DRED

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Dimensional Reduction

Siegel introduced DRED as a variation of DREG to maintain the equality of Bose-Fermi degrees of freedom characteristic of supersymmetry.

$$
\begin{aligned}
x^\mu &\equiv (x^i, 0) \\
p^\mu &\equiv (p^i, 0) \\
W_\mu &\equiv \big(W_i(x^j), W_\sigma(x^j)\big)
\end{aligned}
$$

A (Dirac) fermion represents 4 degrees of freedom as long as we define the Dirac matrix trace to satisfy $\mathrm{Tr}1 = 4$.
It is useful to define hatted quantities with $\mu, \nu \cdots$ indices whose only non-vanishing components are in the D-dimensional subspace; in particular, $\hat{g}_{\mu\nu} = (g_{ij}, 0)$.

Metric tensors

So we define

$$g_{\mu\nu} = \hat{g}_{\mu\nu} + \tilde{g}_{\mu\nu}$$

where

$$\begin{aligned}
g_{\mu\nu} g^{\mu\nu} &= 4 \\
g_{\mu\nu} \hat{g}^{\nu}{}_{\rho} &= \hat{g}_{\nu\rho} \\
\hat{g}_{\mu\nu} \hat{g}^{\mu\nu} &= D \\
g_{\mu\nu} \tilde{g}^{\nu}{}_{\rho} &= \tilde{g}_{\nu\rho} \\
\tilde{g}_{\mu\nu} \tilde{g}^{\mu\nu} &= \epsilon = 4 - D \\
\hat{g}_{\mu\nu} \tilde{g}^{\nu}{}_{\rho} &= 0
\end{aligned}$$

Gauge transformations

The dimensionally reduced form of the gauge transformations:

$$
\begin{aligned}
\delta W_i^a &= \partial_i \Lambda^a + g f^{abc} W_i^b \Lambda^c \\
\delta W_\sigma^a &= g f^{abc} W_\sigma^b \Lambda^c \\
\delta \psi^\alpha &= i g (R^a)^{\alpha\beta} \psi^\beta \Lambda^a
\end{aligned}
$$

show that W_σ^a transform as scalars, called ϵ-scalars. Consequently the interactions

$$
g \overline{\psi} \gamma_\sigma R^a \psi W_\sigma^a \quad \text{and} \quad g^2 f^{abc} f^{ade} W_\sigma^b W_{\sigma'}^c W_\sigma^d W_{\sigma'}^e
$$

are both gauge invariant by themselves.
Moreover, a mass for the ϵ-scalars is itself gauge invariant.

Evanescent Couplings and Masses

We have three classes of theories which behave differently under renormalisation using DRED.

- Supersymmetric Theories:
  The ϵ-scalar interactions remain in step with the corresponding gauge interactions, and its mass remains zero.
- Softly-broken supersymmetric theories:
  Radiative corrections generate a mass for the ϵ-scalar.
- Un-supersymmetric theories:
  Again a mass for the ϵ-scalar, and both its Yukawa coupling and the quartic interaction renormalise differently from the gauge coupling. New quartic group theory structures are generated.

ϵ-scalar quartic couplings

A basis for tensors K^{abcd} in $SU(N)$ is given by

$$\begin{aligned} K_1 &= \delta^{ab}\delta^{cd} \quad K_4 = d^{abe}d^{cde} \quad K_7 = d^{abe}f^{cde} \\ K_2 &= \delta^{ac}\delta^{bd} \quad K_5 = d^{ace}d^{bde} \quad K_8 = d^{ace}f^{bde} \\ K_3 &= \delta^{ad}\delta^{bc} \quad K_6 = d^{ade}d^{bde} \quad K_9 = d^{ade}f^{bce}. \end{aligned}$$

So for ϵ-scalars a natural basis is

$$\begin{aligned} H_1 &= \tfrac{1}{2}K_1, \quad H_2 = \tfrac{1}{2}(K_2 + K_3) \\ H_3 &= \tfrac{1}{2}K_4, \quad H_4 = \tfrac{1}{2}(K_5 + K_6), \end{aligned}$$

reducible in $SU(3)$ since then $H_3 + H_4 = \frac{1}{3}(H_1 + H_2)$.
This makes DRED ponderous in non-susy theories.
Nevertheless, DRED and DREG are equivalent.

DRED ambiguities

Given $d < 4$, one can define $\hat{\epsilon}^{\mu\nu\rho\sigma}$ as follows:

$$\hat{\epsilon}^{\mu\nu\rho\sigma} = \hat{g}^{\mu\alpha}\hat{g}^{\nu\beta}\hat{g}^{\rho\gamma}\hat{g}^{\sigma\delta}\epsilon_{\alpha\beta\gamma\delta}$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is the usual 4-dimensional tensor. Then it is easy to show that

$$\hat{\epsilon}^{\mu\nu\rho\sigma}\hat{\epsilon}^{\alpha\beta\gamma\delta} = \hat{g}^{\mu\alpha}\hat{g}^{\nu\beta}\hat{g}^{\rho\gamma}\hat{g}^{\sigma\delta} - \hat{g}^{\mu\beta}\hat{g}^{\nu\alpha}\hat{g}^{\rho\gamma}\hat{g}^{\sigma\delta} + \cdots$$

and hence by consideration of

$$A^{\mu\nu\rho\sigma} = \hat{\epsilon}^{\mu\nu\rho\sigma}\hat{\epsilon}^{\alpha\beta\gamma\delta}\hat{\epsilon}_{\alpha\beta\gamma\delta}$$

that

$$(d+1)(d-4)(d^2-3d+6)\hat{\epsilon}^{\mu\nu\rho\sigma} = 0$$

γ_5

From

$$\{\gamma_\mu, \gamma^5\} = 0 \quad \text{we have} \quad \{\hat{\gamma}_\mu, \gamma^5\} = 0. \tag{7}$$

and hence that

$$(d-4)\mathrm{Tr}\left[\gamma^5 \hat{\gamma}^\mu \hat{\gamma}^\nu \hat{\gamma}^\rho \hat{\gamma}^\sigma\right] = 0. \tag{8}$$

For $d > 4$, however, Eq. (7) does not hold and so Eq. (8) no longer follows. In that case you can impose

$$[\gamma_\sigma, \gamma^5] = 0, \quad \text{for} \quad 4 < \sigma < d$$

giving an unambiguous DREG derivation of the anomaly.

Living with $\epsilon^{\alpha\beta\gamma\delta}$ and γ^5

To avoid ambiguities we must avoid assuming relations like

$$\epsilon^{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta} = g^{\mu\alpha}g^{\nu\beta}g^{\rho\gamma}g^{\sigma\delta} - g^{\mu\beta}g^{\nu\alpha}g^{\rho\gamma}g^{\sigma\delta} + \cdots \qquad (10)$$

and

$$\{\gamma_\mu, \gamma^5\} = 0.$$

For example, in two dimensional σ models the relation

$$\hat{\epsilon}^{\mu}{}_{\nu}\hat{\epsilon}^{\nu\rho} = (1 + c\epsilon)\hat{g}^{\mu\rho} \qquad (12)$$

can be used without ambiguity; the dependence on the c-parameter can be absorbed into redefinitions of the renormalised metric and torsion. Stöckinger has formalised this by distinguishing "normal" $d = 4$ space, from a quasi $d = 4$ space, $Q4S$, in which Eqs. (10), (12) are not true.

The supersymmetry Ward identity

The $N = 1$ theory

$$L_S = -\frac{1}{4}G^{\mu\nu}G_{\mu\nu} + \frac{i}{2}\overline{\lambda}^a\gamma^\mu D_\mu^{ab}\lambda^b + \frac{1}{2}D^2.$$

is invariant under supersymmetry transformations except for one term:

$$\delta L_S = \frac{g}{2}f^{abc}\overline{\epsilon}\gamma^\mu\lambda^a\overline{\lambda}^b\gamma_\mu\lambda^c,$$

which is zero in *strictly* four dimensions. However, an insertion of δL_S in a Feynman graph of arbitrary complexity depends on a quantity Δ given by

$$\Delta = \mathrm{Tr}(A\gamma^\mu B\gamma_\mu) + \mathrm{Tr}(A\gamma^\mu)\mathrm{Tr}(B\gamma_\mu) - (-1)^k\mathrm{Tr}(A\gamma^\mu B^R\gamma_\mu)$$

Here A, B are strings of γ-matrices. In DREG, Δ is non-zero even at one loop; in *strict* $d = 4$ it is zero, but not in Q4S, since for example if

$$A = \gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_5} \quad \text{and} \quad B = \gamma_{\nu_1}\gamma_{\nu_2}\cdots\gamma_{\nu_5} \tag{16}$$

then

$$\Delta = 48\delta^{[\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\cdots\delta^{\mu_5]}_{\nu_5}$$

which is *not* zero in Q4S. For instance

$$\mathrm{Tr}\Delta = 48d(d-1)(d-2)(d-3)(d-4)$$

There is no problem of principle; the contribution of δL_S ensures that the Ward identity remains satisfied.

The NSVZ β-function

$$\beta_g^{NSVZ} = \frac{g^3}{16\pi^2}\left[\frac{Q - 2r^{-1}\mathrm{Tr}\left[\gamma^{NSVZ}C(R)\right]}{1 - 2C(G)g^2(16\pi^2)^{-1}}\right].$$

β_g and γ calculated using DRED begin to deviate from the NSVZ results at three loops. However, there is an analytic redefinition of g, $g \to g'(g, Y)$ which connects them. It is non-trivial that the redefinition exists; in the abelian case for example, the redefinition consists of a single term, but it affects four distinct terms (with different tensor structure) in the β-functions. Exploiting the fact that $N = 2$ theories are finite beyond one loop it was possible to determine β_g^{DRED} for $N = 1$ at three and four loops by (comparatively) simple calculations. Subsequently some of these results were confirmed by the Karlsruhe group.

The NSVZ↔DRED connection

$$\beta_g^{(3)DRED} = r^{-1} g \left\{3X_1 + 6X_3 + X_4 - 6g^6 Q \mathrm{tr}[C(R)^2]\right\}$$

$$\beta_g^{(3)NSVZ} = r^{-1} g \left\{2X_1 + 4X_3 - 4g^6 Q \mathrm{tr}[C(R)^2]\right\}$$

The coupling constant redefinition linking the two schemes is uniquely determined up to an overall constant:

$$\delta g = -(16\pi^2)^{-2} \tfrac{1}{2} r^{-1} g^3 \mathrm{tr}\left[PC(R)\right]$$

and generates just the right shift in β_g:

$$(16\pi^2)^3 \delta\beta_g = r^{-1} g \left(-X_1 - 2X_3 - X_4 + 2g^6 Q \mathrm{tr}[C(R)^2]\right)$$

Physical quantities and the schemes

A QCD example: in DRED:

$$m_t^{\mathrm{pole}} = m_t^{DRED}(\mu)\left[1 + \frac{\alpha_3^{DRED}(\mu)}{3\pi}\left(5 - 3\ln\frac{m_t^2}{\mu^2}\right)\right]$$

whereas in DREG:

$$m_t^{\mathrm{pole}} = m_t^{DREG}(\mu)\left[1 + \frac{\alpha_3^{DREG}(\mu)}{3\pi}\left(4 - 3\ln\frac{m_t^2}{\mu^2}\right)\right]$$

from which we can deduce that

$$m_t^{DREG}(\mu) = m_t^{DRED}(\mu)\left[1 + \frac{\alpha_3}{3\pi}\right]$$

The DRED SQCD β-function

$$
\begin{aligned}
16\pi^2\beta_g^{(1)} &= \left(N_f - 3N_c\right) g^3, \\
(16\pi^2)^2\beta_g^{(2)} &= \left(\left[4N_c - \tfrac{2}{N_c}\right] N_f - 6N_c^2\right) g^5, \\
(16\pi^2)^3\beta_g^{(3)} &= (\left[\tfrac{3}{N_c} - 4N_c\right] N_f^2 \\
&+ \left[21N_c^2 - \tfrac{2}{N_c^2} - 9\right] N_f - 21N_c^3) g^7, \\
(16\pi^2)^4\beta_g^{(4)} &= \left(-\tfrac{2}{3N_c}N_f^3 + \left[132N_c^3 - 66N_c - \tfrac{8}{N_c} - \tfrac{4}{N_c^3}\right] N_f\right. \\
&+ \left[44 + \tfrac{36\zeta(3)-20}{3N_c^2} - (42 + 12\zeta(3))N_c^2\right] N_f^2 \\
&- \left. 102N_c^4\right) g^9.
\end{aligned}
$$

Higher Order Invariants

Note that the higher order group theory invariants of the form $(\mathrm{Tr} F^a F^b F^c F^d + \cdots)^2$ and $(\mathrm{Tr} R^a R^b R^c R^d + \cdots)^2$ found in the 4 loop QCD calculation (using DREG or DRED) do not appear here; and indeed they cancel in those calculations when the fermion representation R^a is replaced by the adjoint, F^a.

It is possible that β_g^{DRED} for SQCD is free of such structures to all orders (manifestly so for β_g^{NSVZ} in the absence of chiral superfields, of course).

These new terms in QCD cannot be removed by analytic coupling constant redefinitions; it follows that the DRED$\leftrightarrow$DREG$\leftrightarrow$NSVZ linkage does not extend to the QCD β-function ansatz of Ryttov and Sannino.

DRED and soft breaking

In DRED the ϵ-scalar mass mixes with the physical masses of genuine particles under renormalisation:

$$\begin{aligned} \beta_{\tilde{m}^2} &= A(g,Y)\tilde{m}^2 + \sum_i B_i(g,Y)m_i^2 + \cdots, \\ \beta_{m_i^2} &= C_i(g,Y)m_i^2 + D_i\tilde{m}^2 + \cdots, \end{aligned}$$

where the $+\cdots$ denotes terms involving gaugino masses and A-parameters.
By an analytic redefinition of the form

$$m_i^2|_{\mathrm{DRED}'} = m_i^2|_{\mathrm{DRED}} - C_i(g)\tilde{m}^2 + \cdots$$

we can make $\beta_{m_i^2}$ is independent of $\tilde{m}^2$.

The Soft β-functions

Using DRED$'$, we can prove that:

$$\begin{aligned}
\beta_h^{ijk} &= \gamma_l^{(i} h^{jk)l} - 2\Gamma_l^{(i} Y^{jk)l} \\
\beta_b^{ij} &= \gamma_l^{(i} b^{j)l} - 2\Gamma_l^{(i} \mu^{j)l} \\
\beta_M &= 2\mathcal{O}\left(\frac{\beta_g}{g}\right)
\end{aligned}$$

where

$$\mathcal{O} = \left(Mg^2\frac{\partial}{\partial g^2} - h\frac{\partial}{\partial Y}\right), \quad (\Gamma)^i{}_j = \mathcal{O}\gamma^i{}_j$$

The soft scalar mass β-function

$$\beta_{m^2} = \left[2\mathcal{O}\mathcal{O}^* + 2|M|^2 g^2 \frac{\partial}{\partial g^2} + \left(\tilde{Y}\frac{\partial}{\partial Y} + cc\right) + X\frac{\partial}{\partial g}\right]\gamma$$

where $Y_{lmn} = (Y^{lmn})^*$, and $\tilde{Y}^{ijk} = Y^{l(jk}(m^2)^{i)}{}_l$
Here

$$X_{\text{NSVZ}} = -2\frac{g^3}{16\pi^2}\frac{r^{-1}\mathrm{tr}[m^2 C(R)] - MM^* C(G)}{1 - 2C(G)g^2(16\pi^2)^{-1}}.$$

X_{NSVZ} is known through three loops.

The RS anzatz for QCD

$$\beta_g^{RS} = \frac{g^3}{16\pi^2}\left[\frac{Q - \frac{2}{3}\gamma_m N_f T(R)}{1 - 2C(G)g^2(16\pi^2)^{-1}\left(1 + 2\frac{N_f T(R) - C(G)}{Q}\right)}\right]$$

where γ_m is the fermion mass anomalous dimension. In the special case of a single fermion adjoint multiplet (corresponding to $N = 1$ susy) they equate this to

$$\beta_g^{NSVZ} = -3\frac{g^3 C(G)}{16\pi^2}/(1 - 2C(G)g^2(16\pi^2)^{-1})$$

to deduce that then

$$\gamma_m = -6\frac{g^2 C(G)}{16\pi^2}/(1 - 2C(G)g^2(16\pi^2)^{-1})$$

But in this case the γ_m is the β-function for the gaugino mass; a soft breaking term. Consequently it is given in the NSVZ scheme by the formula

$$\beta_M = 2\mathcal{O}\left(\frac{\beta_g}{g}\right),$$

i.e.

$$\gamma_m^{NSVZ} = -6\frac{g^2C(G)}{16\pi^2}\frac{1}{(1-2C(G)g^2(16\pi^2)^{-1})^2}$$

So: I can calculate in the DRED and DREG schemes, and relate the results to each other and to the NSVZ scheme; but I don't know how to calculate in the RS scheme.

Summary

- In supersymmetric theories use DRED.
- In softly-broken supersymmetric theories use DRED’.
- In non-supersymmetric theories use DREG.