

Decomposing Feynman rules

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Joint work with **Spencer Bloch**, arXiv:1007.0338,
and **Francis Brown**, arXiv:1112.1180.

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Hopf algebra of graphs $H = \mathbb{Q}1 \oplus \bigoplus_{j=1}^{\infty} H^j$

► The coproduct

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \overbrace{\sum_{\gamma=\cup_i \gamma_i, \omega_4(\gamma_i) \geq 0} \gamma \otimes \Gamma/\gamma}^{\Delta'(\Gamma)} \quad (1)$$



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► The renormalized Feynman rules

$$\Phi_R = m(S_R^\Phi \otimes \Phi)\Delta$$

BCFW and the core Hopf algebra

- ▶ a sequence of quotient Hopf algebras by looking at short distance singularities in $2k$ dimensions

$$H_0 \subset H_2 \subset H_4 \subset \cdots \subset H_{2k} \subset \cdots H_\infty \quad (6)$$

$$\begin{aligned}\Delta'_4 \left(\text{Diagram } A \right) &= \text{Diagram } B \otimes \text{Diagram } C \\ \Delta'_{\infty} \left(\text{Diagram } A \right) &= 2 \text{Diagram } D \otimes \text{Diagram } E + \text{Diagram } F \otimes \text{Diagram } G\end{aligned} \quad (7)$$



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- ▶ the primitives are all one-loop
- ▶ quantum gravity: $H_{\text{ren}} = H_\infty$, $\omega_4(\Gamma) = 2|\Gamma| + 2$



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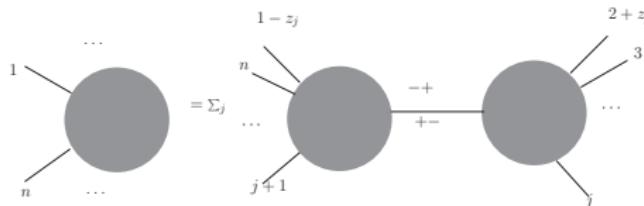
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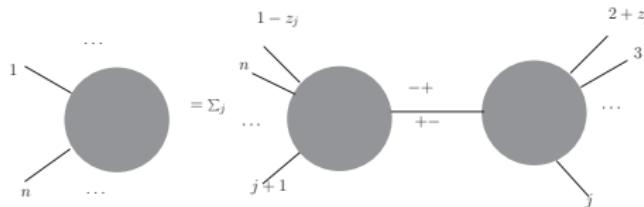
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- ▶ KLT relations or kinematic STU ↔ co-ideal respected



Kinematics and Cohomology

- ▶ Exact co-cycles

$$[B_+^{r;j}] = B_+^{r;j} + b\phi^{r;j} \quad (8)$$

with $\phi^{r;j} : H \rightarrow \mathbb{C}$



Kinematics and Cohomology

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- ▶ Variation of momenta

$$G^R(\{g\}, \ln s, \{\Theta\}) = 1 \pm \Phi_{\ln s, \{\Theta\}}^R(X^r(\{g\})) \quad (9)$$

with $X^r = 1 \pm \sum_j g^j B_+^{r;j}(X^r Q^j(g))$, $bB_+^{r;j} = 0$. Then, for kinematic renormalization schemes:

$$\{\Theta\} \rightarrow \{\Theta'\} \Leftrightarrow B_+^{r;j} \rightarrow B_+^{r;j} + b\phi^{r;j}.$$

$$\Phi_{L_1+L_2, \{\Theta\}}^R = \Phi_{L_1, \{\Theta\}}^R * \Phi_{L_2, \{\Theta\}}^R.$$

$$\Phi^R(\ln s, \{\Theta\}, \{\Theta_0\}) = \Phi_{\text{fin}}^{-1}(\{\Theta_0\}) * \Phi_{1-\text{scale}}^R(\ln s) * \Phi_{\text{fin}}(\{\Theta\}).$$



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Graph Polynomials



$$0 \rightarrow H^1(\Gamma) \rightarrow \mathbb{Q}^{E_\Gamma} \rightarrow \mathbb{Q}^{V_\Gamma, 0} \rightarrow 0. \quad (10)$$

$\{h_i\}$ basis of homology (loops!)



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- $(q_0, q_1, q_2, q_3)^T \rightarrow q_0 \cdot 1 + q_1 \cdot i + q_2 \cdot j + q_3 \cdot k$ quaternionic embedding



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$$N := \begin{pmatrix} N_0 := \left(\sum_{e \in h_i \cap h_j} A_e \right)_{ij} \mathbb{I} & \sum_{e \in h_j} \mu_e A_e \\ \sum_{e \in h_j} \bar{\mu}_e A_e & \sum_{e \in \Gamma^{[1]}} \bar{\mu}_e \mu_e A_e \end{pmatrix}$$



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- ▶ $|N_0| = \psi(\Gamma),$
 $|N| = \phi(\Gamma).$

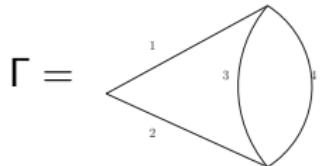


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Example



$$N_\Gamma = \begin{pmatrix} A_1 + A_2 + A_3 & A_1 + A_2 & A_1\mu_1 + A_2\mu_2 + A_3\mu_3 \\ A_1 + A_2 & A_1 + A_2 + A_4 & A_1\mu_1 + A_2\mu_2 + A_4\mu_4 \\ A_1\bar{\mu}_1 + A_2\bar{\mu}_2 + A_3\bar{\mu}_3 & A_1\bar{\mu}_1 + A_2\bar{\mu}_2 + A_4\bar{\mu}_4 & \sum_{i=1}^4 A_i\bar{\mu}_i\mu_i \end{pmatrix}$$

$$\begin{aligned}\psi_\Gamma &= (A_1 + A_2)(A_3 + A_4) + A_3A_4 = \sum_{\text{sp. Tr. } T} \prod_{e \notin T} A_e \\ \phi_\Gamma &= (A_3 + A_4)A_1A_2p_a^2 + A_2A_3A_4p_b^2 + A_1A_3A_4p_c^2 = \\ &\sum_{\text{sp. 2-Tr. } T_1 \cup T_2} Q(T_1) \cdot Q(T_2) \prod_{e \notin T_1 \cup T_2} A_e.\end{aligned}$$

The Feynman rules in projective space

First, $\phi_\Gamma \rightarrow \phi_\Gamma + \psi_\Gamma(\sum_e m_e^2 A_e)$.

$$\Phi_\Gamma^R(S, S_0, \{\Theta, \Theta_0\}) = \int_{\mathbb{P}^{E-1}(\mathbb{R}_+)} \overbrace{\sum_f}^{\text{forestsum}} (-1)^{|f|}$$
$$\ln \frac{\frac{S}{S_0} \phi_{\Gamma/f} \psi_f + \phi_f^0 \psi_{\Gamma/f}}{\frac{\phi_{\Gamma/f}^0 \psi_f + \phi_f^0 \psi_{\Gamma/f}}{\psi_{\Gamma/f}^2 \psi_f^2}}$$
$$\underbrace{\Omega_\Gamma}_{(E-1)\text{-form}}$$

Note: for 1-scale graphs, $\phi_\Gamma = \psi_\Gamma^\bullet$.

The polylog as a Hodge structure

Iterated integrals: obvious Hopf algebra structure

$$\begin{pmatrix} 1 & 0 & 0 \\ -Li_1(z) & 2\pi i & 0 \\ -Li_2(z) & 2\pi i \ln(z) & (2\pi i)^2 \end{pmatrix} = (C_1, C_2, C_3) \quad (11)$$

$$\text{Var}(\Im Li_2(z) - \ln |z| \Im Li_1(z)) = 0 \quad (12)$$

Hodge structure from Hopf algebra structure: branch cut
ambiguities columnwise

Griffith transversality \Leftrightarrow differential equation



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Limiting mixed Hodge structures

- ▶ Hopf algebra from flags

$$f := \gamma_1 \subset \gamma_2 \subset \dots \subset \Gamma, \Delta'(\gamma_{i+1}/\gamma_i) = 0 \quad (13)$$

The set of all such flags $F_\Gamma \ni f$ determines Hopf algebra structure, $|F_\Gamma|$ is the length of the flag.



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- ▶ It also determines a column vector $v = v(F_\Gamma)$ and a nilpotent matrix $(N) = (N(|F_\Gamma|))$, $(N)^{k+1} = 0$, $k = \text{corad}(\Gamma)$ such that

$$\lim_{t \rightarrow 0} (e^{-\ln t(N)}) \Phi_R(v(F_\Gamma)) = (c_1^\Gamma(\Theta) \ln s, c_2^\Gamma(\Theta), c_k^\Gamma(\Theta) \ln^k s)^T \quad (14)$$

where k is determined from the co-radical filtration and t is a regulator say for the lower boundary in the parametric representation.



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The Feynman graph as a Hodge structure

Hopf algebra structure as above

$$\left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) = (C_1, C_2, C_3, C_4, C_5)$$
$$\text{Var} \left(\mathfrak{I} - \left[\Re \cdot \mathfrak{I} \cdot \mathfrak{I} \right] + \dots \right) = 0$$

Hodge structure: cut-reconstructability: from Hopf algebra structure:
branch cut ambiguities columnwise
Griffith transversality \Leftrightarrow differential equation?



$$\zeta(s_1, \dots, s_k) = \sum_{n_i < n_{i+1}} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}$$

► counting over \mathbb{Q}

$$1 - \frac{x^3y}{1-x^2} + \frac{x^{12}y^2(1-y^2)}{(1-x^4)(1-x^6)} = \prod_{n \geq 3} \prod_{k \geq 1} (1-x^n y^k)^{D_{n,k}} \quad (15)$$

→ first irreducible MZV from planar graphs at 7 loops in scalar field theory (integrability???)

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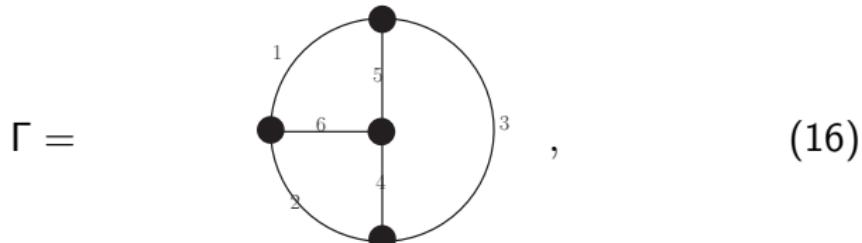
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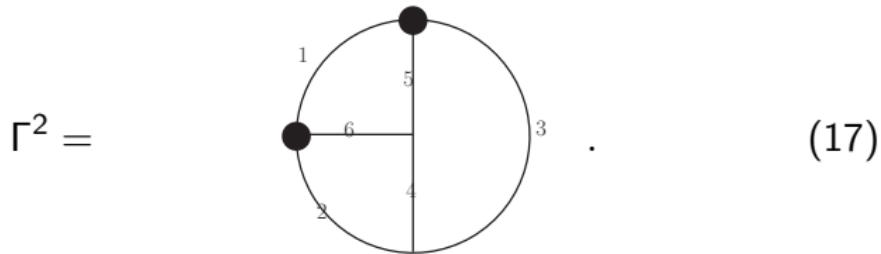
- ▶ When is a graph reducible to MZVs? Francis Brown: when it has vertex width three.
- ▶ Caution! Non-MZVs at eight loops from non-planar graphs, at nine loops from planar graphs ('A $K3$ in ϕ^4 ', Brown and Schnetz). Proof from counting points $[X_\Gamma]$ on graph hypersurfaces X_Γ over \mathbb{F}_q , defined by vanishing of the first Symanzik polynomial. If the graph gives a MZV, $[X_\Gamma]$ better is polynomial in the prime power $q = p^n$. Alas, it is not in general, with counterexamples relating graphs to elliptic curves with complex multiplication, and point-counting function a modular form.

Decomposing scales and angles

Consider



and



We let $S = p_1^2 + p_2^2 + p_3^2 + 2p_1 \cdot p_2 + 2p_2 \cdot p_3 + 2p_3 \cdot p_1$ (which defines the variable angles $\Theta^{ij} = p_i \cdot p_j / S$, $\Theta^e = m_e^2 / S$) and subtract symmetrically say at S_0 , $\Theta_0^{ij} = \frac{1}{3}(4\delta_{ij} - 1)$ and $\Theta_0^e = m_e^2 / S_0$, which specifies the fixed angles Θ_0 .



$$\Phi_{\Gamma}^R = \frac{\ln \frac{S}{S_0} \phi_{\Gamma}(\Theta)}{\psi_{\Gamma}^2} \Omega_{\Gamma}. \quad (18)$$

To find the desired decomposition, we use

$$\Delta^2(\Gamma) = \Gamma \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes \Gamma. \quad (19)$$

We then have

$$\Phi_{\Gamma}^R = \Phi_{\text{fin}}^{-1}(\Theta_0)(\Gamma) + \Phi_{1-s}^R(S/S_0)(\Gamma) + \Phi_{\text{fin}}(\Theta)(\Gamma). \quad (20)$$

We have

$$\Phi_{\text{fin}}^{-1}(\Theta_0)(\Gamma) = -\frac{\ln \frac{\phi_\Gamma(\Theta_0)}{\psi_{\Gamma^2}^\bullet}}{\psi_\Gamma^2} \Omega_\Gamma, \quad (21)$$

$$\Phi_{1-\text{s}}^R(S/S_0)(\Gamma) = \frac{\ln \frac{S}{S_0}}{\psi_\Gamma^2} \Omega_\Gamma, \quad (22)$$

which integrates to the renormalized value

$\Phi_{1-\text{s}}^R(S/S_0)(\Gamma) = 6\zeta(3) \ln \frac{S}{S_0}$. Finally,

$$\Phi_{\text{fin}}(\Theta)(\Gamma) = \frac{\ln \frac{\phi_\Gamma(\Theta)}{\psi_{\Gamma^2}^\bullet}}{\psi_\Gamma^2} \Omega_\Gamma. \quad (23)$$

These integrands indeed all converge, which is synonymous for us to say that they can be integrated against $\mathbb{P}^{E-1}(\mathbb{R}_+)$.