Complete Integrability in QCD:

Applications and Directions

Vladimir Braun

University of Regensburg

thanks to

Sergey Derkachov, Gregory Korchemsky and Alexander Manashov

Outline

In this talk:

- Renormalization in conformally-covariant form
- Baryon wave functions
- Twist-three effects in polarized deep-inelastic scattering
- High energy scattering in QCD
- Some open problems

not included:

- *SUSY extensions: From* $\mathcal{N} = 0$ *to* $\mathcal{N} = 4$
- $N = 4$ *SUSY and gauge/string correspondence*

From spin lattice models to QCD and beyond...

Complete integrability

$\Updownarrow$

Number of degrees of freedom = number of conservation laws

1926: Heisenberg spin chain
1981-83: Generalizations for arbitrary spin

Mathematical methods:

1932: Algebraic Bethe Ansatz
1971: Baxter $\mathbb{Q}$–operator
1985: Separation of variables

1994: Compound states of reggeized gluons:
Lipatov; Faddeev, Korchemsky

1997: "Possible that evolution equations in SUSY theories are integrable"
Lipatov

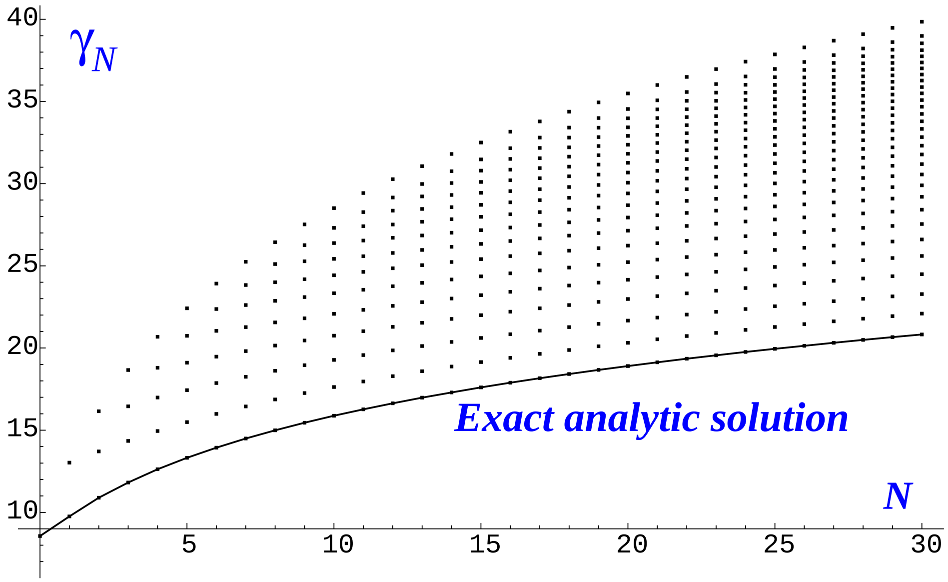
1998: Three-particle evolution equations in QCD:
Braun, Derkachov, Manashov

1999– Applications to QCD phenomenology:
Braun, Derkachov, Korchemsky, Manashov
Belitsky

2001– Spectrum of multireggeon states:
Lipatov, de Vega
Derkachov, Korchemsky, Kotansky, Manashov

2003– N=4 SUSY and AdS/CFT
Kotikov, Lipatov
Minahan, Zarembo
Beisert, Staudacher; . . .

Twist–3 structure function $g_2(x, Q^2)$: quark distribution in transversely polarized nucleon



$$\mathcal{M}_N(Q^2) = \int_0^1 dx\, x^{N-1} g_2(x, Q^2)$$

$N-1$ independent operators contribute

1983: *Bukhvostov, Kuraev, Lipatov*

One-loop mixing matrix

1991: *Ali, Braun, Hiller* $N_c \to \infty$ **limit**

- **Exact analytic expression for the lowest γ_N**
- **All other operators decouple from $g_2(x, Q^2)$**

One nonperturbative parameter for each N

1998: *Braun, Derkachov, Manashov* **Open $SL(2,\mathbb{R})$ Heisenberg spin chain**

Example:

Twist-three operators $\Leftrightarrow$ **Leading-twist three-particle parton distributions**

E.g. baryon distribution amplitudes $B = N, \Delta, \ldots$

$$\langle 0|q(z_1)q(z_2)q(z_3)|B(p,\lambda)\rangle \quad = \quad \ldots \int_0^1 dx_1 dx_2 dx_3\, \delta(\textstyle\sum x_i - 1)\, e^{-ip(x_1 z_1 + x_2 z_2 + x_3 z_3)} \varphi_B(x_i, \mu^2$$

$$q^{\uparrow}\, q^{\uparrow}\, q^{\uparrow} \quad \Rightarrow \quad \varphi_{\Delta}^{\lambda=3/2}(x_i, \mu^2)\,,$$

$$q^{\uparrow}\, q^{\downarrow}\, q^{\uparrow} \quad \Rightarrow \quad \begin{cases} \varphi_N^{\lambda=1/2}(x_i, \mu^2) \\ \varphi_{\Delta}^{\lambda=1/2}(x_i, \mu^2) \end{cases}$$

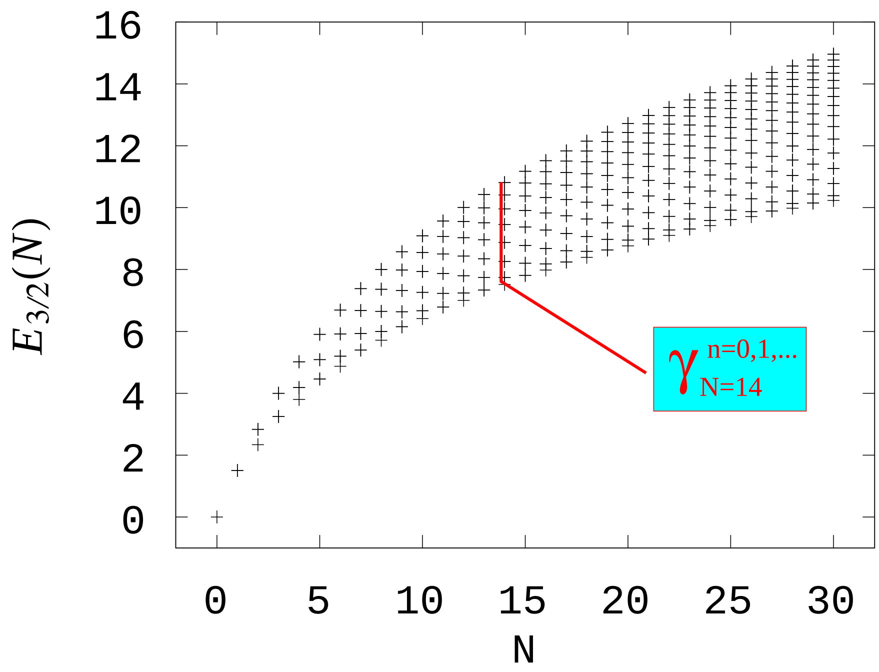
- quark fields “live” on a light-ray $z^2 = 0$

Moments of distribution amplitudes ⇔ local operators:

$$\varphi(x_i) \to \varphi(k_i) = \int \mathcal{D}x_i \, x_1^{k_1} x_2^{k_2} x_3^{k_3} \, \varphi(x_i, \mu^2)$$

$$q(z_1)q(z_2)q(z_3) \to (D_+^{k_1} q)\,(D_+^{k_2} q)\,(D_+^{k_3} q)$$

Mixing matrix: $N = k_1 + k_2 + k_3$



Example:
qqq
$\lambda = 3/2$

Rich spectrum of anomalous dimensions reflects complexity of genuine degrees of freedom

$$z \to z' = \frac{az+b}{cz+d}\,, \qquad ad-bc=1 \qquad \Phi(z) \to \Phi'(z) = \Phi\left(\frac{az+b}{cz+d}\right)\cdot(cz+d)^{-2j_\Phi}$$

$j_q = 1$, $j_g = 3/2$ is conformal spin of the field

Generators obey the $SL(2)$ algebra

$$\begin{aligned} L_-\,\Phi(z) &= -\frac{d}{dz}\Phi(z) \\ L_+\,\Phi(z) &= \left(z^2\frac{d}{dz} + 2j_\Phi z\right)\Phi(z) \\ L_0\,\Phi(z) &= \left(z\frac{d}{dz} + j_\Phi\right)\Phi(z) \end{aligned}$$

Casimir operators

$$L^2 = L_0^2 - L_0 + L_+L_- \qquad L^2\Phi(z) = j(j-1)\Phi(z)$$

Summation of spins

$$L_{ik}^2 = \sum_{\alpha=0,1,2}(L_{i,\alpha}+L_{k,\alpha})^2 \qquad L_{123}^2 = \sum_{\alpha=0,1,2}(L_{1,\alpha}+L_{2,\alpha}+L_{3,\alpha})^2$$

- second-order differential operators on the space of (z_1, z_2, z_3)

Light-ray operators

$$\left\{\mu\,\frac{\partial}{\partial\mu}+\beta(g)\,\frac{\partial}{\partial g}\right\}B=\mathcal{H}\cdot B\,,\qquad B(z_1,z_2,z_3)\simeq q(z_1)q(z_2)q(z$$

Two-particle structure:

$$\mathcal{H}_{qqq}=\mathcal{H}_{12}+\mathcal{H}_{23}+\mathcal{H}_{13}$$

Renormalization = Displacement along the light-cone:

Balitsky, Braun '88

$$\mathcal{H}_{12}B^{\uparrow\uparrow\uparrow}(z_1,z_2,z_3)\quad=\quad\int_0^1\frac{\alpha d\alpha}{1-\alpha}\Big[2B(z_1,z_2)-B(\alpha z_1+\bar\alpha z_2,z_2)-B(z_1,\alpha z_2+\bar\alpha z_1)\Big]$$

$$\bar\alpha=1-\alpha$$

- **Symmetry transformations explicit**
- **Straightforward generalization to SUSY**

Belitsky, Derkachov, Korchemsky, Manashov ’04

- **Super-light-cone:** $z \to Z = \{z, \theta_1 \dots \theta_{\mathcal{N}}\}$
- **Superconformal algebra**
- **Light-cone superfield formulation** **Mandelstam, Brink et al. ’83**

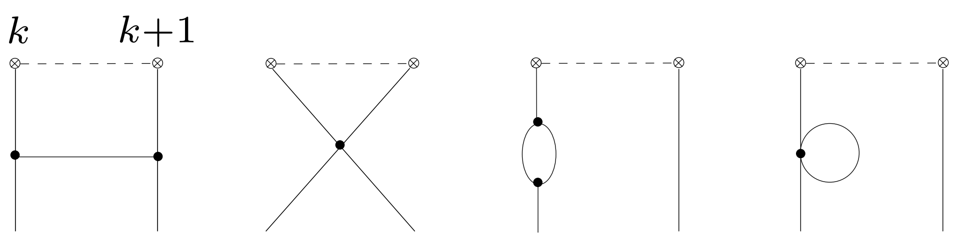
Complex scalar $\mathcal{N} = 4$ chiral superfield

$$\Phi = \partial_+^{-1} A + \theta^A \partial_+^{-1} \bar{\lambda}_A + \tfrac{i}{2!} \theta^A \theta^B \bar{\phi}_{AB} - \tfrac{1}{3!} \varepsilon_{ABCD} \theta^A \theta^B \theta^C \lambda^D - \tfrac{1}{4!} \varepsilon_{ABCD} \theta^A \theta^B \theta^C \theta^D \partial_+ \bar{A}$$

- **Method of truncation for $\mathcal{N} < 4$**

A two particle kernel

$\mathcal{H}_{k,k+1} =$



$= \mathbb{V}_{k,k+1} \left(1 - \Pi_{k,k+1}\right)$

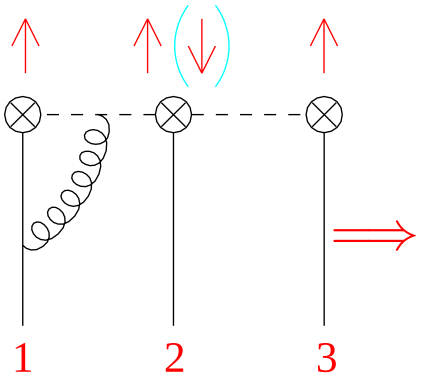
acts as a displacement in the light-cone superspace

$$\mathbb{V}_{12} \mathbb{O}(Z_1, Z_2) = \int_0^1 \frac{d\alpha}{(1-\alpha)\alpha^2} \Big[2\alpha^2 \, \mathbb{O}(Z_1, Z_2) - \mathbb{O}(\alpha Z_1 + \bar{\alpha} Z_2, Z_2) - \mathbb{O}(Z_1, \alpha Z_2 + \bar{\alpha} Z_1) \Big]$$

- **same expression as in QCD, apart from a power of α;** *[For quarks $j_q = 1$, for chiral superfield $j_\Phi = -1$*

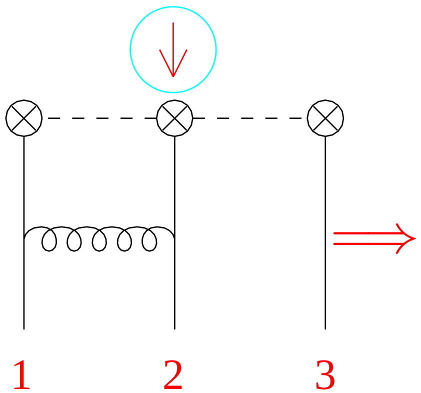
Bukhvostov, Frolov, Kuraev, Lipatov ’85

write $\mathcal{H}$ as a function of Casimir operators $L_{ik}^2 = -\frac{\partial}{\partial z_i}\frac{\partial}{\partial z_k}(z_i - z_k)^2 \equiv \widehat{J}_{ik}(\widehat{J}_{ik} - 1)$



$$\mathcal{H}_{12}^{v} = 2[\psi(J_{12}) - \psi(2)]$$

$$\mathcal{H}_{qqq}^{\lambda=3/2} = \mathcal{H}_{12}^{v} + \mathcal{H}_{23}^{v} + \mathcal{H}_{13}^{v}$$



$$\mathcal{H}_{1/2}^{(e)} = \frac{1}{L_{12}^2} = \frac{1}{J_{12}(J_{12}-1)}$$

$$\mathcal{H}_{qqq}^{\lambda=1/2} = \mathcal{H}_{qqq}^{\lambda=3/2} - \mathcal{H}_{12}^{e} - \mathcal{H}_{23}^{e}$$

Have to solve $\mathcal{H}\Psi_{N,n} = \mathcal{E}_{N,n}\Psi_{N,n}$

—A Schr ödinger equation with Hamiltonian $\mathcal{H}$

Hamiltonian approach — *continued*

Hilbert space?

- **Polynomials in interquark separation**
- **Polynomials in covariant derivatives (local operators)**

—related by a duality transformation BDKM

- **The Hamiltonian is hermitian w.r.t. the conformal scalar product:**

♡ **Eigenvalues (anomalous dimensions) are real numbers**

♡ **Systematic $1/N_c$ expansion**

$$\mathcal{E}_{N,n} = N_c E_{N,n} + N_c^{-1} \delta E_{N,n} + \ldots$$

$$\Psi_{N,n} = \Psi_{N,n}^{(0)} + N_c^{-2} \delta \Psi_{N,n} + \ldots$$

with the usual quantum-mechanical expressions

$$\delta E_{N,n} = \|\Psi_{N,n}^{(0)}\|^{-2} \langle \Psi_{N,n}^{(0)} | \mathcal{H}^{(1)} | \Psi_{N,n}^{(0)} \rangle$$

etc.

- Conformal symmetry implies existence of two conserved quantities: $[\mathcal{H}, L^2] = [\mathcal{H}, L_0] = 0$

- For qqq and GGG states with maximum helicity and for qGq states at $N_c \to \infty$ there exists an additional conserved charge

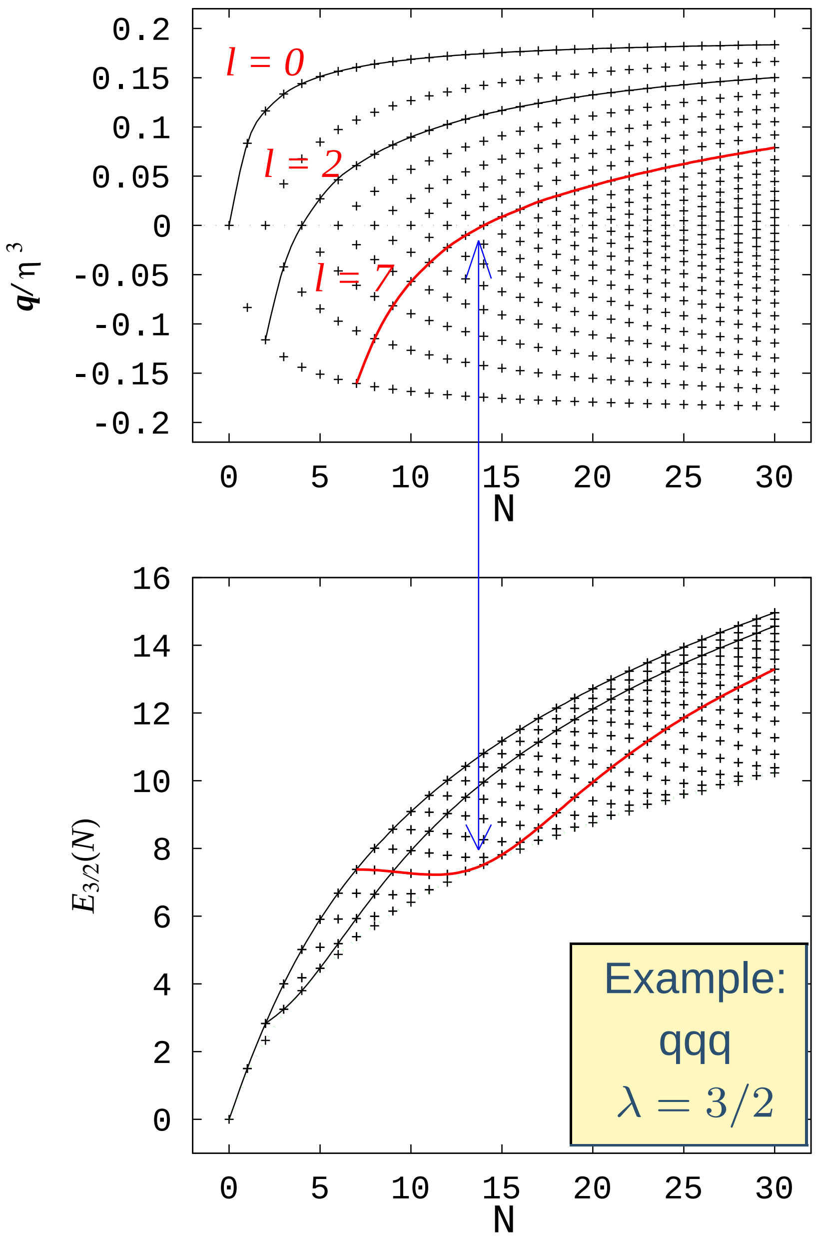
qqq, GGG $\overset{\lambda=max}{\Longrightarrow}$ $$Q = [L_{12}^2, L_{23}^2]$$ $$[\mathcal{H}, Q] = 0$$

BDM '98

qGq $\overset{N_c\to\infty}{\Longrightarrow}$ $$Q = \{L_{qG}^2, L_{Gq}^2\} + c_1 L_{qG}^2 + c_2 L_{Gq}^2$$ $$[\mathcal{H}, Q] = 0$$

- Anomalous dimensions can be classified by values of the charge Q: $\mathcal{H}\Psi = \mathcal{E}\Psi$, $Q\Psi = q\Psi$ $\Longrightarrow$ $\mathcal{E} = \mathcal{E}(q)$

♢ **A new quantum number**



Double degeneracy:

$$\mathcal{E}(N,q) = \mathcal{E}(N,-q)$$

$$q(N,\ell) = -q(N,N-\ell)$$

BDKM ’98

WKB expansion of the Baxter equation $\Rightarrow$ 1/N expansion

G. Korchemsky '95-'97

- **equation $Q\Psi = q\Psi$ is much simpler as $\mathcal{H}\Psi = \mathcal{E}\Psi$**
- **non-integrable corrections are suppressed at large N**
- **can use some techniques of integrable models**

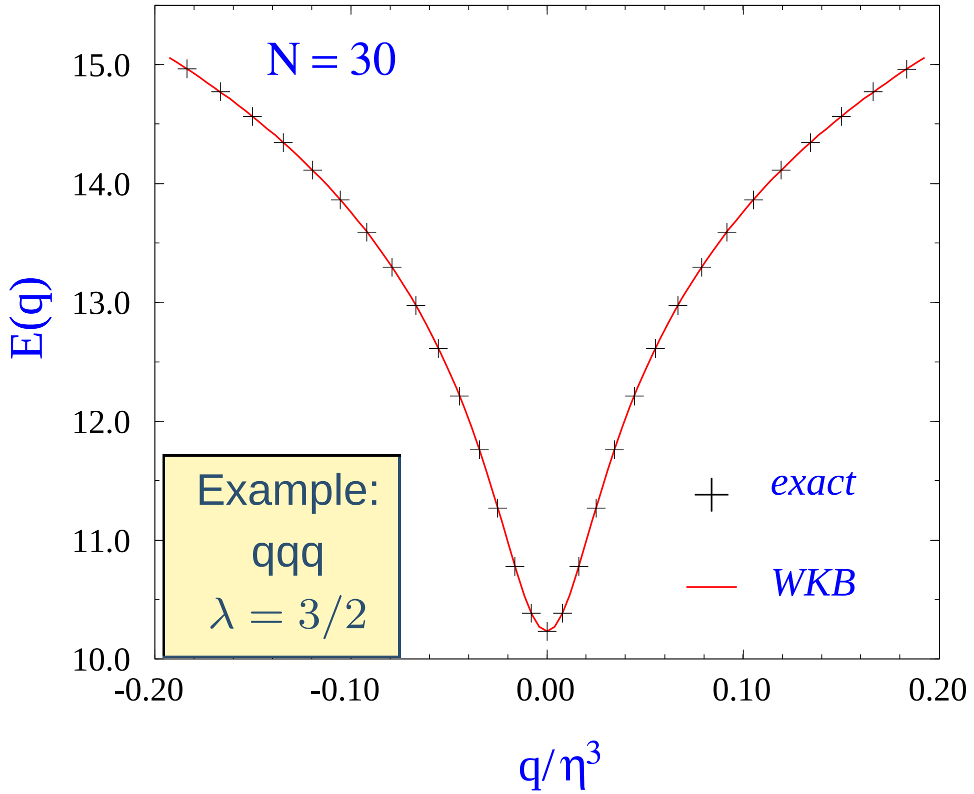
$$\begin{aligned}\mathcal{E}(N,\ell) \quad = \quad & 6\ln\eta - 3\ln 3 - 6 + 6\gamma_E \\ & -\frac{3}{\eta}(2\ell+1) \\ & -\frac{1}{\eta^2}\left(5\ell^2+5\ell-7/6\right) \\ & -\frac{1}{72\eta^3}\left(464\ell^3+696\ell^2-802\ell-517\right) \\ & +\ldots\end{aligned}$$

$$\eta = \sqrt{(N+3)(N+2)}$$

An integer ℓ numerates the trajectories:
(semiclassically quantized solitons: Korchemsky, Krichever, '97)

- **Integrability imposes a nontrivial analytic structure**

The ‘dispersion curve’ $\mathcal{E}(q)$



$$\mathcal{E}(q) = 2\ln 2 - 6 + 6\gamma_E + 2\mathrm{Re}\sum_{k=1}^{3}\psi(1 + i\eta^3\delta_k) + \mathcal{O}(\eta^{-6})$$

δ_k are defined as roots of the cubic equation:

$$2\delta_k^3 - \delta_k - q/\eta^3 = 0$$

$$\eta = \sqrt{(N+3)(N+2)}$$

Simplest case:

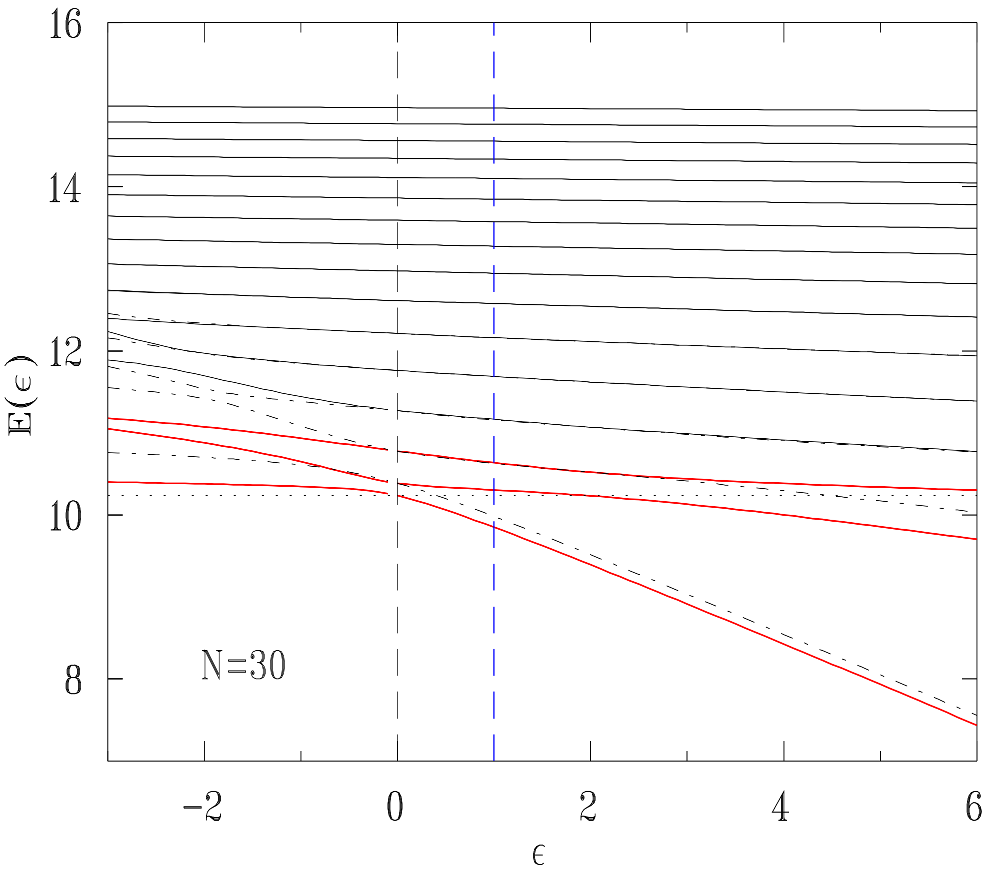
Difference between $\Delta^{\lambda=3/2}$ and $N(\Delta)^{\lambda=1/2}$

$$\mathcal{H}(\varepsilon) = \mathcal{H}_{3/2} - \varepsilon\left(\frac{1}{L_{12}^2} + \frac{1}{L_{23}^2}\right)$$

$$\mathcal{H}(\varepsilon = 1) = \mathcal{H}_{1/2}$$

Flow of energy levels

(anomalous dimensions):



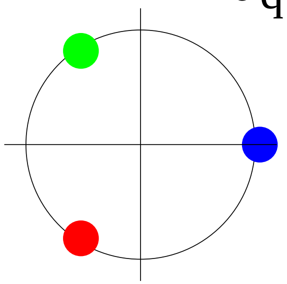
Breakdown of integrability: Mass gaps and bound states —*"theory"*

In lower part of the qqq spectrum

'Perturbation'	$\langle q' \vert \frac{1}{L_{12}^2} + \frac{1}{L_{23}^2} \vert q \rangle \sim \frac{1}{\ln N}$
Level splitting	$\sim \frac{1}{\ln^2 N}$

$\Longrightarrow$ **Of order $\sim \ln N$ lower levels have to be rediagonalized**

$$\langle q' \vert \frac{1}{L_{12}^2} + \frac{1}{L_{23}^2} \vert q \rangle \quad \sim \frac{1}{\ln N} \cos(\theta_q - \theta_{q'})$$



Phases of the cyclic permutations:

$$(123) \xrightarrow{\mathcal{P}} (312)$$
$$\mathcal{P} \vert q \rangle = \theta_q \vert q \rangle$$

Matrix ($\ln N \times \ln N$):

$$\frac{1}{\ln N} \begin{pmatrix} 1 & -1/2 & -1/2 & 1 & \dots \\ -1/2 & 1 & -1/2 & -1/2 & \dots \\ -1/2 & -1/2 & 1 & -1/2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \xrightarrow{\text{diag}} \frac{1}{\ln N} \begin{pmatrix} \ln N & 0 & 0 & 0 & \dots \\ 0 & \ln N & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

! A mass gap !

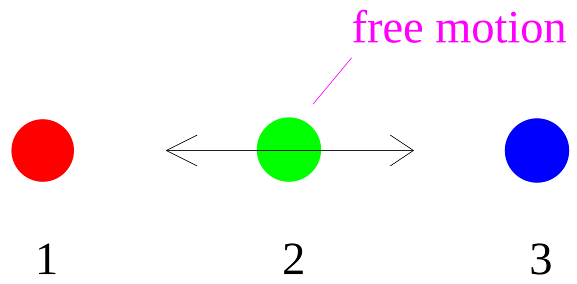
$\Delta^{\lambda=3/2}$ wave function

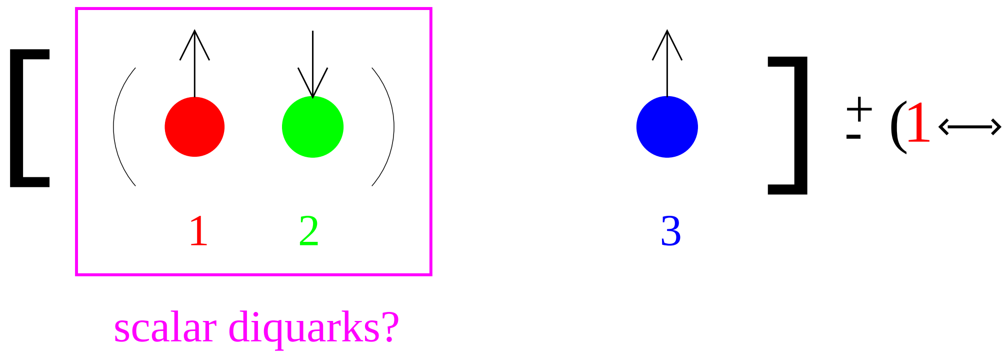
Nucleon and $\Delta^{1/2}$ wave functions

$$\varphi_{\Delta^{3/2}}(x_i)^\mu = \sum_{N=0}^{\infty} \varphi^{\mu_0}_{N,n=0} \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\gamma_{N,n=0}} \Big\{ x_1(1-x_1)C^{3/2}_{N+1}(1-2x_1) + x_2(1-x_2)C^{3/2}_{N+1}(1-2x$$

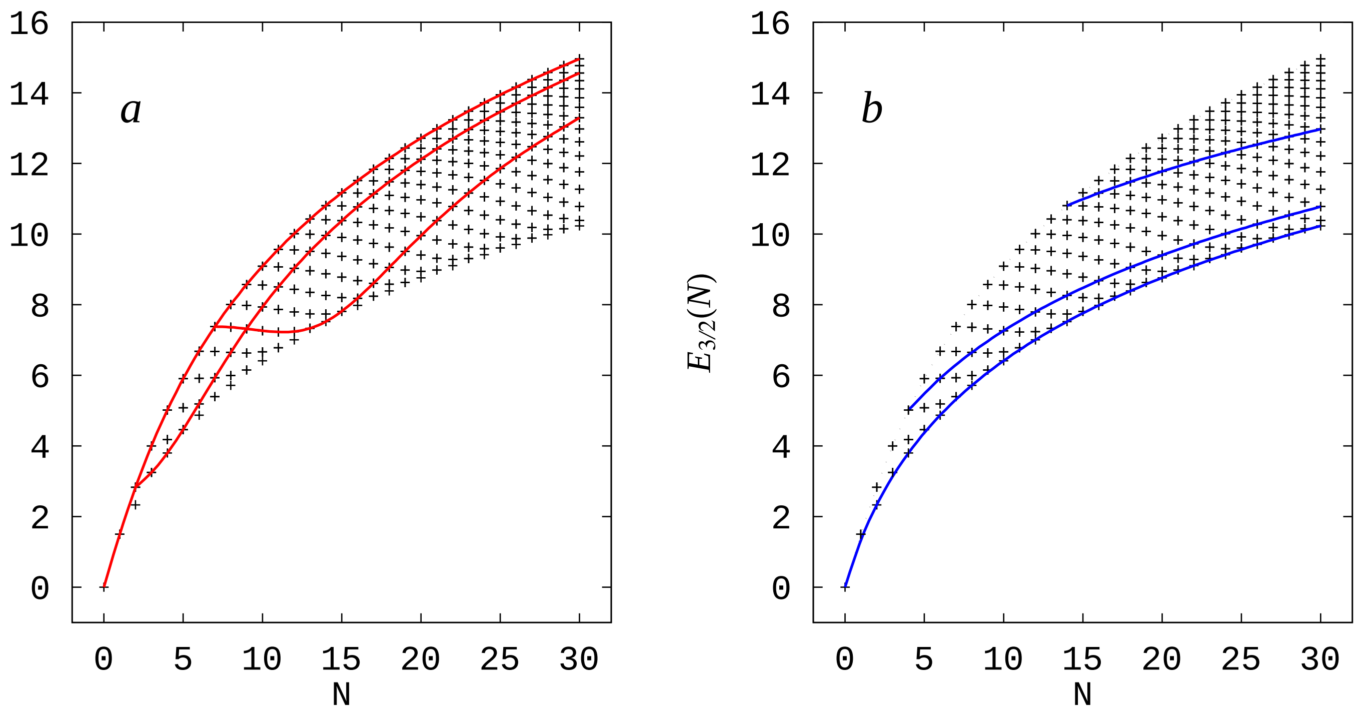
$$+x_3(1-x_3)C^{3/2}_{N+1}(1-2x_3)\Big\}$$

$$\varphi_{\Delta^{1/2}}(x_i)^\mu = x_1x_2x_3 \sum_{N=0}^{\infty} \varphi^{\mu_0}_{N,n=0} \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\gamma_{N,n=0}} \Big\{ P^{(1,3)}_N(1-2x_3) \pm P^{(1,3)}_N(1-2x_1) \Big\}$$





three quarks, $\lambda = 3/2$



large $x_B \to 1$ behavior: each red trajectory grows as $\sim 6 \ln N$

each blue trajectory grows as $\sim 4 \ln N$

small $x_B \to 0$ behavior: singularities in the complex N plane?

2004: *Feretti, Heise, Zarembo*

2005: *Beisert, Feretti, Heise, Zarembo*

Embedding $SL(2,\mathbb{R})$ in $SO(4,2)$

Example: Twist-4 nucleon distribution amplitudes

$$\langle 0|\, u_+^\uparrow(a_1 z) u_+^\downarrow(a_2 z) d_-^\uparrow(a_3 z)\,|N(P,\lambda)\rangle \quad \sim \quad N_-^\uparrow(P) \int \mathcal{D}\xi e^{-ipz\sum \xi_i a_i} \Phi_4^{\parallel}(\xi_i)$$

$$\langle 0|\, u_+^\uparrow(a_1 z) u_-^\downarrow(a_2 z) d_+^\downarrow(a_3 z)\,|N(P,\lambda)\rangle \quad \sim \quad N_+^\uparrow(P) \int \mathcal{D}\xi e^{-ipz\sum \xi_i a_i} \Phi_4^{\perp}(\xi_i)$$

$$\langle 0|\, u_-^\uparrow(a_1 z) u_+^\uparrow(a_2 z) d_+^\uparrow(a_3 z)\,|N(P,\lambda)\rangle \quad \sim \quad N_+^\uparrow(P) \int \mathcal{D}\xi e^{-ipz\sum \xi_i a_i} \Phi_4^{T}(\xi_i)$$

involve a 'minus' component of one of the quark fields

Braun, Manashov, Rohrwild; work in progress

More distant future: Twist-4 corrections to DIS

Advanced Example: Polarized deep-inelastic scattering

Cross section: $\nu = pq/M = E_l - E_{l'}$

$$\frac{d^2\sigma}{d\Omega dE_{l'}}(\downarrow\uparrow + \uparrow\downarrow) = \frac{8\alpha^2 E_{l'}^2}{Q^4\nu}\left[F_2(x,Q^2)\cos^2\frac{\theta}{2} + \frac{2\nu}{M}F_1(x,Q^2)\sin^2\frac{\theta}{2}\right],$$

$$\frac{d^2\sigma}{d\Omega dE_{l'}}(\downarrow\uparrow - \uparrow\downarrow) = \frac{8\alpha^2 E_{l'}}{Q^4}x\left[g_1(x,Q^2)\left(1+\frac{E_{l'}}{E_l}\cos\theta\right) - \frac{2Mx}{E_l}g_2(x,Q^2)\right]$$

At tree level:

$$g_1(x) = \int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}\,\mathrm{e}^{i\lambda x}\langle p, s_z|\bar{q}(0)\not{n}\gamma_5 q(\lambda n)|p, s_z\rangle$$

$$g_T(x) = \frac{1}{2M}\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}\,\mathrm{e}^{i\lambda x}\langle p, s_\perp|\bar{q}(0)\gamma_\perp\gamma_5 q(\lambda n)|p, s_\perp\rangle$$

— quark distributions in longitudinally and transversely polarized nucleon, respectively

Beyond the tree level

$$\langle N(p,s)|\bar{q}(z_1)G(z_2)q(z_3)|N(p,\lambda)\rangle =$$
$$= \ldots \int_{-1}^{1} dx_1 dx_2 dx_3\, \delta(x_1+x_2+x_3)\, e^{-ip(x_1 z_1 + x_2 z_2 + x_3 z_3)} D_q(x_i,\mu^2)$$

$$\bar{q}\,G\,q \quad \Rightarrow \quad g_2^{p-n}(x_B,\mu^2)$$
$$\left.\begin{array}{c}\bar{q}\,G\,q\\ G\,G\,G\end{array}\right\} \quad \Rightarrow \quad g_2^{p+n}(x_B,\mu^2)$$

quark-gluon correlations in the nucleon

$$\mathcal{H}_{qGq} = N_c \mathcal{H}^{(0)} - \frac{2}{N_c}\mathcal{H}^{(1)}\,,$$

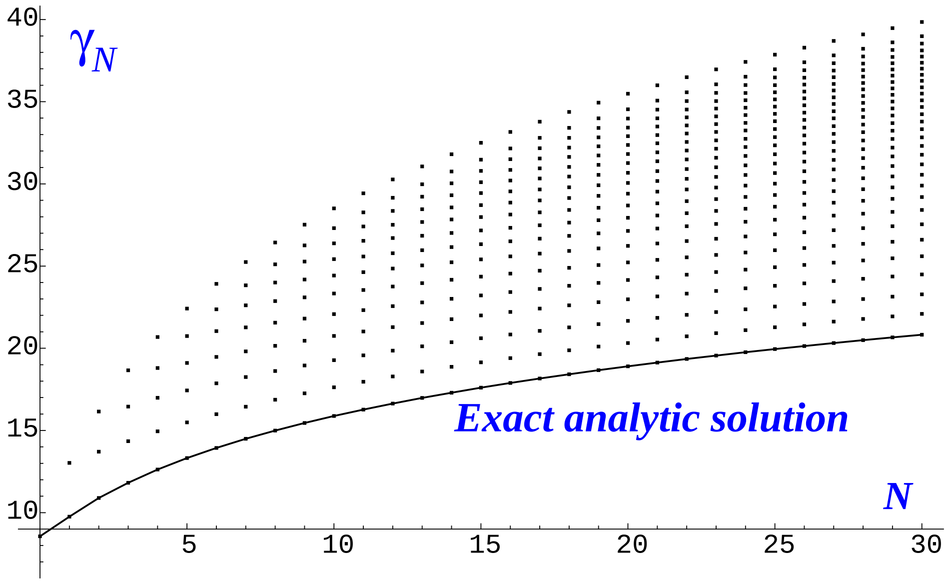
$$\mathcal{H}^{(0)} = V_{qg}^{(0)}(J_{12}) + U_{qg}^{(0)}(J_{23})\,,$$

$$\mathcal{H}^{(1)} = V_{qg}^{(1)}(J_{12}) + U_{qg}^{(1)}(J_{23}) + U_{qq}^{(1)}(J_{13})\,.$$

where

$$\begin{aligned}
V_{qg}^{(0)}(J) &= \psi(J+3/2) + \psi(J-3/2) - 2\psi(1) - 3/4\,,\\
U_{qg}^{(0)}(J) &= \psi(J+1/2) + \psi(J-1/2) - 2\psi(1) - 3/4\,,\\
V_{qg}^{(1)}(J) &= \frac{(-1)^{J-5/2}}{(J-3/2)(J-1/2)(J+1/2)}\,,\\
U_{qg}^{(1)}(J) &= -\frac{(-1)^{J-5/2}}{2(J-1/2)}\,,\\
V_{qq}^{(1)}(J) &= \psi(J) - \psi(1) - 3/4\,,\\
U_{qq}^{(1)}(J) &= \frac{1}{2}\left[\psi(J-1) + \psi(J+1)\right] - \psi(1) - 3/4\,.
\end{aligned}$$

Open inhomogeneous spin chain



◊ The lowest level is special and is separated from the rest of the spectrum by a **"mass gap"**

BDM '98

◊ This special level was found by ABH 91' and it determines the evolution of $g_2(x, Q^2)$ to the leading logarithmic accuracy

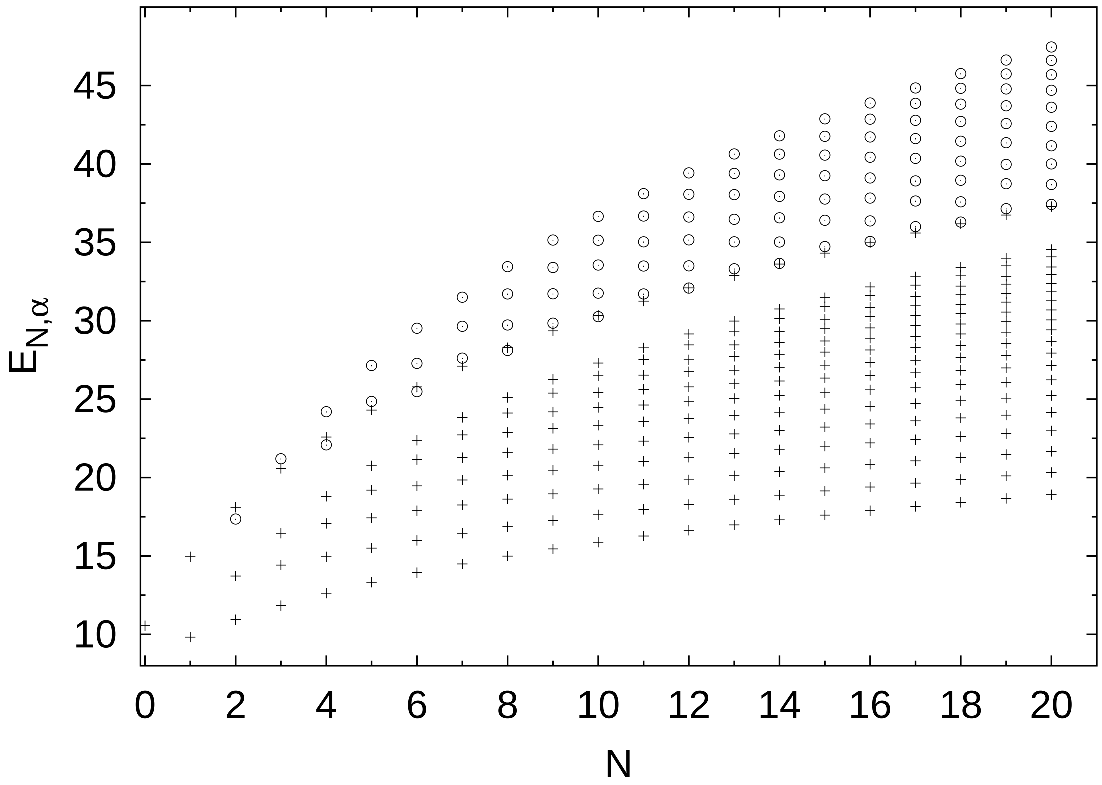
◊ It corresponds to the particular combination of quark-antiquark-gluon operators that can be reduced to the quark-antiquark twist-three operator using equations of motion

Detailed study:

Belitsky; Derkachov, Korchemsky, Manashov '00

Flavor singlet qGq and GGG operators

BDM '01



The highest qGq level is separated from the rest by a finite gap and is almost degenerate with the lowest GGG state

qGq levels: crosses

GGG levels: open circles

- **Interacting open and closed spin chains**

Introduce flavor-singlet quark and gluon transverse spin distributions

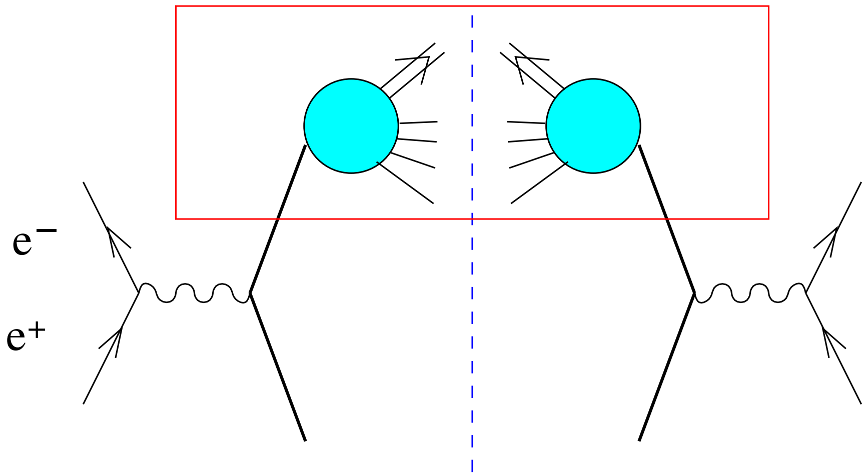
$$
\begin{aligned}
Q^2 \frac{d}{dQ^2} \Delta q_T^{+,\,\mathrm{S}}(x;Q^2) &= \frac{\alpha_s}{4\pi} \int_x^1 \frac{dy}{y} \left[P_{qq}^T(x/y) \Delta q_T^{+,\,\mathrm{S}}(y;Q^2) + P_{qg}^T(x/y) \Delta g_T(y;Q^2) \right] \\
Q^2 \frac{d}{dQ^2} \Delta g_T(x;Q^2) &= \frac{\alpha_s}{4\pi} \int_x^1 \frac{dy}{y} P_{gg}^T(x/y) \Delta g_T(y;Q^2)
\end{aligned}
$$

with the splitting functions

$$
\begin{aligned}
P_{qq}^T(x) &= \left[\frac{4C_F}{1-x} \right]_+ + \delta(1-x) \left[C_F + \frac{1}{N_c} \left(2 - \frac{\pi^2}{3} \right) \right] - 2C_F \,, \\
P_{gg}^T(x) &= \left[\frac{4N_c}{1-x} \right]_+ + \delta(1-x) \left[N_c \left(\frac{\pi^2}{3} - \frac{1}{3} \right) - \frac{2}{3} n_f \right] \\
&\quad + N_c \left(\frac{\pi^2}{3} - 2 \right) + N_c \ln \frac{1-x}{x} \left(\frac{2\pi^2}{3} - 6 \right) , \\
P_{qg}^T(x) &= -4n_f \left[x - 2(1-x)^2 \ln(1-x) \right] .
\end{aligned}
$$

Belitsky, Kuraev '97

$$e^+e^- \to H(\zeta) + X$$



Fragmentation function

$$\mathcal{D}(\zeta) = \sum_X \int \frac{d\lambda}{2\pi} e^{i\lambda\zeta} \langle 0|\psi(\lambda n)|h, X\rangle\langle h, X|\bar{\psi}(0)|0\rangle$$

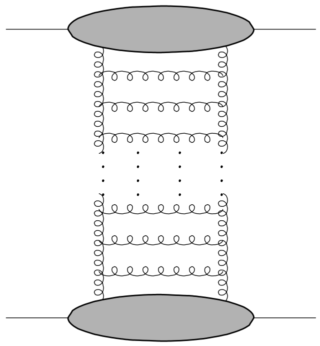
has autonomous evolution for $N_c \to \infty$, i.e. does not mix with functions involving gluon field:

$$\mathcal{Z}(\zeta, \zeta') = \sum_X \int \frac{d\lambda}{2\pi}\frac{d\lambda'}{2\pi} e^{i\lambda\zeta - i\lambda'\zeta'} \langle 0|\gamma_\rho^\perp \gamma^+ \psi(\lambda n)|h, X\rangle\langle h, X|\bar{\psi}(0) B_\perp^\rho(\lambda' n)|0\rangle$$

Integrability?

- **short-distance expansion not applicable —no relation to local operators**
- **Operator language exists: Balitsky, Braun '91**

High Energy Scattering: The Regge Limit of QCD



1976-78: Balitsky-Fadin-Kuraev-Lipatov (BFKL) pomeron:

$$\mathcal{A}(s,t) \sim i s^{1+\alpha_s/\pi N_c 4\ln 2} \qquad s \to \infty, \quad t \sim \mathit{const}$$

Partial waves with the complex angular momentum

$$\mathcal{A}(s,t) \sim i s \alpha_s^2 \int_{\delta - i\infty}^{\delta + i\infty} \frac{d\omega}{2\pi i} s^{\omega} \widetilde{\mathcal{A}}(\omega, t)$$

can be writtten in the impact parameter representation

$$\widetilde{\mathcal{A}}(\omega,t) = \int d^2b\, e^{iqb} \int d^2b_k d^2b'_k \Phi(b_1 - b, b_2 - b) T_\omega(b_1, b_2; b'_1, b'_2) \Phi(b'_1, b'_2) = \int d^2b\, e^{iqb} \langle \Phi(b) | \mathbb{T}_\omega | \Phi(0$$

where the kernel satisfies Bethe-Salpeter equation

$$\omega \mathbb{T}_\omega = \mathbb{T}_\omega^{(0)} + \frac{\alpha_s N_c}{\pi} \mathbb{H}_{\mathrm{BFKL}} \mathbb{T}_\omega \quad \to \quad \mathbb{T}_\omega = \left(\omega - \frac{\alpha_s N_c}{\pi} \mathbb{H}_{\mathrm{BFKL}} \right)^{-1} \mathbb{T}_\omega^{(0)}$$

Hence singularities of $\mathbb{T}_\omega^{(0)}$ in the ω-plane are determined by eigenvalues of $\mathbb{H}_{\mathrm{BFKL}}$

$$[\mathbb{H}_{\mathrm{BFKL}} \cdot \psi_\alpha(b_1, b_2) = E_\alpha \psi_\alpha(b_1, b_2)$$

The largest E_α corresponds to right-most singularity alias leading large-s behavior

$\mathbb{H}_{\mathrm{BFKL}}$ **has a number of remarkable properties** **Lipatov**

- **Holomorphic separability** $\vec{b}_{1,2} \to z_{1,2}$

$$\mathbb{H}_{\mathrm{BFKL}} = \mathcal{H}_2 + \bar{\mathcal{H}}_2 \qquad \mathcal{H}_2 = \partial_{z_1}^{-1} \ln(z_{12}) \partial_{z_1} + \partial_{z_2}^{-1} \ln(z_{12}) \partial_{z_2} + \ln(\partial_{z_1} \partial_{z_2}) - 2\Psi(1)$$

- **Invariance under $SL(2,\mathbb{C})$ conformal transformations of the transverse plane**

$\longrightarrow$ **can be rewritten in terms of two-particle Casimir operators**

$$\mathbb{H}_{\mathrm{BFKL}} = \frac{1}{2}\left[H(J_{12}) + H(\bar{J}_{12})\right], \qquad H(j) = 2\Psi(1) - \Psi(j) - \Psi(1-j)$$

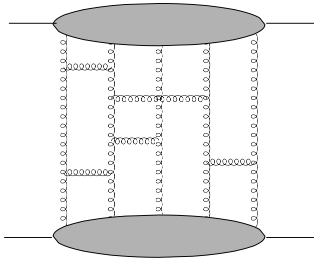
the solutions can be characterized by a pair of complex conformal spins

$$h = \frac{1+n}{2} + i\nu\,, \quad \bar{h} = \frac{1-n}{2} + i\nu$$
$$E_{n,\nu} = 2\Psi(1) - \Psi\left(\frac{1+n}{2} + i\nu\right) - \Psi\left(\frac{1+n}{2} - i\nu\right)$$

The maximum value corresponds to $n = \nu = 0$, $E_{0,0} = 4\ln 2$

Unitarity?

Generalization to N interacting reggeized gluons in the $N_c \to \infty$ limit:



$$\mathbb{H}_N = \frac{1}{4} \sum_{k=1}^{N} \left[H(J_{k,k+1}) + H(\bar{J}_{k,k+1}) \right]$$

- **Complete Integrability:** $\mathbb{H}_N$ **(separately in holomorphic and antiholomorphic sector) is the Hamiltonian of the** $SL(2,\mathbb{C})$ **closed spin chain**

  Lipatov '94, Faddeev, Korchemsky '95

$$[q_k, \mathbb{H}_N] = [q_k, q_n] = 0 \qquad k, n = 2, \ldots, N$$

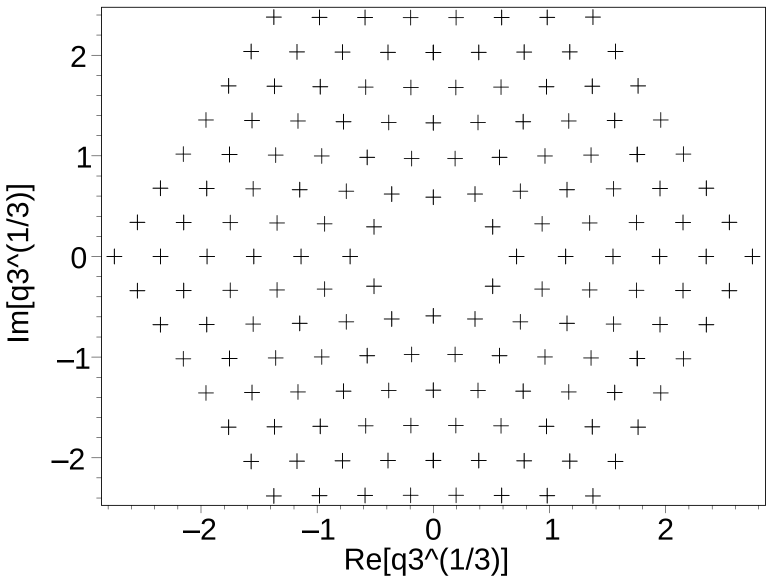
- ◇ **Algebraic Bethe Ansatz not applicable (no pseudovacuum state)**
- ◇ **Quantization conditions?**

Methods:

($N = 3$) eigenvalue problem for the transfer matrices Janik, Wosiek '97, Bartels, Lipatov, Vacca '00

($N > 3$) construct the Baxter $\mathbb{Q}$ operator Derkachov '99, Derkachov, Korchemsky, Manashov '01

The spectrum of q_3 ($N = 3$) DKKM '02



States with maximum energy DKKM '02

N	iq_3	q_4	iq_5	q_6	iq_7	E_N
2						2.7
3	.2053					-.24
4	0	.1536				.67
5	.2677	.0395	.0602			-.12
6	0	.2818	0	.0705		.39
7	.3131	.0710	.1285	.0085	.0195	-.08

For odd N there are also solutions with $E_N = 0, q_2 = \ldots q_N = 0$

WKB approximation

$$[q_3(n_1, n_2)]^{1/3} = \frac{\Gamma^3(2/3)}{2\pi}\left(\frac{1}{2}n_1 + i\frac{\sqrt{3}}{2}n_2\right)$$

Unitarity?

Outlook

- **One-loop dilatation operator (evolution equations) in QCD is/are integrable in some sectors**
  - *Classical bremsstrahlung:* $$\mathcal{A} \sim \frac{d\omega}{\omega}\frac{d\theta}{\theta}$$
  - *Cusp anomalous dimension:* *for* $N \to \infty$
    $$\gamma_N \sim \Gamma_{\rm cusp}(\alpha_s) \log N$$
  - *Integrability appears as a consequence of the existence of massless vector particles*
- **Offers powerful machinery**
- **Open problems in QCD context:**
  - *Analytic structure of the spectrum — Parton interpretation in higher twists*
  - *From* $SL(2,\mathbb{R})$ *to* $SO(4,2)$*: Rethinking of the role of non-quasipartonic operators; properties of anomalous dimensions in all twists*
  - *Beyond one loop: Formal conformal limit*
    $$\mathcal{A} = \mathcal{A}^{\rm conformal} + \frac{\beta(g)}{g}\Delta\mathcal{A}$$
    Integrability? Particular models?
  - *Breaking of integrability vs. breaking of conformal symmetry: physics issues?*