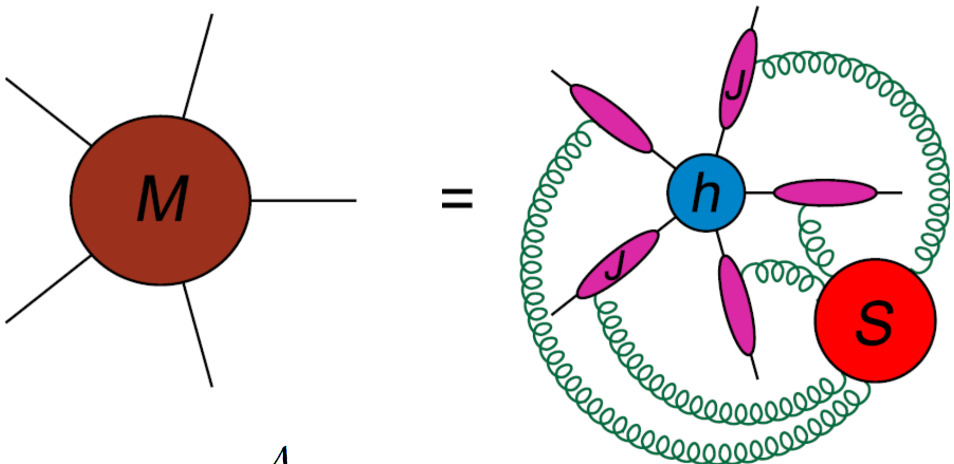
$$\gamma_j \equiv -\int_0^1 dx\, x^{j-1}\, P_{aa}(x)$$

$$P_{aa}(x) = \frac{1}{2}\frac{\gamma_K(g)}{(1-x)_+} + B(g)\,\delta(1-x)+\cdots$$

→ important for soft gluon resummations



Soft/Collinear Factorization



Magnea, Sterman (1990)
Sterman, Tejeda-Yeomans, hep-ph/0210130

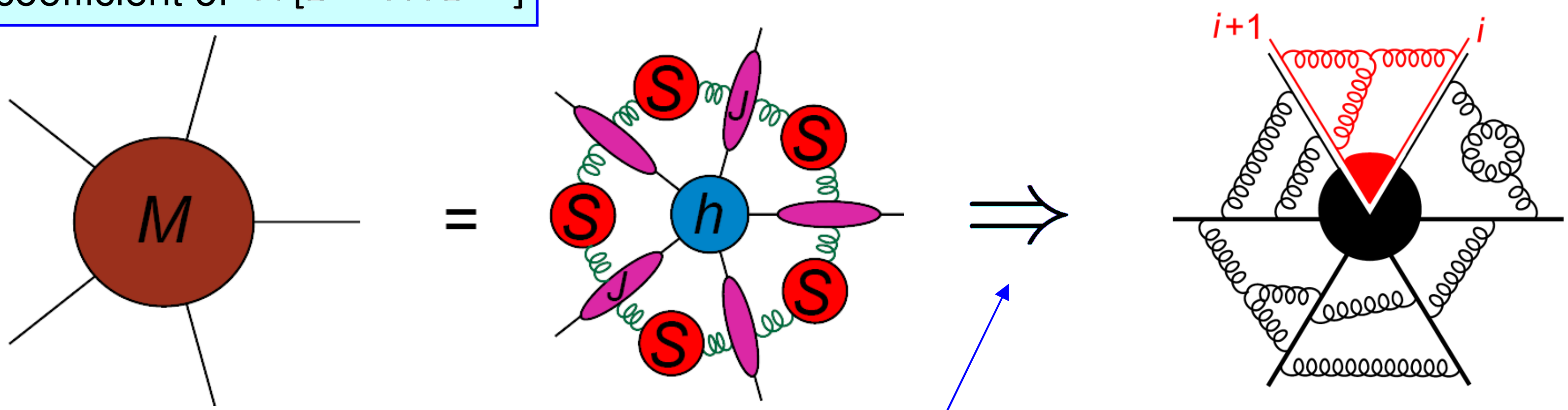
$$\mathcal{M}_n \equiv \frac{\mathcal{A}_n}{\mathcal{A}_n^{\text{tree}}}$$

$$\boxed{\mathcal{M}_n = S(k_i,\mu,\alpha_s(\mu),\epsilon) \times \left[\prod_{i=1}^{n} J_i(\mu,\alpha_s(\mu),\epsilon)\right] \times h_n(k_i,\mu,\alpha_s(\mu),\epsilon)}$$

- *S = soft function (only depends on color of i^{th} particle)*
- *J = jet function (color-diagonal; depends on i^{th} spin)*
- *h_n= hard remainder function (finite as $\epsilon \to 0$)*

Simplification at Large N_c (Planar Case)

coefficient of Tr$[T^{a_1} \ldots T^{a_n}]$



- *Soft function* only defined up to a multiple of the identity matrix in color space
- Planar limit is color-trivial; can absorb S into J_i
- If all n particles are identical, say gluons, then each "wedge" is the square root of the "*gg → 1*" process (Sudakov form factor):

$$\mathcal{M}_n = \prod_{i=1}^{n} \left[\mathcal{M}^{[gg\to 1]} \left(\frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon \right) \right]^{1/2} \times h_n(k_i, \mu, \alpha_s, \epsilon)$$

Sudakov form factor

- Factorization → differential equation for form factor

Mueller (1979); Collins (1980); Sen (1981); Korchemsky, Radyushkin (1987); Korchemsky (1989); Magnea, Sterman (1990)

$$\frac{\partial}{\partial \ln Q^2} \mathcal{M}^{[gg\to 1]}(Q^2/\mu^2, \alpha_s(\mu), \epsilon)$$
$$= \frac{1}{2}\Big[K(\epsilon, \alpha_s) + G(Q^2/\mu^2, \alpha_s(\mu), \epsilon)\Big] \times \mathcal{M}^{[gg\to 1]}(Q^2/\mu^2, \alpha_s(\mu), \epsilon)$$

finite as $\varepsilon \to 0$; contains all Q^2 dependence

Pure counterterm (series of $1/\varepsilon$ poles);
like $\beta(\varepsilon, \alpha_s)$, single poles in ε determine K completely

K, G also obey differential equations (ren. group):

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g}\right)(K+G) = 0 \qquad \left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g}\right)K = -\gamma_K(\alpha_s)$$

cusp anomalous dimension

General amplitude in planar N=4 SYM

- Solve differential equations for K, G. **Easy** because coupling doesn't run.
- Insert result for Sudakov form factor into *n*-point amplitude

$$\Rightarrow \quad \mathcal{M}_n = 1 + \sum_{L=1}^{\infty} a^L M_n^{(L)} = \exp\Big[\underbrace{-\frac{1}{8} \sum_{l=1}^{\infty} a^l \Big(\frac{\hat{\gamma}_K^{(l)}}{(l\epsilon)^2} + \frac{2\hat{\mathcal{G}}_0^{(l)}}{l\epsilon} \Big) \sum_{i=1}^{n} \Big(\frac{\mu^2}{-s_{i,i+1}} \Big)^{l\epsilon}} \Big] \times h_n$$

loop expansion parameter:

$$a \equiv \frac{N_c \alpha_s}{2\pi} (4\pi e^{-\gamma})^{\epsilon} = \frac{\lambda}{8\pi^2} (4\pi e^{-\gamma})^{\epsilon}$$

looks like the one-loop amplitude, but with ε shifted to ($l\,\varepsilon$), up to finite terms

$\hat{\gamma}_K^{(l)}, \hat{\mathcal{G}}_0^{(l)}$ are l-loop coefficients of $\gamma_K(a), \mathcal{G}_0(a)$

Rewrite as

$$\boxed{\mathcal{M}_n = \exp\Big[\sum_{l=1}^{\infty} a^l \Big(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + h_n^{(l)}(\epsilon, s_{i,i+1}) \Big) \Big]}$$

$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon\, f_1^{(l)} + \epsilon^2\, f_2^{(l)}$$

collects **3 series of constants**:

$$f_0^{(l)} = \frac{1}{4} \hat{\gamma}_K^{(l)} \qquad f_1^{(l)} = \frac{l}{2} \hat{\mathcal{G}}_0^{(l)} \qquad f_2^{(l)} = (???)$$

Exponentiation in planar N=4 SYM

• For planar N=4 SYM, propose that the finite terms also exponentiate. That is, the hard remainder function $h_n^{(l)}$ defined by

$$\mathcal{M}_n = \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + h_n^{(l)}(\epsilon, s_{i,i+1}) \right)\right]$$

is also a series of constants, $C^{(l)}$ [for **MHV** amplitudes]:

$$\mathcal{M}_n = \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + \mathcal{O}(\epsilon) \right)\right]$$

$$\Rightarrow \quad \mathcal{M}_4|_{\text{finite}} = \exp\left[\frac{1}{8}\gamma_K(a) \ln^2\left(\frac{s}{t}\right) + \text{const.}\right]$$

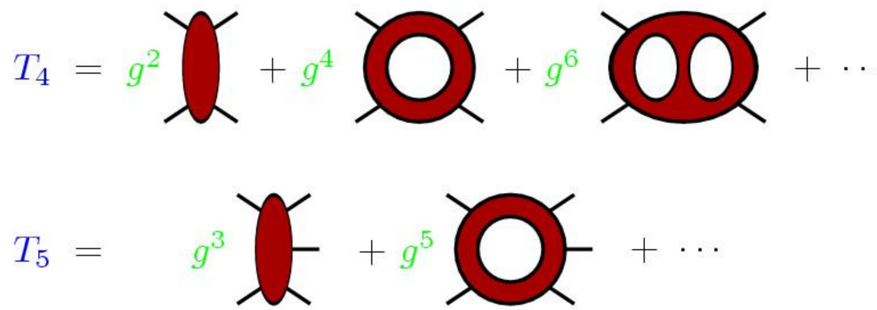
Anastasiou, Bern, LD, Kosower, hep-th/0309040; Cachazo, Spradlin, Volovich, hep-th/0602228; Bern, Czakon, Kosower, Roiban, Smirnov, hep-th/0604074

Evidence based on two loops (*n=4,5*, plus collinear limits)
and three loops (for *n=4*) Bern, LD, Smirnov, hep-th/0505205
and now strong coupling (*n=4,5 only?*) Alday, Maldacena, 0705.0303 [hep-th]

In contrast, for QCD, and non-planar N=4 SYM, two-loop amplitudes have been computed, and hard remainders are a mess of polylogarithms in *t/s*

Evidence: from amplitudes computed via perturbative unitarity

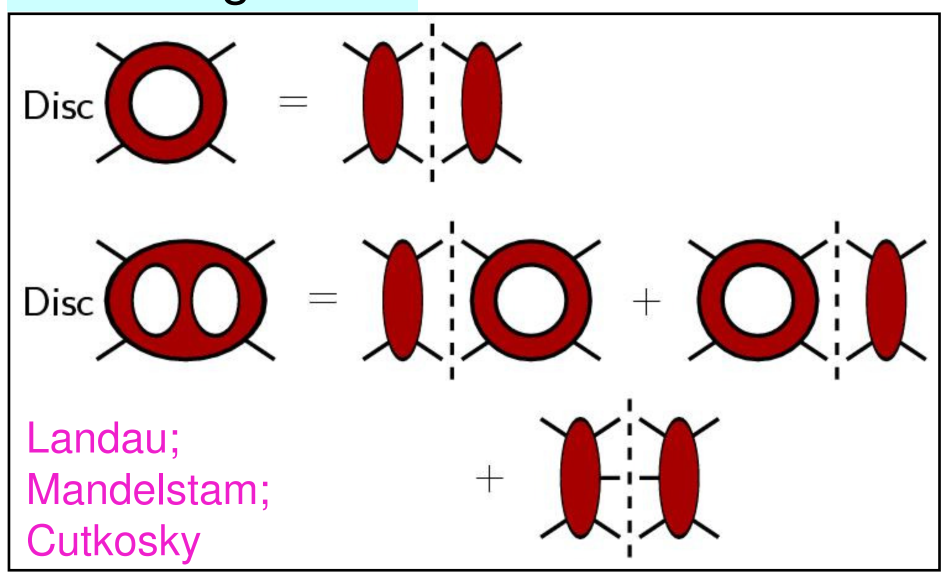
Expand scattering matrix T in coupling g



Insert expansion into unitarity relation

$$2\,\mathrm{Im}\,T = T^\dagger T$$

→ cutting rules:



Find representations of amplitudes in terms of different loop integrals, matching all the cuts

Very efficient – especially for N=4 SYM – due to simple structure of tree helicity amplitudes, plus manifest N=4 SUSY

Bern, LD, Dunbar, Kosower (1994);
Dunbar, this workshop

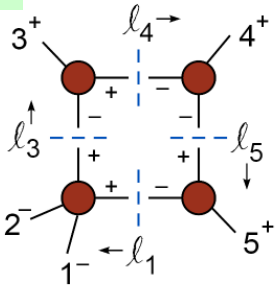
Generalized unitarity

If one cut is good, surely more must be better

Multiple cut conditions connected with leading singularities

Eden, Landshoff, Olive, Polkinghorne (1966)

At one loop, efficiently extract coefficients of triangle integrals & especially box integrals from products of trees



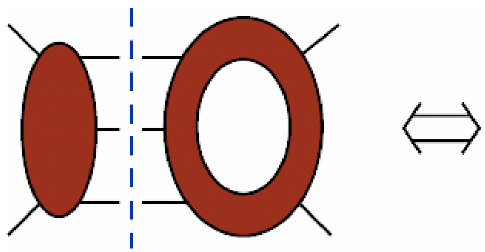
Bern, LD, Kosower (1997); Britto, Cachazo, Feng (2004)

Generalized unitarity at multi-loop level

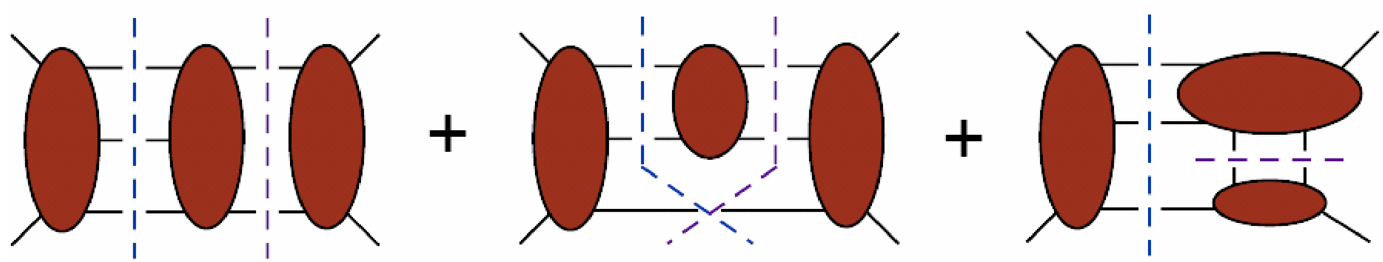
Bern, LD, Kosower (2000); BCDKS (2006); BCJK (2007)

In matching loop-integral representations of amplitudes with the cuts, it is convenient to work with tree amplitudes only.

For example, at 3 loops, one encounters the product of a 5-point tree and a 5-point one-loop amplitude:



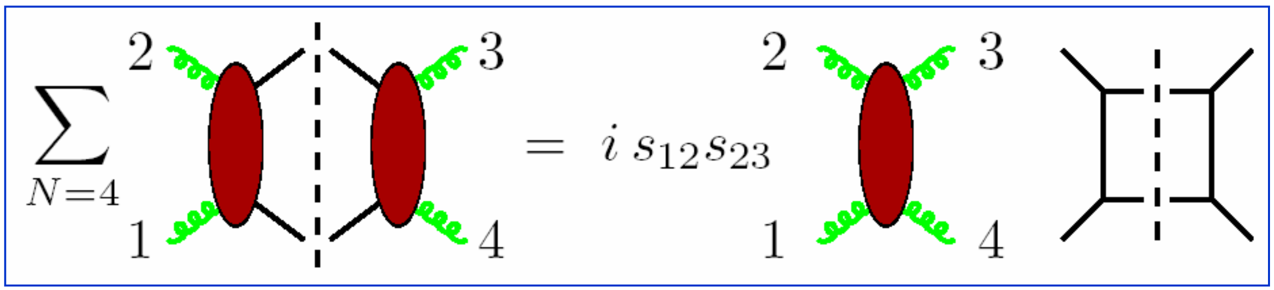
Cut 5-point loop amplitude further, into (4-point tree) x (5-point tree), in all inequivalent ways:



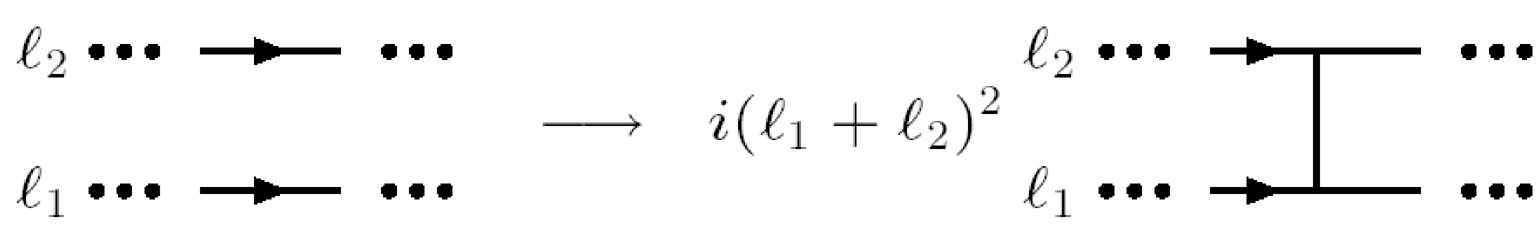
The rung rule

Many higher-loop contributions to *gg* → *gg* scattering deduced from a simple property of the 2-particle cuts at one loop

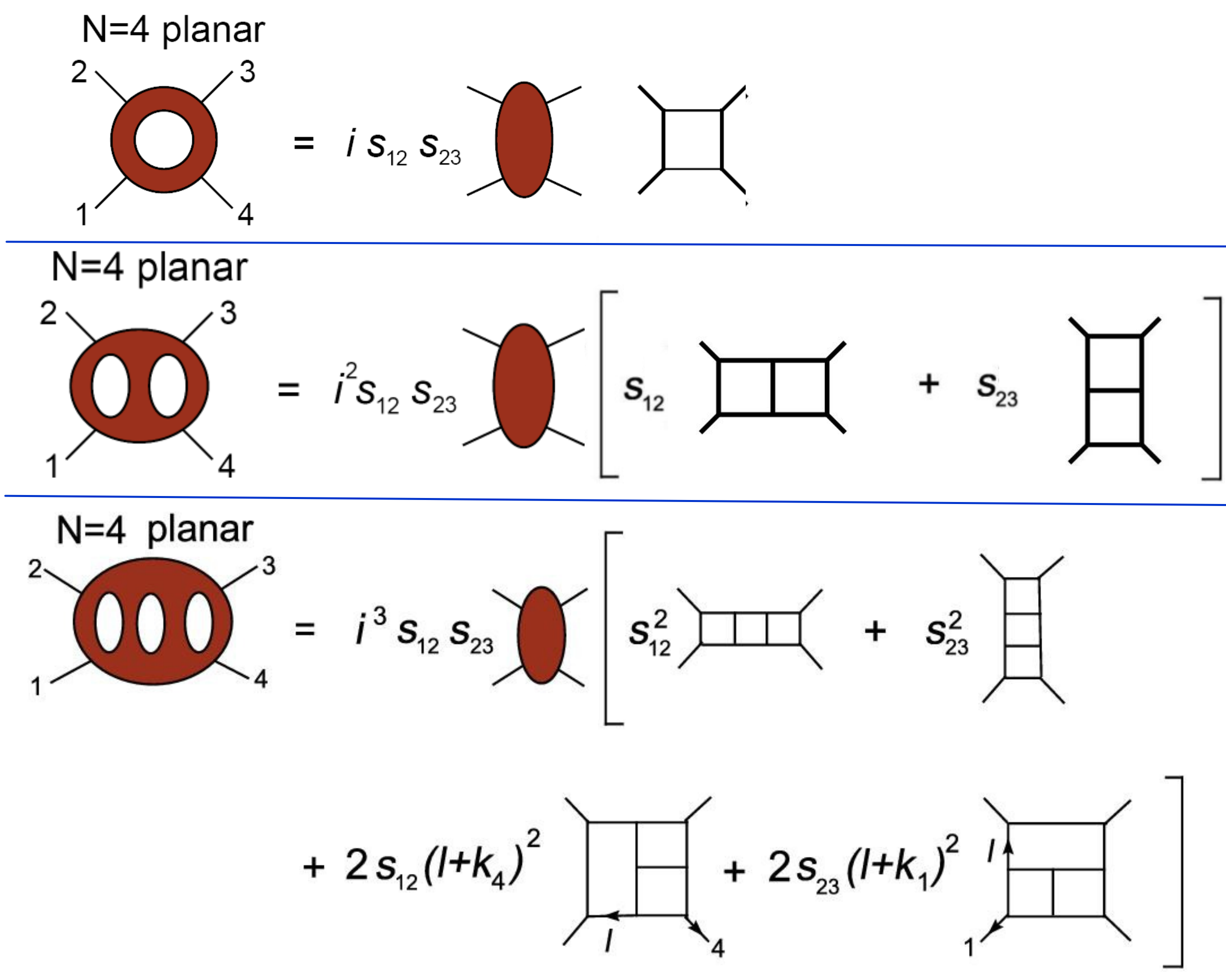
Bern, Rozowsky, Yan (1997)



Leads to "rung rule" for easily computing all contributions which can be built by iterating 2-particle cuts



Planar amplitudes from 1 to 3 loops



Green, Schwarz, Brink (1982)

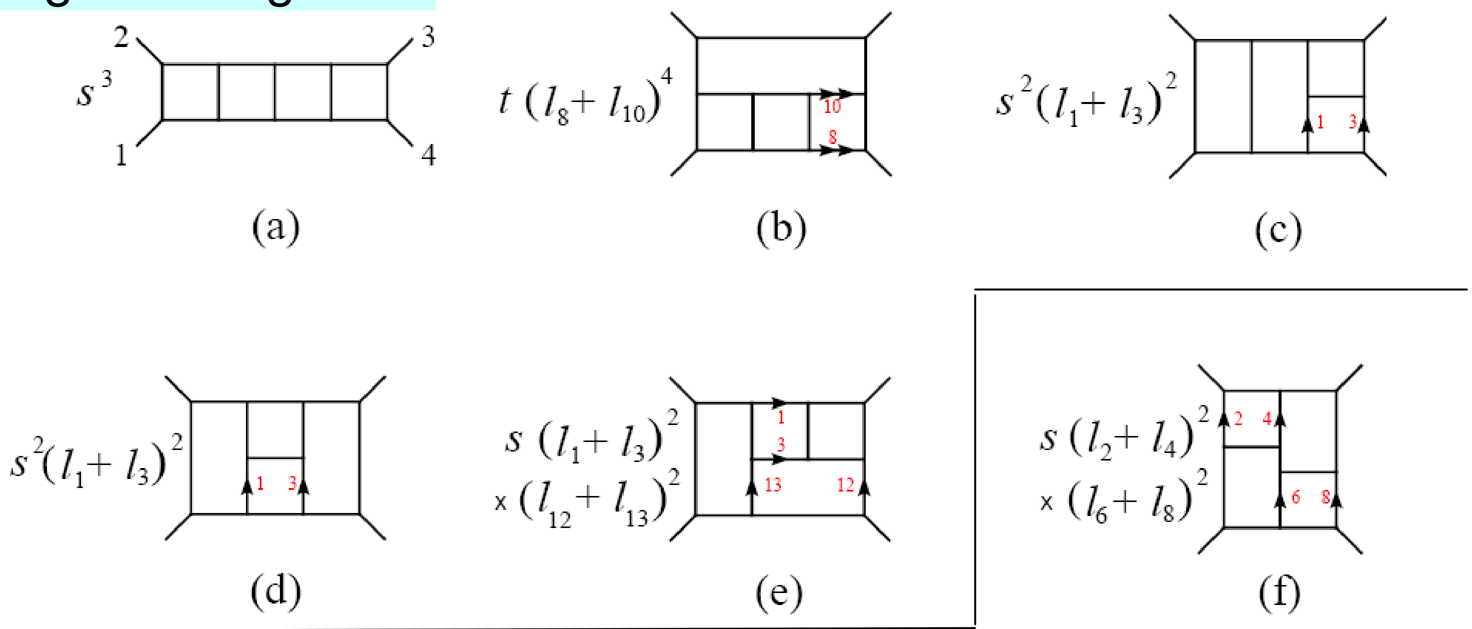
Bern, Rozowsky, Yan (1997)

all follow from rung rule

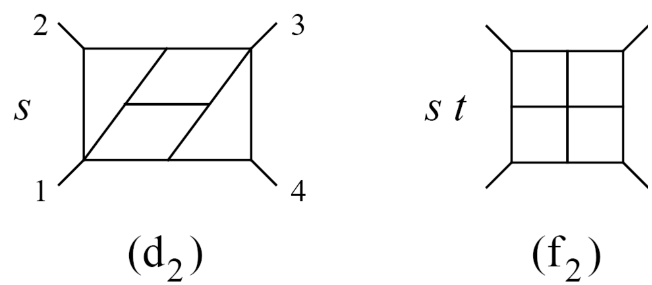
Integrals for planar amplitude at 4 loops

Bern, Czakon, LD, Kosower, Smirnov, hep-th/0610248

rung-rule diagrams



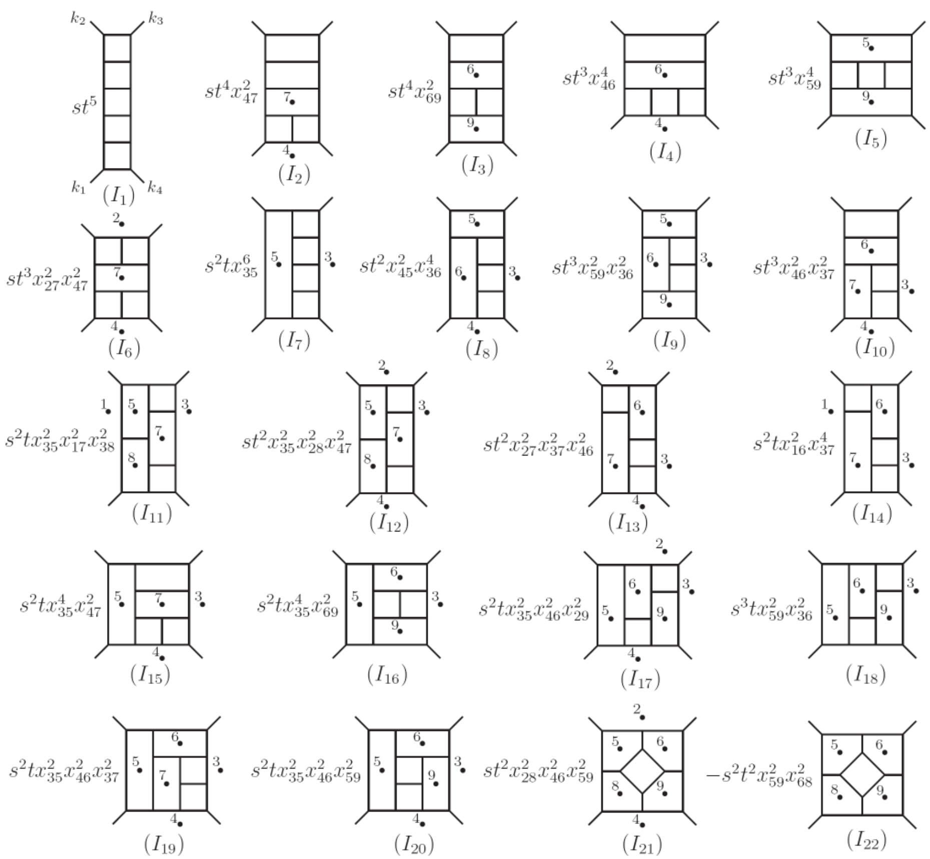
non-rung-rule diagrams,
no 2-particle cuts



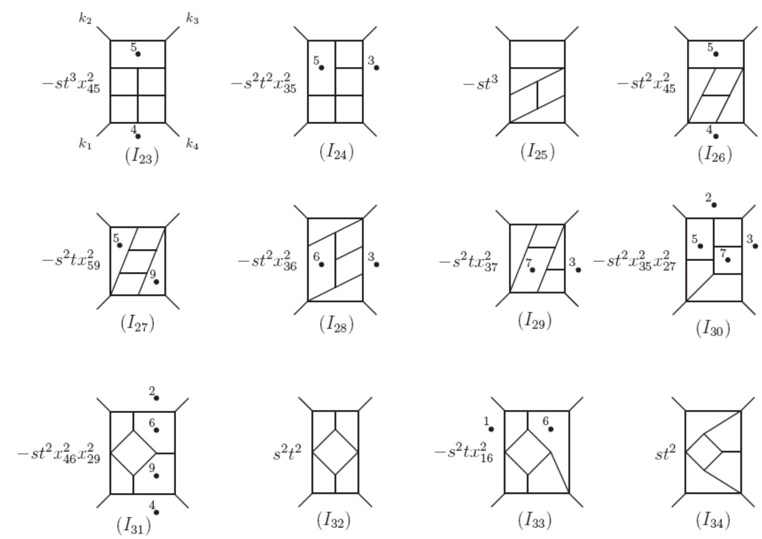
Integrals for planar amplitude at 5 loops

Bern, Carrasco, Johansson, Kosower, 0705.1864[th]

only cubic vertices (22)



also quartic vertices (12)

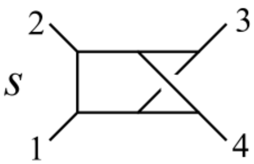


Subleading in $1/N_c$ terms

- Additional non-planar integrals are required
- Coefficients are known through 3 loops:

2 loops

s



Bern, Rozowsky, Yan (1997)

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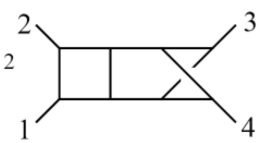
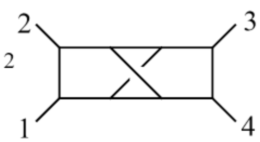
3 loops

Bern, Carrasco, LD, Johansson, Kosower, Roiban, hep-th/0702112

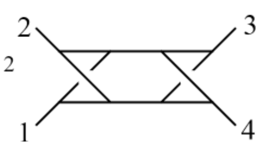
s^2



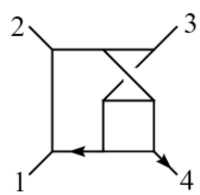
s^2



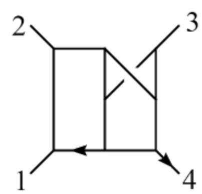
s^2



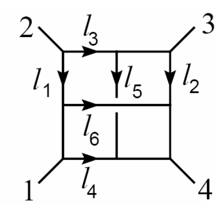
$s\,(l + k_4)^2$



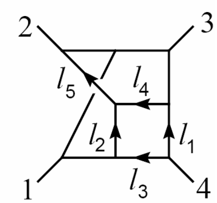
$s\,(l + k_4)^2$



$$s\,(l_1 + l_2)^2 + t\,(l_3 + l_4)^2 - s\, l_5^2 - t\, l_6^2 - s\, t$$

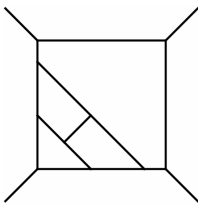


$$s\,(l_1 + l_2)^2 - t\,(l_3 + l_4)^2 - \frac{1}{3}(s - t)\, l_5^2$$

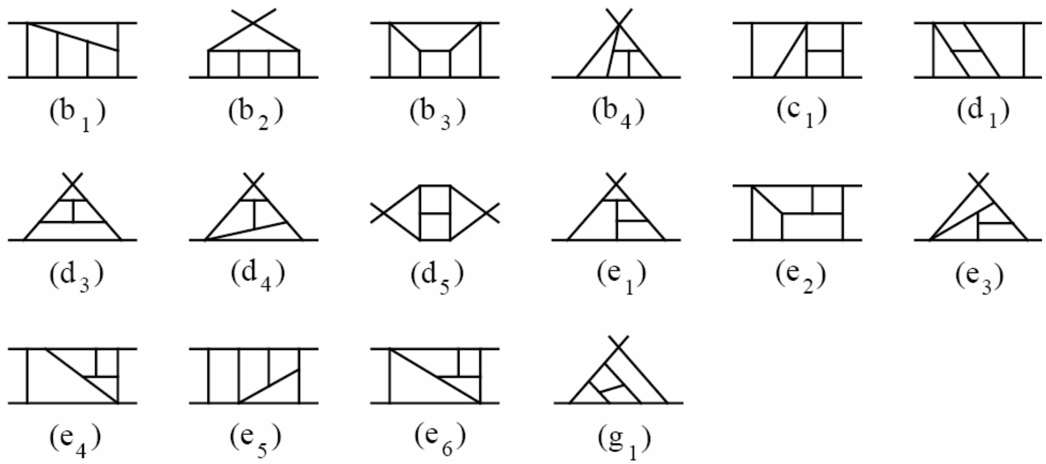


Patterns in the planar case

• At four loops, if we assume there are no triangle sub-diagrams, then besides the 8 contributing rung-rule & non-rung-rule diagrams, there are over a dozen additional possible integral topologies:



(g)



• Why do none of these topologies appear?
• What distinguishes them from the ones that do appear?

Surviving diagrams all have “dual conformal invariance”

- Although amplitude is evaluated in *D=4-2ε*,
all non-contributing no-triangle diagrams can be eliminated by requiring *D=4* “dual conformal invariance” and finiteness.
- Take $k_i^2 \neq 0$ to regulate integrals in *D=4*.
- Require inversion symmetry on dual variables x_i^μ : $\boxed{x_i^\mu \to \frac{x_i^\mu}{x_i^2}}$

Lipatov (2d) (1999); Drummond, Henn, Smirnov, Sokatchev, hep-th/0607160

- No explicit $x_{i-1,i}^2 = k_i^2$ allowed (so $k_i^2 \to 0$ OK)

$$x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}, \qquad d^4 x_i \to \frac{d^4 x_i}{x_i^8}$$

Requires 4 (net) lines out of every internal dual vertex,
1 (net) line out of every external one.
Dotted lines = numerator factors

Two-loop example

$k_1 = x_{41}$

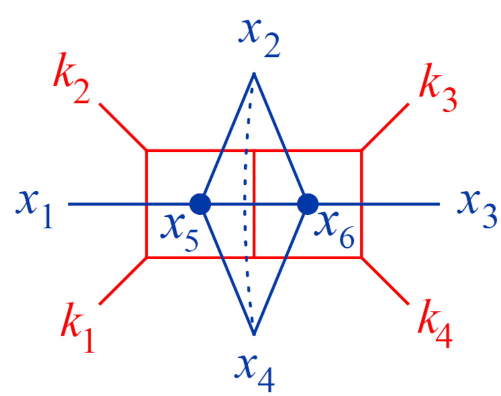
$k_2 = x_{12}$

$k_3 = x_{23}$

$k_4 = x_{34}$

$p = x_{45}$

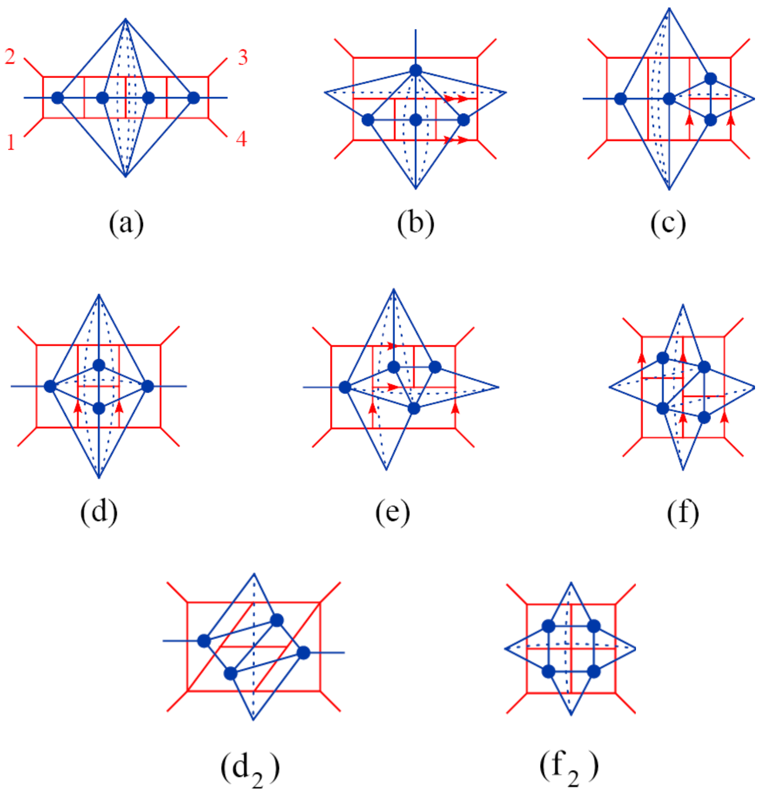
$q = x_{65}$



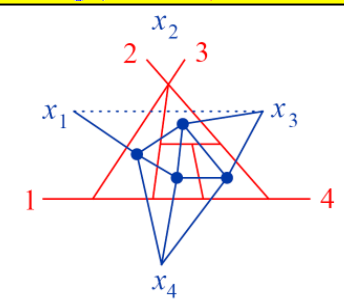
numerator: $x_{42}^2 = (k_1 + k_2)^2 = s$

Dual diagrams at four loops

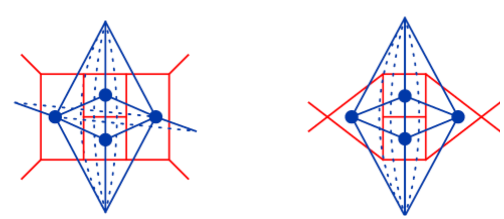
Present in the amplitude



Not present:
Requires $x_{34}^2 = k_4^2 = 0$ *on shell*



- 2 diagrams possess dual conformal invariance and a smooth $k_i^2 \to 0$ limit, yet are **not present** in the amplitude.
- But they are not finite in *D=4*



Drummond, Korchemsky, Sokatchev, 0707.0243[th]

Dual conformal invariance at five loops

Bern, Carrasco, Johansson, Kosower, 0705.1864[th]

59 diagrams possess dual conformal invariance and a smooth on-shell limit ($k_i^2 \to 0$)

Only 34 are present in the amplitude

The other 25 are not finite in *D=4*

Drummond, Korchemsky, Sokatchev, 0707.0243[th]

- Through 5 loops, only finite dual conformal integrals enter the planar amplitude.
- All such integrals do so with weight ± 1.

It's a pity, but there does not (yet) seem to be a good notion of dual conformal invariance for nonplanar integrals…

Back to exponentiation: the 3 loop case

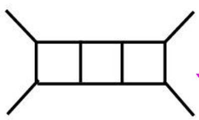
• *L*-loop formula:

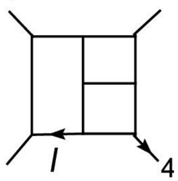
$$\mathcal{M}_n = \exp\Big[\sum_{l=1}^{\infty} a^l \Big(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + \mathcal{O}(\epsilon) \Big)\Big]$$

implies at 3 loops:

$$M_n^{(3)}(\epsilon) = -\frac{1}{3}\big[M_n^{(1)}(\epsilon)\big]^3 + M_n^{(1)}(\epsilon) M_n^{(2)}(\epsilon) + f^{(3)}(\epsilon) M_n^{(1)}(3\epsilon) + C^{(3)} + \mathcal{O}(\epsilon)$$

• To check exponentiation at $\mathcal{O}(\epsilon^0)$ for $n=4$, need to evaluate just 4 integrals:

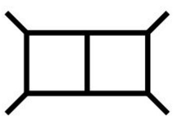




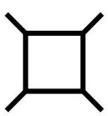
$\frac{1}{\epsilon^6}, \frac{1}{\epsilon^5}, \frac{1}{\epsilon^4}, \frac{1}{\epsilon^3}, \frac{1}{\epsilon^2}, \frac{1}{\epsilon}, \epsilon^0$

Smirnov, hep-ph/0305142

Use Mellin-Barnes integration method



$\frac{1}{\epsilon^4}, \frac{1}{\epsilon^3}, \frac{1}{\epsilon^2}, \frac{1}{\epsilon}, \epsilon^0, \epsilon, \epsilon^2$



$\frac{1}{\epsilon^2}, \frac{1}{\epsilon}, \epsilon^0, \epsilon, \epsilon^2, \epsilon^3, \epsilon^4$ elementary

Exponentiation at 3 loops (cont.)

• Inserting the values of the integrals (including those with $s \leftrightarrow t$) into

$$M_4^{(3)}(\epsilon) \;=\; -\frac{1}{3}\left[M_4^{(1)}(\epsilon)\right]^3 + M_4^{(1)}(\epsilon)M_4^{(2)}(\epsilon) + f^{(3)}(\epsilon)M_4^{(1)}(3\epsilon) + C^{(3)} + E_4^{(3)}(\epsilon)$$

using harmonic polylogarithm identities, etc., relation was verified, and constants extracted:

BDS, hep-th/0505205

$$f_0^{(3)} = \frac{11}{5}(\zeta_2)^2 \qquad f_1^{(3)} = 6\zeta_5 + 5\zeta_2\zeta_3 \qquad f_2^{(3)} = c_1\zeta_6 + c_2\zeta_3^2$$

$$C^{(3)} = \left(\frac{341}{216} + \frac{2}{9}c_1\right)\zeta_6 + \left(-\frac{17}{9} + \frac{2}{9}c_2\right)\zeta_2$$

n-point information still required to separate

Confirmed result for 3-loop cusp anomalous dimension from maximum transcendentality
Kotikov, Lipatov, Onishchenko, Velizhanin, hep-th/0404092

Four-loop anomalous dimensions

$$M_4^{(4)}(\epsilon) = \frac{1}{4}\Big[M_4^{(1)}(\epsilon)\Big]^4 - \Big[M_4^{(1)}(\epsilon)\Big]^2 M_4^{(2)}(\epsilon) + M_4^{(1)}(\epsilon) M_4^{(3)}(\epsilon) + \frac{1}{2}\Big[M_4^{(2)}(\epsilon)\Big]^2 + f^{(4)}(\epsilon) M_4^{(1)}(4\epsilon) + C^{(4)} + \mathcal{O}(\epsilon)$$

- $\gamma_K^{(4)}(\lambda)$ and $\mathcal{G}_0^{(4)}(\lambda)$ can be extracted from $1/\varepsilon^2$ and $1/\varepsilon$ coefficients in four-loop amplitude.
- Also need lower-loop integrals. For $\gamma_K^{(4)}(\lambda)$ only to same order that they were already evaluated analytically for the ε^0 coefficient of the three-loop amplitude

- Four-loop integrals evaluated semi-numerically, using computer programs which automate extraction of $1/\varepsilon$ poles from Mellin-Barnes integrals, and set up numerical integration over the multiple inversion contours. (Collider physics technology!)

Anastasiou, Daleo, hep-ph/0511176; Czakon, hep-ph/0511200;
AMBRE [Gluza, Kajda, Riemann], 0704.2423[ph]

Four-loop cusp anomalous dimension

BCDKS, hep-th/0610248

- Working at $s = t = -1$, we found

$$f_0^{(4)} = -29.335 \pm 0.052$$

- Existing prediction based on integrability (with no dressing factor) was:

$$f_0^{(4)}\Big|_{\rm ES} = -\frac{73}{2520}\pi^6 + \zeta_3^2 = -26.4048255233906609 65\ldots$$

Eden, Staudacher, th/0603157;
Beisert, this workshop

- Assuming discrepancy to be of "leading transcendentality", can write:

$$f_0^{(4)} = f_0^{(4)}\Big|_{\rm ES} + r\,\zeta_3^2$$

$$r = -2.028 \pm 0.036$$

$r = -2$ flipped the sign of the $\zeta_3{}^2$ term in the ES prediction, leaving the π^6 term alone

- Later, precision on r improved significantly to:

$$r = -2.00002 \pm 0.00003$$

Cachazo, Spradlin, Volovich, hep-th/0612309

Weak/strong-coupling interpolation

• Kotikov, Lipatov and Velizhanin (KLV), hep-ph/0301021 proposed the formula:

$$\boxed{\hat{a}^n = \sum_{r=n}^{2n} C_r \, [\tilde{f}_0(\hat{a})]^r}$$

$$\hat{a} \equiv \frac{N_c \alpha_s}{2\pi} = \frac{\lambda}{8\pi^2}$$

to interpolate between $f_0 \sim \hat{a}$ at weak coupling and $f_0 \sim \sqrt{\hat{a}}$ at strong coupling.

Strong-coupling prediction from AdS/CFT (energy of spinning folded string):

$$f_0 = \sqrt{\frac{\hat{a}}{2}} - \frac{3\ln 2}{4\pi} + \mathcal{O}(\hat{a}^{-1/2})$$

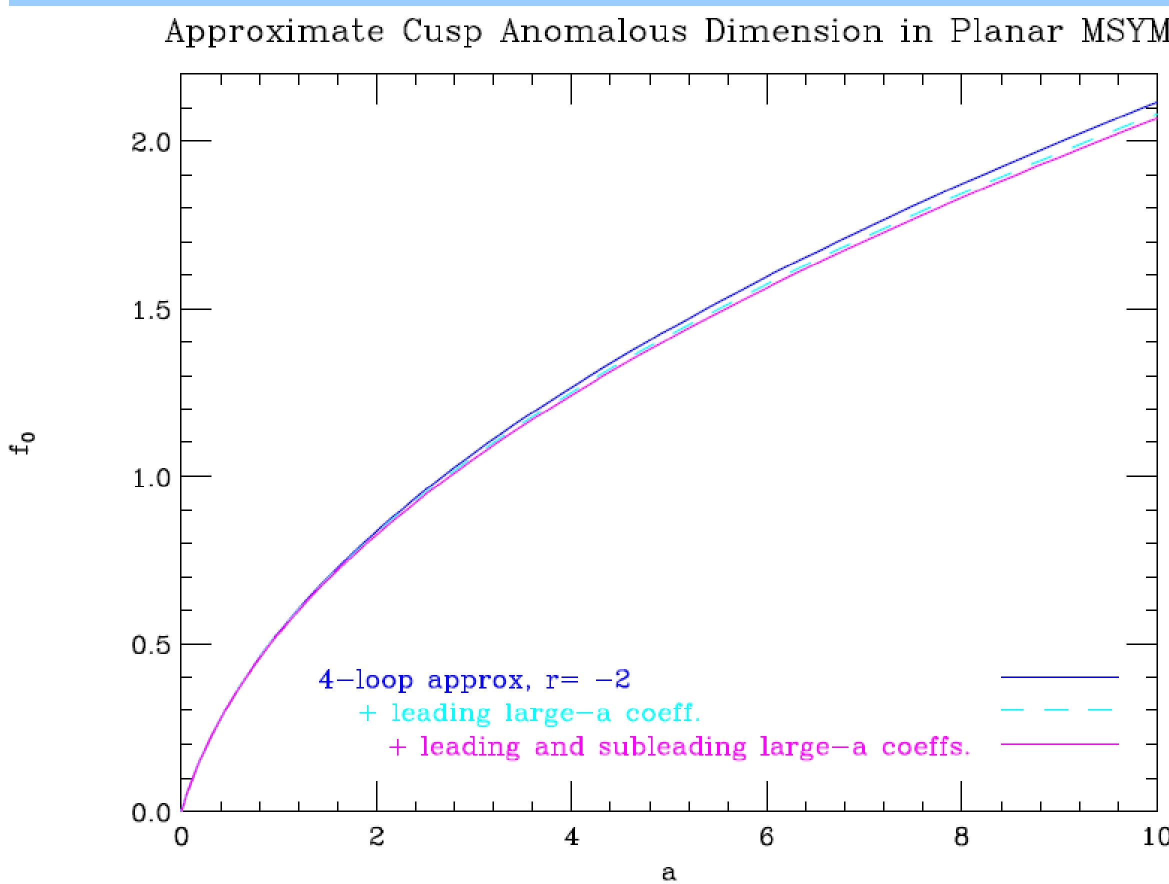
Gubser Klebanov, Polyakov, hep-th/0204051

Frolov, Tseytlin, hep-th/0204226

More recently, Roiban, Tseytlin, 0709.0681[th] computed $\hat{a}^{-1/2}$ term

Interpolation (cont.)

Using 4 weak-coupling coefficients, plus [0,1,2] strong-coupling coefficients gave very consistent results:



These curves predict the five-loop coefficient:

$$f_0^{(5)} = 167.03$$
$$166.34$$
$$165.25$$

ES predicted 131.22 – but flipping signs of odd-zeta terms gives 165.65 !

Independently…

- At the same time as BCDKS, Beisert, Eden, Staudacher [hep-th/0610251] were investigating strong-coupling properties of the dressing factor

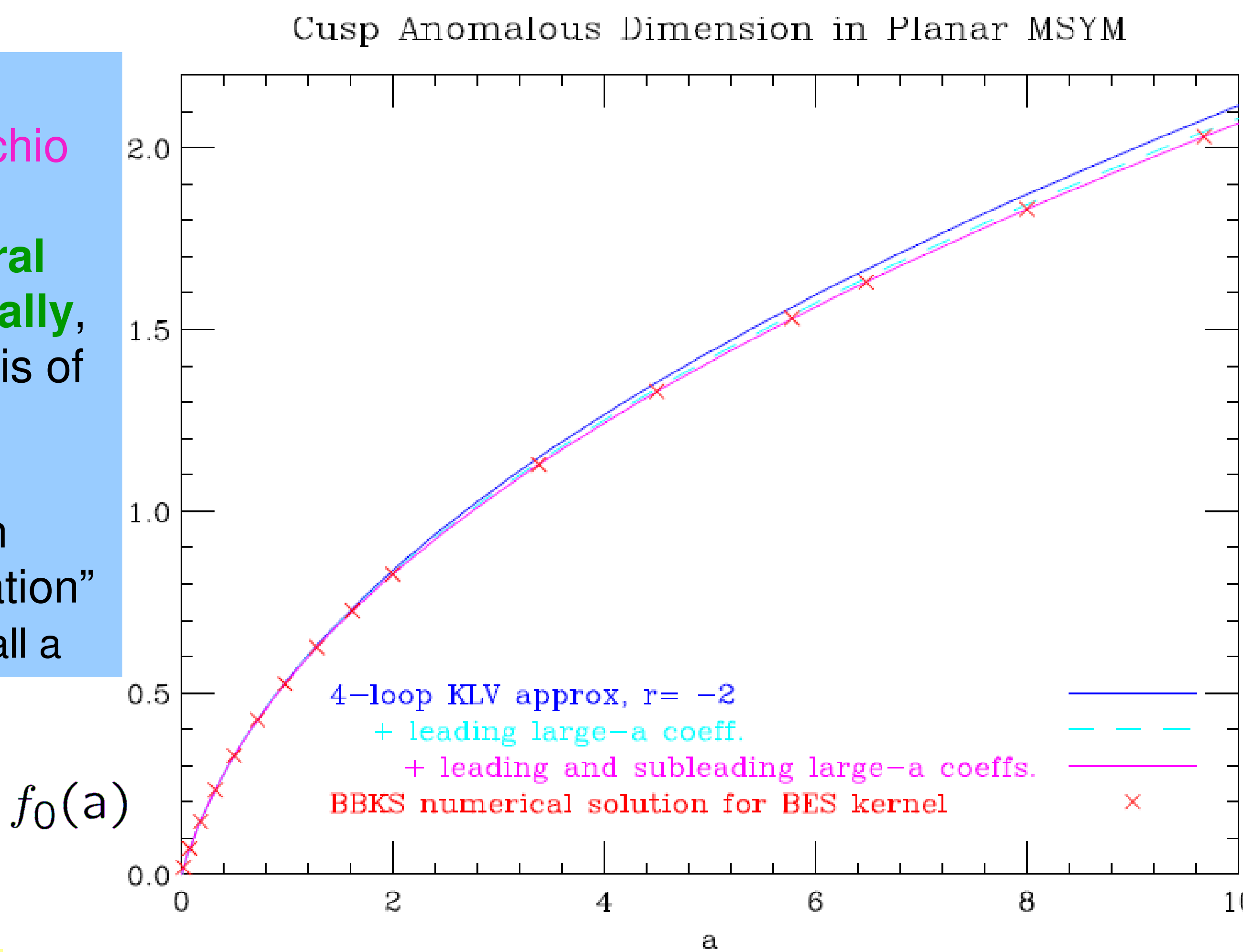
Arutyunov, Frolov, Staudacher, hep-th/0406256;
Hernández, López, hep-th/0603204; …

- BES were led to propose a new integral equation, whose only effect at weak coupling was to flip signs of odd-zeta terms in the ES prediction (actually, $\zeta_{2k+1} \rightarrow i\,\zeta_{2k+1}$)
– in precise agreement with our simultaneous 4-loop calculation and 5-loop estimate

Soon thereafter …

Benna, Benvenuti, Klebanov, Scardicchio [hep-th/0611135] **solved BES integral equation numerically**, expanding in a basis of Bessel functions. Solution agrees stunningly well with "best KLV interpolation" – to within 0.2% for all a

Very recently, full strong-coupling expansion given by Basso, Korchemsky, Kotański, 0708.3933[th]



Pinning down $\mathcal{G}_0(\lambda)$

Cachazo, Spradlin, Volovich, 0707.1903 [hep-th]

- CSV recently computed the four-loop coefficient numerically by expanding the same integrals to one higher power in ε

$$\mathcal{G}_0(\lambda) = -\zeta_3 \left(\frac{\lambda}{8\pi^2}\right)^2 + \frac{2}{3}(6\zeta_5 + 5\zeta_2\zeta_3)\left(\frac{\lambda}{8\pi^2}\right)^3 - (77.69 \pm 0.06)\left(\frac{\lambda}{8\pi^2}\right)^4 + \cdots$$

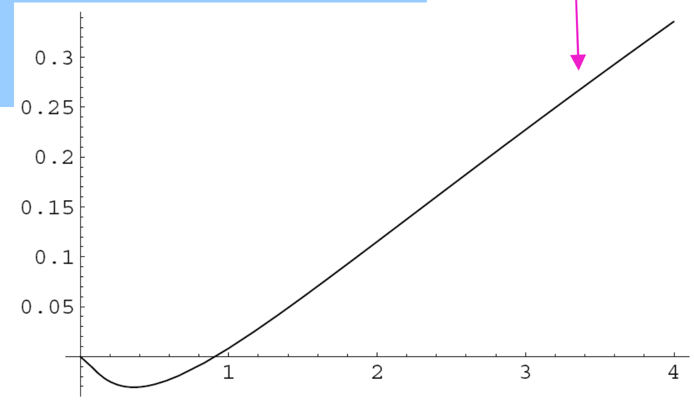
- They also compared this number to the prediction of a KLV-type approximation interpolating between weak and strong coupling: -83.55

The two results agree to within 7%.

strong coupling from Alday, Maldacena

And they gave a [3/2] Padé approximant for $\mathcal{G}_0(\lambda)$ incorporating all data

$\mathcal{G}_0(\lambda)$



$\lambda/(16\pi^2)$

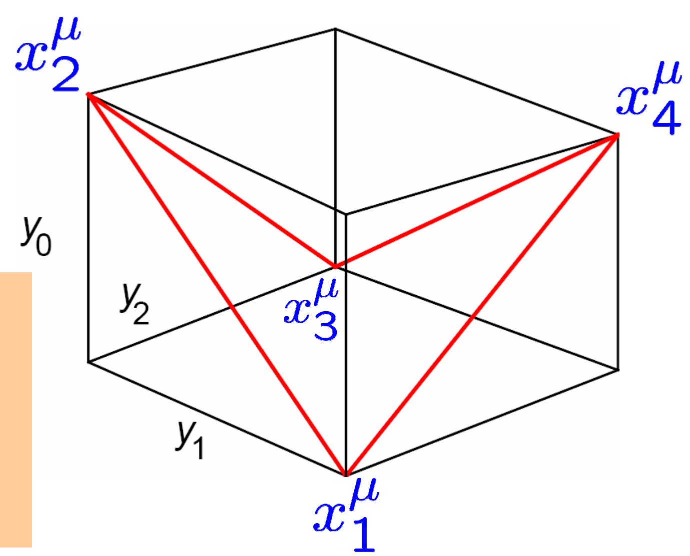
Dual variables and strong coupling

• T-dual variables y^μ introduced by Alday, Maldacena

• Boundary values for world-sheet are light-like segments in y^μ :

$\Delta y^\mu = 2\pi k^\mu$ for gluon with momentum k^μ

• For example, for *gg* → *gg* 90-degree scattering, *s = t = -u/2*, the boundary looks like:



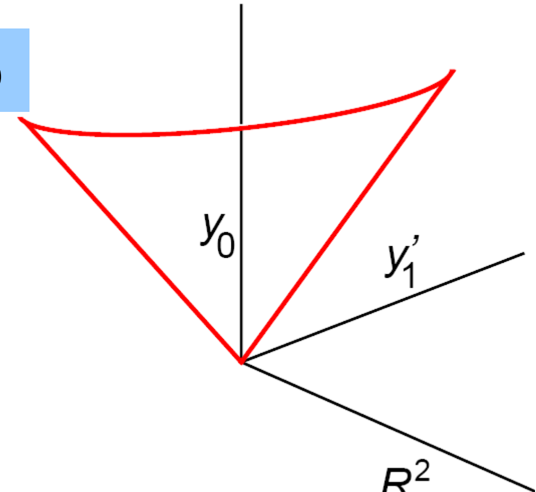
Corners (cusps) are located at x_i^μ
– same dual momentum variables
introduced above for discussing
dual conformal invariance of integrals!!

Cusps in the solution

• Near each corner, solution has a cusp

Kruczenski, hep-th/0210115

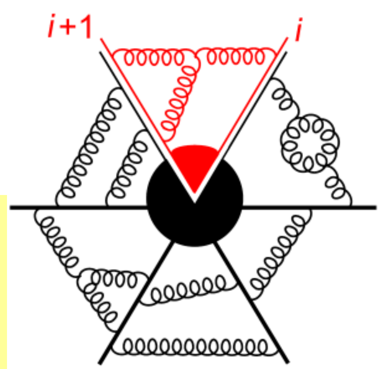
$$r = \sqrt{(2+\epsilon)(y_0^2 - y_1'^2)} \equiv \sqrt{(2+\epsilon)y^+ y^-}$$



• Classical action divergence is regulated by ϵ

$$iS = -S_E = -\frac{R^2}{4\pi}\int d\sigma d\tau$$

$$\to \; -R^2 \int_0 \frac{dy^+ dy^-}{(y^+ y^-)^{1+\epsilon/2}} \sim -\frac{\sqrt{\lambda}}{\epsilon^2} \sim -\frac{\gamma_K(\lambda)}{\epsilon^2}$$



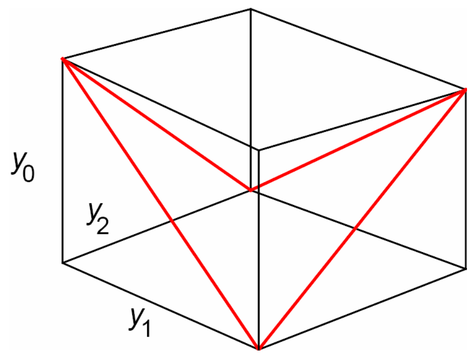
• Cusp in (*y*,*r*) is the strong-coupling limit of the red wedge; i.e. the Sudakov form factor.

• See also Buchbinder, 0706.2015 [hep-th]

The full solution

- Divergences only come from corners; can set *D=4* in interior.
- Evaluating the action as $\varepsilon \to 0$ gives:

Alday, Maldacena, 0705.0303 [hep-th]



$$A_4 = \exp(-S_E)$$

$\gamma_K(\lambda)$ $\quad$ $\mathcal{G}_0(\lambda)$

$$-S_E = \left(-\frac{1}{\epsilon^2}\frac{\sqrt{\lambda}}{2\pi} - \frac{1}{\epsilon}\frac{\sqrt{\lambda}}{4\pi}(1-\ln 2)\right)\left[\left(\frac{\mu^2}{-s}\right)^{\epsilon} + \left(\frac{\mu^2}{-t}\right)^{\epsilon}\right]$$
$$+\frac{\sqrt{\lambda}}{4\pi}\left[\ln^2\frac{s}{t} + \tilde{C}\right]$$

combination of $f_2(\lambda) \oplus C(\lambda)$

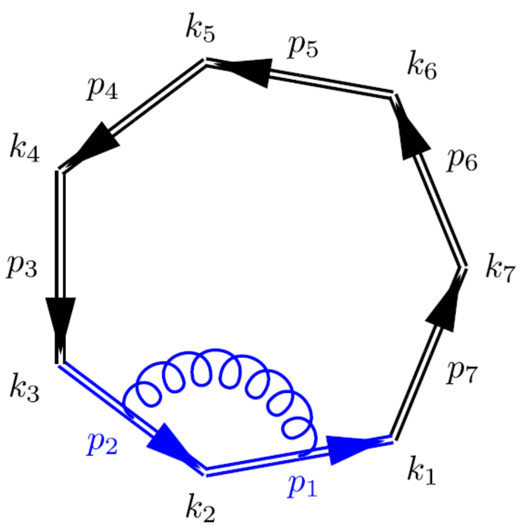
$\gamma_K(\lambda) \times M_4^{(1)}(s,t)$

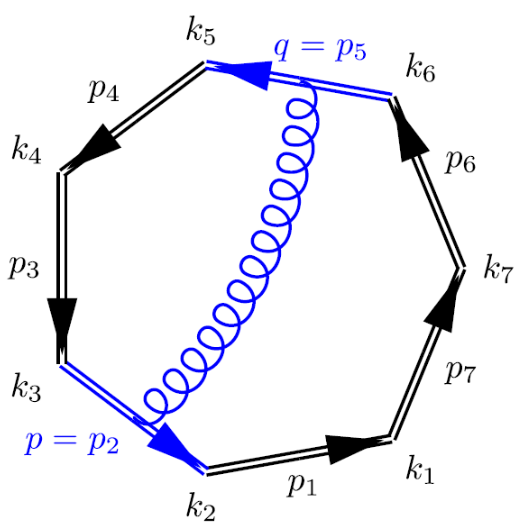
Dual variables and Wilson lines at weak coupling

• Inspired by Alday, Maldacena, there has been a sequence of recent computations of Wilson-line configurations with same "dual momentum" boundary conditions:

• One loop, $n=4$ Drummond, Korchemsky, Sokatchev, 0707.0243[th]

• One loop, any n Brandhuber, Heslop, Travaglini, 0707.1153[th]

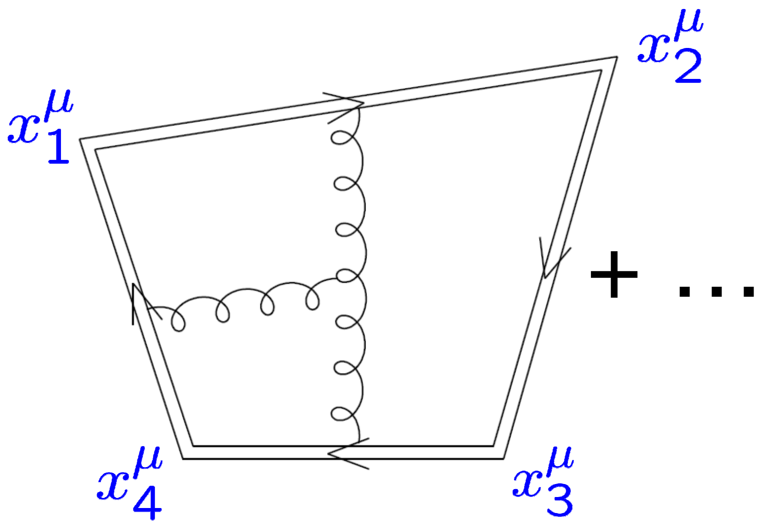




Dual variables and Wilson lines at weak coupling (cont.)

- Two loops, *n=4*

Drummond, Henn, Korchemsky, Sokatchev, 0709.2368[th]



- In all 3 cases, Wilson-line results match the full scattering amplitude [the MHV case for *n>5*] !?!
– up to an additive constant in the 2-loop case.

DHKS also remark that the one-loop MHV N=4 SYM amplitudes obey an “anomalous” (due to IR divergences) dual conformal Ward identity they propose, which totally fixes their structure for *n=4,5*.

Non-MHV very different even at 1 loop

MHV:

$$A_n^{1-\text{loop}} = A_n^{\text{tree}}\, L \qquad \Rightarrow \qquad \mathcal{A}_n = A_n^{\text{tree}} \exp[aL]$$

power law

$$\frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle\cdots\langle n\, 1\rangle}$$

logarithmic

Non-MHV:

??

$$A_n^{1-\text{loop}} = \sum_i T_i\, L_i \qquad \Rightarrow \qquad \mathcal{A}_n = \sum_i T_i \exp[aL_i]$$

$$A_n^{\text{tree}} = \sum_i T_i$$

Conclusions & Open Questions

- Through a number of approaches, especially integrability, an exact solution for the cusp anomalous dimension in planar N=4 SYM certainly seems in hand.
- Remarkably, finite terms in MHV planar N=4 SYM amplitudes exponentiate in a very similar way to the IR divergences. Full amplitude seems to depend on just 4 functions of λ alone, so MHV problem may be at least "1/4" solved! [Pending resolution, for $n > 5$, of issue raised by Alday, Maldacena]
- What is the AdS/operator interpretation of the other 3 functions? Can one find integral equations for them?
- How is exponentiation/iteration related to AdS/CFT, integrability, [dual] conformality, and Wilson lines?
- What happens for non-MHV amplitudes? From structure of 1-loop amplitudes, answer must be more complex.

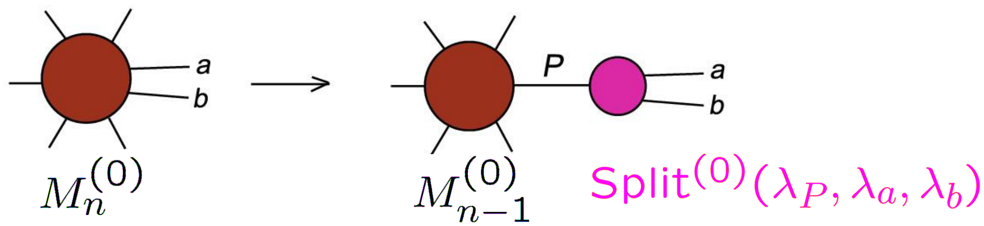
Extra Slides

Two-loop exponentiation & collinear limits

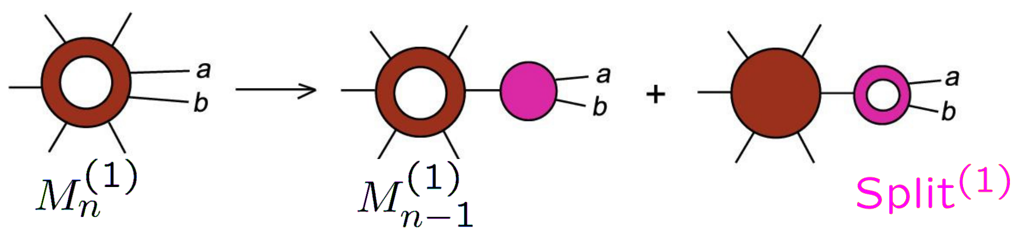
- Evidence for *n>4*: Use limits as 2 momenta become **collinear:**

$$k_a \to z k_P$$
$$k_b \to (1-z) k_P$$

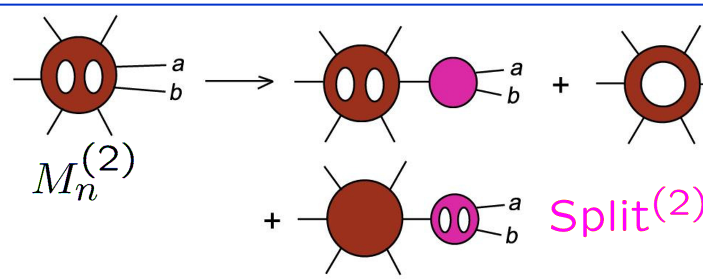
- Tree amplitude behavior:



- One-loop behavior:



- Two-loop behavior:



Two-loop splitting amplitude iteration

- In N=4 SYM, all helicity configurations are equivalent, can write

$$\text{Split}^{(l)}(\lambda_P,\lambda_a,\lambda_b) = r_S^{(l)}(z,s_{ab},\epsilon)\times \text{Split}^{(0)}(\lambda_P,\lambda_a,\lambda_b)$$

- The two-loop splitting amplitude obeys:

Anastasiou, Bern, LD, Kosower, hep-th/0309040

$$r_S^{(2)}(\epsilon) \;=\; \frac{1}{2}\Big[r_S^{(1)}(\epsilon)\Big]^2 + f^{(2)}(\epsilon)\,r_S^{(1)}(2\epsilon) + \mathcal{O}(\epsilon)$$

which is consistent with the *n*-point amplitude ansatz

$$\mathcal{M}_n^{(2)}(\epsilon) \;=\; \frac{1}{2}\Big[M_n^{(1)}(\epsilon)\Big]^2 + f^{(2)}(\epsilon)\,M_n^{(1)}(2\epsilon) + C^{(2)} + E_n^{(2)}(\epsilon)$$

and fixes $f_0^{(2)} = -\zeta_2 \qquad f_1^{(2)} = -\zeta_3 \qquad f_2^{(2)} = -\zeta_4 \qquad C^{(2)} = -\frac{(\zeta_2)^2}{2}$

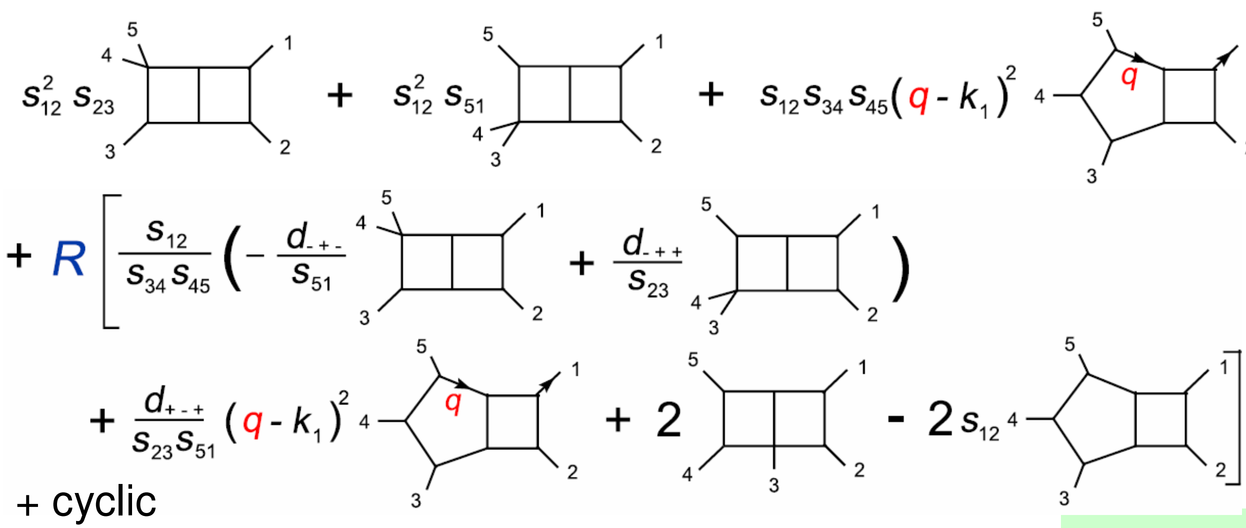
n-point information required to separate these two

Note: by definition $f_0^{(1)} = 1,\; f_1^{(1)} = f_2^{(1)} = C^{(1)} = E_n^{(1)}(\epsilon) = 0$

Two-loop check for *n=5*

Collinear limits are highly suggestive, but not quite a proof.

Using unitarity, first in $D=4$, later in $D=4-2\varepsilon$, the two-loop $n=5$ amplitude was found to be:



Bern, Rozowsky, Yan, hep-ph/9706392

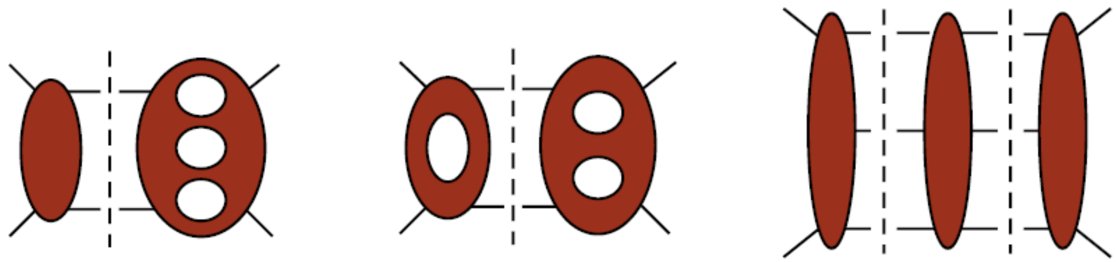
Cachazo, Spradlin, Volovich, hep-th/0602228

$$R = \varepsilon(k_1,k_2,k_3,k_4) \times s_{12}s_{23}s_{34}s_{45}s_{51}/\det(s_{ij})|_{i,j=1,2,3,4}$$

Bern, Czakon, Kosower, Roiban, Smirnov, hep-th/0604074

Even and odd terms checked numerically with aid of Czakon, hep-ph/0511200

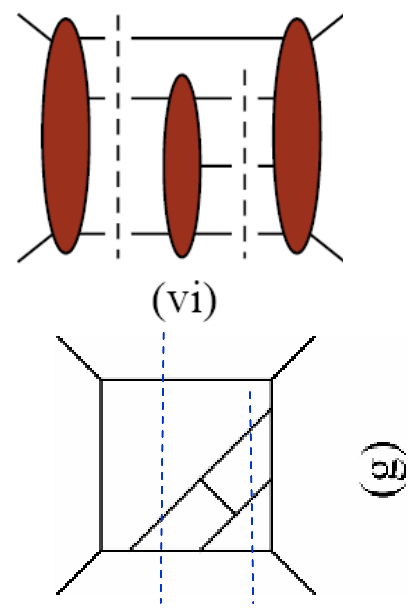
Generalized cuts computed at 4 loops



(i) (ii) (iii)

Graph detection table

Graph	Cuts	Graph	Cuts	Graph	Cuts
(a)	(i), (ii), (iii)	(b_1)	(v), (vi)	(d_5)	(i), (iii), (iv)
(b)	(i), (v)	(b_2)	(i), (vi)	(e_1)	(i), (iii), (iv)
(c)	(i), (ii), (iii), (iv)	(b_3)	(v), (vi)	(e_2)	(iii), (iv)
(d)	(i), (iii), (v)	(b_4)	(vi)	(e_3)	(i), (vi)
(e)	(i), (iii), (iv), (vi)	(c_1)	(i), (iii), (iv)	(e_4)	(i), (vi)
(f)	(iii), (iv), (vi)	(d_1)	(i), (iii), (iv)	(e_5)	(i), (iii), (iv)
(d_2)	(iii), (iv)	(d_3)	(i), (iii), (iv)	(e_6)	(vi)
(f_2)	(iii), (vi)	(d_4)	(i), (iii), (iv)	$\boxed{(g)}$	(i), $\boxed{(vi)}$
				(g_1)	(i), (vi)



(vi)

Iteration in other theories?

Khoze, hep-th/0512194

Two classes of (large N_c) conformal gauge theories "inherit" the same large N_c perturbative amplitude properties from N=4 SYM:

1. Theories obtained by orbifold projection
– product groups, matter in particular bi-fundamental rep's

Bershadsky, Johansen, hep-th/9803249

2. The N=1 supersymmetric "beta-deformed" conformal theory
– same field content as N=4 SYM, but superpotential is modified:

$$ig\,\mathrm{Tr}(\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2) \to ig\,\mathrm{Tr}(e^{i\pi\beta_R}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta_R}\Phi_1\Phi_3\Phi_2)$$

Leigh, Strassler, hep-th/9503121

Supergravity dual known for this case, deformation of $AdS_5 \times S^5$

Lunin, Maldacena, hep-th/0502086

Breakdown of inheritance at five loops (!?) for more general marginal perturbations of N=4 SYM? Khoze, hep-th/0512194

Cusp anomalous dimension in QCD

Computed through 3 loops:

Moch Vermaseren, Vogt (MVV), hep-ph/0403192, hep-ph/0404111

$$\gamma_K(\alpha_s) = \gamma_K^{(1)}\left(\frac{\alpha_s}{2\pi}\right) + \gamma_K^{(2)}\left(\frac{\alpha_s}{2\pi}\right)^2 + \gamma_K^{(3)}\left(\frac{\alpha_s}{2\pi}\right)^3 + \cdots$$

$$\begin{aligned}
\gamma_K^{(1)} &= 4\,C_i \cdot 1 \\
\gamma_K^{(2)} &= 4\,C_i\left[\left(\frac{67}{18} - \zeta_2\right) C_A - \frac{5}{9} n_f\right] \\
\gamma_K^{(3)} &= 4\,C_i\Big[\left(\frac{245}{24} - \frac{67}{9}\zeta_2 + \frac{11}{6}\zeta_3 + \frac{11}{5}\zeta_2^2\right) C_A^2 \\
&\quad + \left(-\frac{208}{108} + \frac{10}{9}\zeta_2 - \frac{7}{3}\zeta_3\right) C_A n_f \\
&\quad + \left(-\frac{55}{24} + 2\zeta_3\right) C_F n_f - \frac{n_f^2}{27}\Big]
\end{aligned}$$

$$\zeta_n \equiv \zeta(n) \equiv \sum_{k=1}^{\infty} \frac{1}{k^n}$$

$C_i = C_A$ (gluons)

$C_i = C_F$ (quarks)

“Leading transcendentality” relation between QCD and N=4 SYM

- KLOV (Kotikov, Lipatov, Onishschenko, Velizhanin, hep-th/0404092) noticed (at 2 loops) a remarkable relation between kernels for
  - BFKL evolution (strong rapidity ordering)
  - DGLAP evolution (pdf evolution = strong collinear ordering)
    → **includes cusp anomalous dimension**

  in QCD and N=4 SYM**:**
- Set fermionic color factor $C_F = C_A$ in the QCD result and keep only the “leading transcendentality” terms. They coincide with the full N=4 SYM result (even though theories differ by scalars)
- Conversely, N=4 SYM results predict pieces of the QCD result

- transcendentality (weight): n for π^n
  n for ζ_n

Similar counting for HPLs and for related harmonic sums used to describe DGLAP kernels at finite j

$\gamma_K(\alpha_s)$ in N=4 SYM through 3 loops:

$$\gamma_K(\alpha_s) = \gamma_K^{(1)}\left(\frac{\alpha_s}{2\pi}\right) + \gamma_K^{(2)}\left(\frac{\alpha_s}{2\pi}\right)^2 + \gamma_K^{(3)}\left(\frac{\alpha_s}{2\pi}\right)^3 + \cdots$$

$$\boxed{\begin{aligned} \gamma_K^{(1)} &= 4\,C_A \cdot 1 \\ \gamma_K^{(2)} &= 4\,C_A^2\,(-\,\zeta_2) \\ \gamma_K^{(3)} &= 4\,C_A^3\left(\frac{11}{5}\zeta_2^2\right) \end{aligned}} \longleftarrow \boxed{\text{KLOV prediction}}$$

$$\boxed{C_A\left(\frac{\alpha_s}{2\pi}\right) = \frac{\lambda}{8\pi^2}}$$

- Finite *j* predictions confirmed (with assumption of integrability)

Staudacher, hep-th/0412188

- Confirmed at infinite *j* using on-shell amplitudes, unitarity

Bern, LD, Smirnov, hep-th/0505205

- and with all-orders asymptotic Bethe ansatz

Beisert, Staudacher, hep-th/0504190

- leading to an integral equation

Eden, Staudacher, hep-th/0603157