

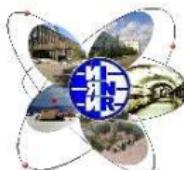
# Z decay in to hadrons and DIS sum rules at $\mathcal{O}(\alpha_s^4)$

Konstantin Chetyrkin, KIT, Karlsruhe & INR, Moscow

in collaboration with

P. Baikov (MSU), J. Kühn (KIT) and J. Rittinger (KIT)

**Loops & Legs 2012, Wernigerode**



# Final results of a large Karlsruhe based project:

- more than 10 years of work:
- + many talks on RADCORS's and LOOPS& LEGS, starting from  
“Five-loop vacuum polarization in pQCD:  $O(\alpha_s^4 N_f^2)$  results”  
( Talk presented at RADCOR/Loops and Legs 2002, Kloster Banz,  
Germany)
- + ...
- + ...
- + recent works Phys.Rev.Lett.101:012002,2008; arXiv:0801.1821  
Phys.Rev.Lett.104:132004,2010; arXiv:1001.3606v1  
Nucl.Phys.B837:186-220,2010; arXiv:1004.1153  
Nucl.Phys.Proc.Supp.205-206:237-241,2010;  
arXiv:1201.5804v2  
P. Baikov, K. Ch., J. Kühn and J. Rittinger, in preparation

# New Results to report

- (**new!**) result for the singlet contribution into the (massless)  $\langle VV \rangle$  corelator  $\implies$  final complete  $\mathcal{O}(\alpha_s^4)$  result for the R(s)
- (**new!**) as spin-off: the QED  $\beta$ -function in five loops
- (**new!**) final complete result for  $Z \rightarrow$  hadrons at  $\mathcal{O}(\alpha_s^4)$  (the latter including in full power-not-suppressed top-mass dependence)
- (**new!**):  
finishing the DIS sum rules: Ellis-Jaffe sum-rule at  $\mathcal{O}(\alpha_s^4)$

(the last from 3 classical: Bjorken and Gross-Llewellyn Smith computed to the same order in 2009-2010)

# Vector and Axial Contributions to Z-decay

Z current to fermion  $i$ :

$$j_{Z_i}^\mu = g_i^V j_{V_i}^\mu + g_i^A j_{A_i}^\mu = g_i^V \overbrace{\overbrace{\bar{\psi}_i \gamma^\mu \psi_i}^{=V_i}} + g_i^A \overbrace{\overbrace{\bar{\psi}_i \gamma^\mu \gamma_5 \psi_i}^{=A_i}}$$

$$\text{QCD}_{\mu}^Z = \sum_i (g_i^V)^2 \text{VV loop diagram} + \sum_{i,j} g_i^V g_j^V \text{VA loop diagram}$$

$$+ \sum_i (g_i^A)^2 \text{AA loop diagram} + \sum_{i,j} g_i^A g_j^A \text{VA loop diagram}$$

$$i, j = \{t, b, c, s, u, d\}$$

- 2 scales:  $\sqrt{s} = M_Z$  and  $m_t$  (but  $M_Z^2/(4m_t^2) \ll 1$ ):
- Decoupling of top (transition to the effective  $n_f = 5$  massless QCD) works only for VV correlator and for non-singlet part of AA one
- singlet AA diagrams displays *power not-suppressed* effects due to mass of the top quark (**B. Kniehl and J. Kühn, 1990**); which must be taken into account!

• VV-correlator (in the massless limit) •

$$R^{NS} = 3 \sum_i Q_i^2 \left( 1 + \frac{\alpha_s}{\pi} + \# \left( \frac{\alpha_s}{\pi} \right)^2 + \# \left( \frac{\alpha_s}{\pi} \right)^3 + \# \left( \frac{\alpha_s}{\pi} \right)^4 + \dots \right)$$

parton  
model

QED  
Källen+  
Sabry  
1955

K. Ch., Kataev, Tkachov  
Dine, Sapirstein;  
Celmaster  
1979

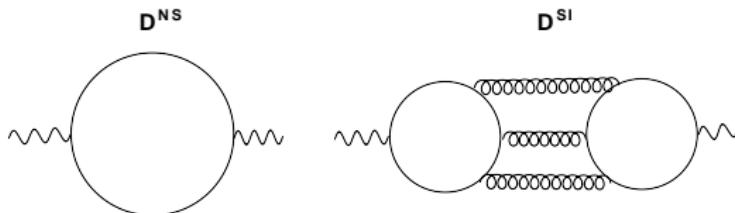
Gorishny, Kataev, Larin;  
Surguladze, Samuel 1991  
K. Ch. /gen. gauge/ 1996

Baikov, K. Ch., Kühn 2008  
Baikov, K. Ch., Kühn 2010  
(Feynman Gauge only)

$$R^{SI} = \left( \sum_i Q_i \right)^2 \left( \# \left( \frac{\alpha_s}{\pi} \right)^3 + \textcolor{red}{??} \left( \frac{\alpha_s}{\pi} \right)^4 + \dots \right)$$



Singlet and non-singlet contributions to the Adler function:



# Singlet contribution to the (vector) Adler function are ready (Last missing term!)

$$D^{SI}(Q^2) = d_R \left( \sum_{i=3}^{\infty} d_i^{SI} a_s^i(Q^2) \right)$$

$$d_3^{SI} = \frac{d^{abc} d^{abc}}{d_R} \left( \frac{11}{192} - \frac{1}{8} \zeta_3 \right), \quad d_4^{SI} = \frac{d^{abc} d^{abc}}{d_R} \left( C_F d_{4,1}^{SI} + C_A d_{4,2}^{SI} + T n_f d_{4,3}^{SI} \right)$$

$$d_{4,1}^{SI} = -\frac{13}{64} - \frac{\zeta_3}{4} + \frac{5\zeta_5}{8}, \quad d_{4,3}^{SI} = \frac{-149}{576} + \frac{13}{32}\zeta_3 - \frac{5}{16}\zeta_5 + \frac{1}{8}\zeta_3^2$$

$$d_{4,2}^{SI} = \frac{3893}{4608} - \frac{169}{128}\zeta_3 + \frac{45}{64}\zeta_5 - \frac{11}{32}\zeta_3^2$$

## Non-trivial check with the Crewther relation

from Conclusion of my talk at Loops and Legs 2010:

- CF  $C^{GLS}(a_s)$  of the Gross-Llewellyn Smith sum rule has been analytically evaluated for a generic gauge group at  $\mathcal{O}(\alpha_s^4)$
- At order  $\mathcal{O}(\alpha_s^4)$  the singlet Adler function  $D^{SI}$  is fixed by the corresponding Crewther relation up to only **one** still unknown constant:

our new result:

$$d_{4,1}^{SI} = -\frac{13}{64} - \frac{\zeta_3}{4} + \frac{5\zeta_5}{8}, \quad d_{4,3}^{SI} = \frac{-149}{576} + \frac{13}{32}\zeta_3 - \frac{5}{16}\zeta_5 + \frac{1}{8}\zeta_3^2$$
$$d_{4,2}^{SI} = \frac{3893}{4608} - \frac{169}{128}\zeta_3 + \frac{45}{64}\zeta_5 - \frac{11}{32}\zeta_3^2$$

prediction of 2010

$$d_{4,1}^{SI} = -\frac{13}{64} - \frac{\zeta_3}{4} + \frac{5\zeta_5}{8}, \quad d_{4,2}^{SI} = -c_{4,2}^{SI} + \frac{11}{12}K_{3,1}^{SI}, \quad d_{4,3}^{SI} = -c_{4,3}^{SI} + \frac{1}{3}K_{3,1}^{SI}$$

→ full agreement! ←

## Phenomenological implications for $\sigma_{tot}(e^+e^- \rightarrow \text{hadrons})$

Numerically the new found term is quite small (but it helps significantly improve the  $\mu$ -dependence of the singlet contribution, we will see below):

$$\begin{aligned} R(s) &= 3 \sum_f Q_f^2 \left\{ 1 + a_s + a_s^2 (1.986 - 0.1153 n_f) \right. \\ &+ a_s^3 (-6.637 - 1.200 n_f - 0.00518 n_f^2) \left. \right\} \\ &- \left( \sum_f Q_f \right)^2 \left( 1.2395 a_s^3 + \frac{(-17.8277 + 0.57489 n_f) a_s^4}{a_s^4} \right) \end{aligned}$$

for  $n_f=5$

$$\frac{11}{3} [1 + a_s + a_s^2 1.409 - 12.767 a_s^3 - 79.98 a_s^4] + \frac{1}{9} [-1.240 a_s^3 - 14.95 a_s^4]$$

**Extra suppression factor**  $\frac{3}{99} \approx 0.03!$

# QED $\beta$ -function in five loops

By a proper change of color factors we arrive at the **full** Adler function  
of QED in five loops  $\Rightarrow$  the QED  $\beta$ -function;  
for a QED with one charged fermion we get ( $A \equiv \frac{e^2}{16\pi^2}$ )

$$\beta^{QED} = \frac{4}{3}A + 4A^2 - \frac{62}{9}A^3 - A^4 \left( \frac{5570}{243} + \frac{832}{9}\zeta_3 \right)$$

**Gorishny, Kataev,  
Larin, Surguladze, 1991**

$$-A^5 \left( \frac{195067}{486} + \frac{800}{3}\zeta_3 + \frac{416}{3}\zeta_4 - \frac{6880}{3}\zeta_5 \right)$$

Numerically ( $A = \frac{\alpha}{4\pi} \approx 5.81 \cdot 10^{-4}$ )

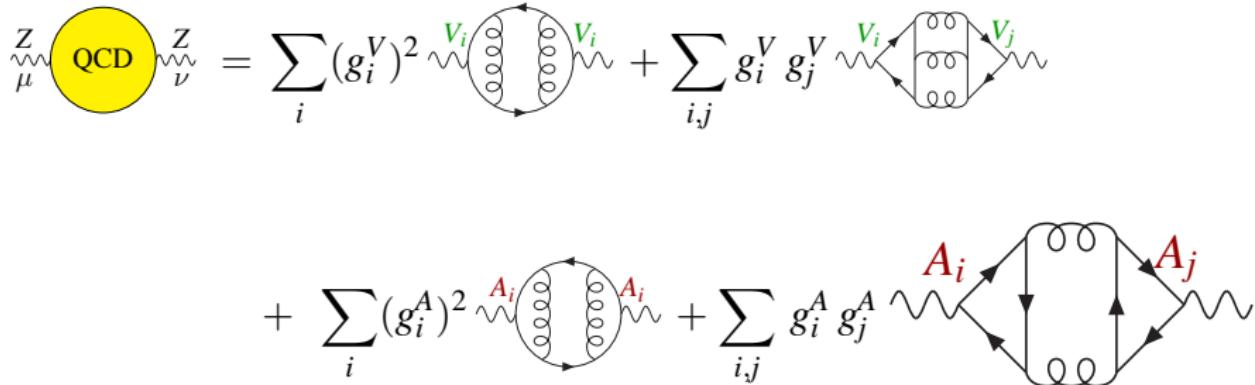
$$\beta^{QED} = \frac{4}{3}A (1 + 3A - 5.1667A^2 - 100.534A^3 + 1129.51A^4)$$

No hope for a non-trivial fixed point solution  $\beta(A^*) = 0$ !

# Tool-box for massless correlators at $\alpha_s^4$ :

- IRR / Vladimirov, (78)/ + IR  $R^*$  -operation /Chetyrkin, Smirnov (1984)/ + resolved combinatorics /Chetyrkin, (1997)/
- reduction to Masters: direct construction of CF's through  $1/D$  expansion within the Baikov's representation for Feynman integrals (Phys. Lett. B385 (1996) 403; B474 (2000) 385;  
Nucl.Phys.Proc.Supp.116:378-381,2003)
- algebraic /Baikov, Chetyrkin (2010); R. Lee, A. Smirnov, V. Smirnov, (2011)/ and numerical /A. Smirnov, M. Tentyukov (2010)/ evaluation of all masters (the former with two absolutely different methods!)
- computing: MPI-based (PARFORM) as well as thread-based (TFORM) versions of FORM  
Vermaseren, Retey, Fliegner, Tentyukov, ... (2000 – . . . )

# Axial Singlet Contributions to the Z-decay



For axial singlet contribution the naive (that is fully anticommuting  $\gamma_5$ ) is not applicable; we use 't Hooft-Veltman-Larin treatment: define the axial vector current as:

$$A_i = \bar{\psi}_i \gamma^\mu \gamma_5 \psi_i = \zeta_A \frac{i}{6} \epsilon^{\mu\mu_1\mu_2\mu_3} \bar{\psi}_i \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} \psi_i$$

$\zeta_A$  effectively restores antisymmetry of  $\gamma_5$  in NS diagrams; it is fixed **uniquely** from the (non-anomalous) Ward identity

# Decoupling of Top from vector and axial currents

/Collins, Wilczek and Zee '78 Chetyrkin, Kühn '93/

Vector current: naive decoupling (due to the Ward Identity)

$$V_t^{(6)} \stackrel{m_t \rightarrow \infty}{=} 0 + \mathcal{O}(1/m_t^2), \quad V_b^{(6)} \stackrel{m_t \rightarrow \infty}{=} V_b^{(5)} + \mathcal{O}(1/m_t^2)$$

Axial vector current: **no** naive decoupling (due to potential anomaly if it would!)

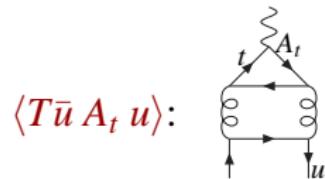
$$A_t^{(6)} \stackrel{m_t \rightarrow \infty}{=} C_h A_S^{(5)} + \mathcal{O}(1/m_t^2), \quad A_b^{(6)} \stackrel{m_t \rightarrow \infty}{=} A_b^{(5)} + C_\psi A_S^{(5)} + \mathcal{O}(1/m_t^2)$$

$$A_S^{(5)} = \sum_l A_l^{(5)}, \quad (A_t^{(6)} - A_b^{(6)}) \stackrel{m_t \rightarrow \infty}{=} (C_h - C_\psi) A_S^{(5)} + \mathcal{O}(1/m_t^2)$$

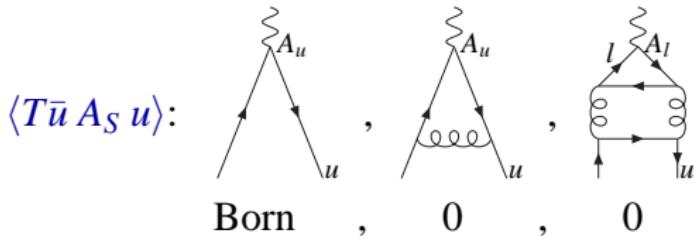
# Decoupling of Top

Calculation of  $C_h$  with method of projectors (Gorishny, Larin '86):

$$A_t^{(6)} \stackrel{m_t \rightarrow \infty}{=} C_h(a^{(6)}(\mu), \mu/m_t) A_S^{(5)}$$



$\langle T\bar{u} A_t u \rangle \Big|_{p=0}$ : TAD's



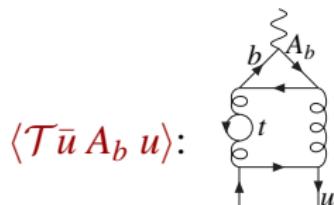
## Calculation of massive tadpoles:

- matching to topologies with EXP (Seidensticker and Steinhauser)
- reduction with Laporta alg. via CRUSHER (Marquard, Seidel)
- all master integrals up to 4 loop are known (Schröder, Vuorinen '05; other contributions: D.J. Broadhurst; S. Laporta; B.A. Kniehl and A.V. Kotikov; Y. Schröder and M. Steinhauser )

# Decoupling of Top

Calculation of  $C_\psi$ :

$$\textcolor{red}{A_b}^{(6)} \stackrel{m_t \rightarrow \infty}{=} A_b^{(5)} + C_\psi(a^{(6)}(\mu), \mu/m_t) A_S^{(5)}$$



# Decoupling of Top

$$A_t^{(6)} \stackrel{m_t \rightarrow \infty}{=} C_h A_S^{(5)}, \quad A_b^{(6)} \stackrel{m_t \rightarrow \infty}{=} A_b^{(5)} + C_\psi A_S^{(5)}.$$

$$\begin{aligned} C_h = & + \left( \frac{\alpha^{(6)}(\mu)}{\pi} \right)^2 \left[ + 0.125 - 0.5 \ln \left( \frac{\mu^2}{m_t^2} \right) \right] \\ & + \left( \frac{\alpha^{(6)}(\mu)}{\pi} \right)^3 \left[ - 0.515 - 0.417 \ln \left( \frac{\mu^2}{m_t^2} \right) - 0.875 \ln^2 \left( \frac{\mu^2}{m_t^2} \right) \right] \\ & + \left( \frac{\alpha^{(6)}(\mu)}{\pi} \right)^4 \left[ - 18.335 + 15.563 \ln \left( \frac{\mu^2}{m_t^2} \right) - 0.767 \ln^2 \left( \frac{\mu^2}{m_t^2} \right) - 1.531 \ln^3 \left( \frac{\mu^2}{m_t^2} \right) \right] \end{aligned}$$

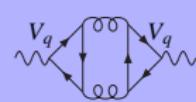
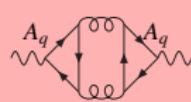
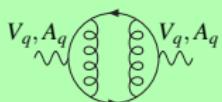
$$\begin{aligned} C_\psi = & + \left( \frac{\alpha^{(6)}(\mu)}{\pi} \right)^3 \left[ - 0.265 - 0.014 \ln \left( \frac{\mu^2}{m_t^2} \right) - 0.084 \ln^2 \left( \frac{\mu^2}{m_t^2} \right) \right] \\ & + \left( \frac{\alpha^{(6)}(\mu)}{\pi} \right)^4 \left[ - 1.282 + 0.393 \ln \left( \frac{\mu^2}{m_t^2} \right) - 0.406 \ln^2 \left( \frac{\mu^2}{m_t^2} \right) - 0.132 \ln^3 \left( \frac{\mu^2}{m_t^2} \right) \right] \end{aligned}$$

- 2 loop Collins, Wilczek and Zee '78
- 3 loop log enhanced terms Chetyrkin, Kühn '93
- 3 loop Larin, Ritbergen, Vermaseren '93; Chetyrkin, Tarasov '94
- 4 loop is new

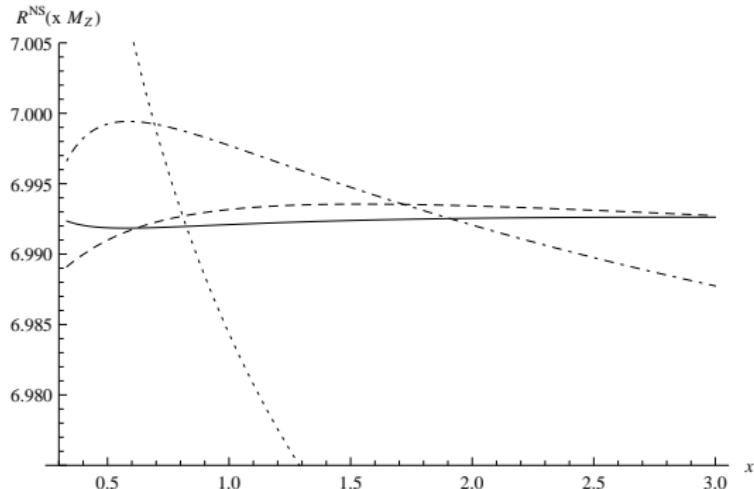
# Results

$$R = \sum_q (g_q^V)^2 R^{V,NS} + \left( \sum_q g_q^V \right)^2 R^{V,S} \\ + 5 R^{A,NS} + (C_h - C_\psi)^2 5(R^{A,NS} + 5R^{A,S}) \\ - 2(C_h - C_\psi)(R^{A,NS} + 5R^{A,S}) + R^{A,S}$$

$$R = + 6.7315 + 6.7315 \frac{\alpha_s}{\pi} \\ ( + 9.4862 - 4.3525 + 0 ) \left( \frac{\alpha_s}{\pi} \right)^2 \\ ( - 85.9415 - 17.6245 - 0.7069 ) \left( \frac{\alpha_s}{\pi} \right)^3 \\ ( - 538.39 + 87.7962 - 8.5271 ) \left( \frac{\alpha_s}{\pi} \right)^4$$

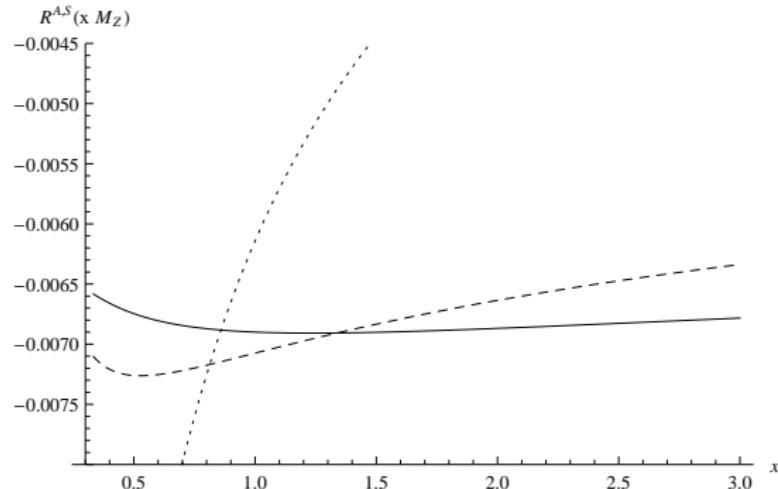


# Scale Dependence - Non Singlet



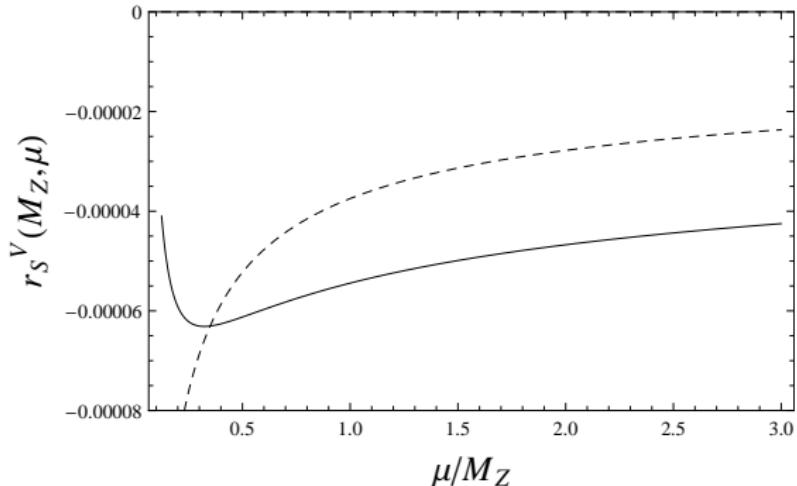
$$R = + 6.7315 + 6.7315 \frac{\alpha_s}{\pi} \left( + 9.4862 - 4.3525 + 0 \right) \left( \frac{\alpha_s}{\pi} \right)^2 \\ \left( - 85.9415 - 17.6245 - 0.7069 \right) \left( \frac{\alpha_s}{\pi} \right)^3 \\ \left( - 538.39 + 87.7962 - 8.5271 \right) \left( \frac{\alpha_s}{\pi} \right)^4$$

# Scale Dependence - Axial Vector Singlet



$$R = + 6.7315 + 6.7315 \frac{\alpha_s}{\pi} \left( + 9.4862 - 85.9415 - 538.39 \right) \left( - 4.3525 - 17.6245 + 87.7962 \right) + 0 - 0.7069 - 8.5271 \left( \frac{\alpha_s}{\pi} \right)^2 \left( \frac{\alpha_s}{\pi} \right)^3 \left( \frac{\alpha_s}{\pi} \right)^4$$

# Scale Dependence - Vector Singlet



$$\begin{aligned}
 R = & + 6.7315 \\
 & + 6.7315 \frac{\alpha_s}{\pi} \\
 & \left( + 9.4862 \right. & - 4.3525 & + 0 & ) \left( \frac{\alpha_s}{\pi} \right)^2 \\
 & \left( - 85.9415 \right. & - 17.6245 & - 0.7069 & ) \left( \frac{\alpha_s}{\pi} \right)^3 \\
 & \left( - 538.39 \right. & + 87.7962 & - 8.5271 & ) \left( \frac{\alpha_s}{\pi} \right)^4
 \end{aligned}$$

# Impact on $\alpha_s$

OLD Calculation: 3 loop QCD

$$\Gamma_Z^{3\text{loop}} = \Gamma_0 \left( R_{QCD}^{3\text{loop}}(\alpha_s) + R_{EW}(\alpha, \alpha \cdot \alpha_s) + R_{mass}(m_b^2/M_Z^2) \right)$$

Fitting to experimental value

$$\Gamma_Z^{3\text{loop}} = \Gamma_Z^{\text{exp}} \quad \Rightarrow \quad \alpha_s^{\text{old}}(M_Z) = 0.1185 (\pm 0.0026)$$

NEW Calculation: 4 loop QCD

$\alpha_s$  dependence of  $R_{EW}$  and  $R_{mass}$  is strongly suppressed and therefore can be treated as constant.

$$R_{QCD}^{3\text{loop}}(\alpha_s^{\text{old}}) + \text{const.} = \Gamma_Z^{\text{exp}}/\Gamma_0 = R_{QCD}^{4\text{loop}}(\alpha_s^{\text{new}}) + \text{const.}$$

$$\Rightarrow \alpha_s^{\text{new}}(M_Z) = (0.1185 + \underbrace{0.0005}_{-} - \underbrace{0.0001}_{+}) = 0.1189$$



# Ellis-Jaffe sum rule

- Polarized Deep-inelastic lepton-hadron scattering ( $e^\pm p$ ,  $e^\pm n$ ,  $\nu p$ ,  $\bar{\nu} p$ , ... - collisions)



$$\begin{aligned}
 W_{\mu\nu} &= \frac{1}{4\pi} \int d^4z e^{iqz} \langle p, s | J_\mu(z) J_\nu(0) | p, s \rangle \\
 &= \left( -g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) F_1(x, Q^2) + \left( p_\mu - \frac{p \cdot q}{q^2} q_\mu \right) \left( p_\nu - \frac{p \cdot q}{q^2} q_\nu \right) \frac{1}{p \cdot q} F_2(x, Q^2) \\
 &\quad + i\epsilon_{\mu\nu\rho\sigma} q_\rho \left( \frac{s_\sigma}{p \cdot q} \boxed{g_1(x, Q^2)} + \frac{s_\sigma p \cdot q - p_\sigma q \cdot s}{(p \cdot q)^2} g_2(x, Q^2) \right)
 \end{aligned}$$

$$J_\mu = \sum_{i=1}^{n_f} e_i \bar{\psi}_i \gamma_\mu \psi_i \equiv \bar{\psi} E \gamma_\mu \psi \quad x = Q^2/(2p \cdot q) \text{ and } Q^2 = -q^2$$

OPE of 2 EM currents:

$$i \int dz e^{iqz} T\{J_\mu(z) J_\nu(0)\} \stackrel{Q^2 \rightarrow \infty}{=} \epsilon_{\mu\nu\rho\sigma} \frac{q_\rho}{q^2} \left[ C^{NS}(L, a_s(\mu)) \sum_a C^a J_\sigma^{5,a}(0) + C^s(L, a_s(\mu)) J_\sigma^5(0) \right] + \dots \text{(higher twists)}$$

$$a_s = \alpha_s/\pi, \quad L = \log(\frac{\mu^2}{Q^2}), \quad \text{flavour CF: } C^a = \text{Tr}(E^2 \cdot T^a)$$

non-singlet axial current:  $J_\sigma^{5,a}(x) = \bar{\psi} \gamma_\sigma \gamma_5 t^a \psi(x)$

singlet axial current:  $J_\sigma^5(x) = \sum_{i=1}^{n_f} \bar{\psi}_i \gamma_\sigma \gamma_5 \psi_i(x)$

OPE of 2 EM currents:

$$i \int dz e^{iqz} T\{J_\mu(z) J_\nu(0)\} \stackrel{Q^2 \rightarrow \infty}{=} \epsilon_{\mu\nu\rho\sigma} \frac{q\rho}{q^2} \left[ C^{NS} \sum_a C^a J_\sigma^{5,a}(0) + C^s J_\sigma^5(0) \right] + \dots \text{(higher twists)}$$

The OPE results to Ellis-Jaffe sum-rule:

$$\int_0^1 dx g_1^{p(n)}(x, Q^2) = C^{\text{ns}}(L, a_s(\mu)) (\pm \frac{1}{12}|g_A| + \frac{1}{36}a_8) + C^{\text{s}}(L, a_s(\mu)) \frac{1}{9}a_0(\mu^2)$$

$$\begin{aligned} |g_A| s_\sigma &= 2\langle p, s | J_\sigma^{5,3} | p, s \rangle = (\Delta u - \Delta d) s_\sigma, \quad / \Delta u \equiv \bar{\psi}_u \gamma_\sigma \gamma_5 \psi_u(x), \text{ etc.} / \\ a_8 s_\sigma &= 2\sqrt{3} \langle p, s | J_\sigma^{5,8} | p, s \rangle = (\Delta u + \Delta d - 2\Delta s) s_\sigma, \\ a_0(\mu^2) s_\sigma &= \langle p, s | J_\sigma^5 | p, s \rangle = (\Delta u + \Delta d + \Delta s) s_\sigma = \Delta \Sigma(\mu^2) s_\sigma. \end{aligned}$$

$g_A = 1.270 \pm 0.003$  from neutron beta decays

$a_8 = 0.58 \pm 0.03$  from hyperon beta decays (assuming SU(3))

$a_0 = 0.33 \pm 0.03$  from EJ sum-rule (COMPASS Collab., Phys. Lett. B647 (2007) 8)

(in the naive parton model  $a_0$  interpreted as the fraction of the proton's spin carried by its quarks (and antiquarks) and was assumed to be between one and a half!

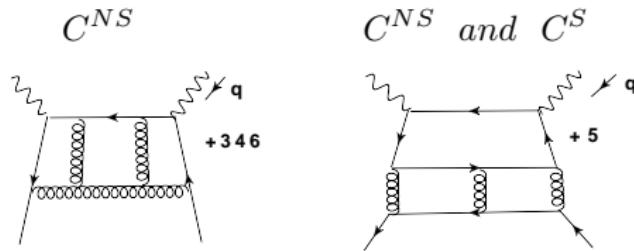
⇒ "proton spin crisis"!

$$\int_0^1 dx g_1^{p(n)}(x, Q^2) = C^{\text{ns}}(L, a_s(\mu)) (\pm \frac{1}{12} |g_A| + \frac{1}{36} a_8) + C^{\text{s}}(L, a_s(\mu)) \frac{1}{9} a_0(\mu^2)$$

If one consider a difference  $g_1^p - g_1^n$  then only first term survives:  
 $\Rightarrow$  Bjorken sum-rule

$$\int_0^1 dx g_1^{(p-n)}(x, Q^2) = C^{\text{ns}}(L, a_s(\mu)) \frac{1}{6} |g_A|$$

Typical diagrams at  $\alpha_s^3$  (known from [/Larin, van Ritbergen, Vermaseren /1997/](#) ).



$C^{\text{NS}}$  contributes to the Bjorken sum rule and has been recently computed at order  $\alpha_s^4$  in the FSB framework ([P. Baikov, K.Ch. J.Kühn, PLR 104 \(2010\) 132004](#)).

After the RG improvement: the choice  $\mu = Q$  (the *non-singlet* axial vector current  $J_\sigma^{5,3}$  is scale invariant<sup>1</sup>)

$$C^{\text{ns}}(0, a_s(Q))|_{n_f=3} = 1 - a_s(Q) - 3.583 a_s^2(Q) - 20.215 a_s^3(Q)$$
$$\quad \quad \quad - 175.75 a_s^4(Q)$$

---

<sup>1</sup> Better to say it could (and, in fact, must) renormalized in such a way to be scale-invariant

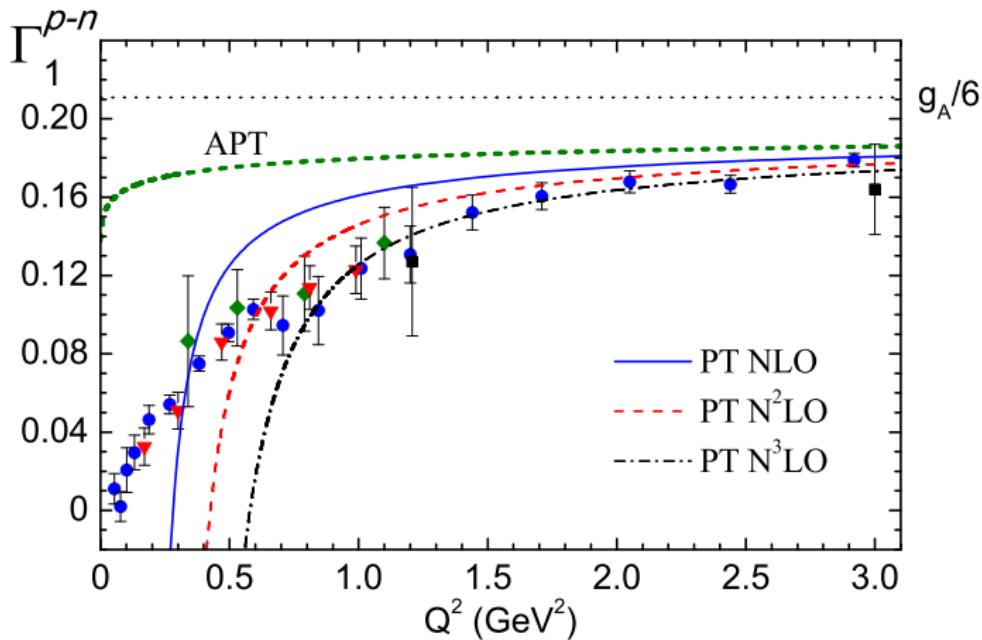


Figure 1: Perturbative part of the BSR as a function of the momentum transfer squared  $Q^2$  in different orders in both the APT and standard PT approaches against the combined set of the Jefferson Lab (taken from V.L. Khandramai, R.S. Pasechnik, D.V. Shirkov, O.P. Solovtsova, O.V. Teryaev, *Four-loop QCD analysis of the Bjorken sum rule vs data*, Phys.Lett.B706:340-344,2012).

OPE of 2 EM currents:

$$i \int dz e^{iqz} T\{J_\mu(z) J_\nu(0)\} \stackrel{Q^2 \rightarrow \infty}{=} \epsilon_{\mu\nu\rho\sigma} \frac{q_\rho}{q^2} \left[ C^{NS}(L, a_s(\mu)) \sum_a C^a J_\sigma^{5,a}(0) + C^s(L, a_s(\mu)) J_\sigma^5(0) \right] + \dots (\text{higher twists})$$

$$a_s = \alpha_s/\pi, \quad L = \log\left(\frac{\mu^2}{Q^2}\right), \quad \text{flavour CF: } C^a = \text{Tr}(E^2 \cdot T^a)$$

$$\text{non-singlet axial current: } J_\sigma^{5,a}(x) = \bar{\psi} \gamma_\sigma \gamma_5 t^a \psi(x)$$

$$\text{singlet axial current: } J_\sigma^5(x) = \sum_{i=1}^{n_f} \bar{\psi}_i \gamma_\sigma \gamma_5 \psi_i(x)$$

Note that within DR in OPE of a T-product of the *vector currents* there is no place for  $\gamma_5$ !

In reality one encounters on the rhs of the above OPE objects like

$$\bar{\psi} \gamma^{[\mu\nu\rho]} \psi q_\rho \quad \text{which one could identify with}$$

$$\epsilon_{\mu\nu\rho\sigma} q_\rho \bar{\psi} \gamma_\sigma \gamma_5 \psi(x) \quad \text{in} \quad D = 4 \text{ only! / we will employ}$$

Larin's formalism, see later/

## Renormalization of Axial Vector Current

I. Larin '93: treat  $\epsilon^{\mu\mu_1\mu_2\mu_3}$  4 dimensional and separate from the  $d$ -dimensional amplitude:

$$J_\sigma^5(x) = \bar{\psi}_i \gamma^\sigma \gamma_5 \psi_i = \frac{i}{6} \epsilon^{\sigma\mu_1\mu_2\mu_3} [\bar{\psi}_i \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} \psi_i]_{\overline{MS}}$$



II. Choose the (finite renorm.) factor  $\zeta_S = 1 + O(a_s)$  so that to nullify the anom. dim. of the combination  $\hat{J}_\sigma^5 = \zeta_S J_\sigma^5(x)$ , explicitly ( $\gamma^s(a'_s)$  is the anom. dim of the  $J_\sigma^5$ , which starts from order  $\alpha_s^2$ ):

$$\zeta_S = \exp \left( - \int^{a_s(\mu^2)} da'_s \frac{\gamma^s(a'_s)}{\beta(a'_s)} \right)$$

Important:  $J_\sigma^5$  is scale invariant **and independent** from any scheme object

$\Rightarrow$  the CF  $\hat{C}^S = C_{\overline{MS}}^S / \zeta_S$  will share the same good properties

The Ellis-Jaffe sum-rule after RG-improvement:

$$\int_0^1 dx g_1^{p(n)}(x, Q^2) = C^{\text{NS}}(0, a_s(Q^2))(\pm \frac{1}{12}|g_A| + \frac{1}{36}a_8) + \hat{C}^{\text{S}}(0, a_s(Q^2))\frac{1}{9}\hat{a}_0$$

$$\hat{a}_0 = \exp \left( - \int^{a_s(\mu^2)} da'_s \frac{\gamma^s(a'_s)}{\beta(a'_s)} \right) a_0(\mu^2)$$

We have analytically computed

$C_{MS}^S$  at  $\mathcal{O}(\alpha_s^4)$  (4 loops) and  $\gamma^s$  at  $\mathcal{O}(\alpha_s^5)$  (5 loops!)

Our results (in numerical form and for  $n_f = 3$ ; /preliminary!/) )

$$C_{\overline{MS}}^S(0, a_s(Q)) = 1. - 2.333 a_s + 0.14023 a_s^2 + 4.79185 a_s^3 - 29.791 a_s^4$$

$$\gamma^s = -4.5 a_s^2 - 16.896 a_s^3 + 1.375 a_s^4 - 49.01 a_s^5$$

And, finally,

$$\hat{C}^S(0, a_s(Q)) = 1. - 0.3333 a_s - 0.5496 a_s^2 - 4.4473 a_s^3 - \textcolor{blue}{36.814 a_s^4}$$

cmp. to the non-singlet case:

$$C^{\text{ns}}(0, a_s(Q))|_{n_f=3} = 1 - a_s - 3.583 a_s^2 - 20.215 a_s^3 - \textcolor{red}{175.75 a_s^4}$$

## Phenomenological Implications

COMPASS Collaboration (2007) have measured the *deuteron* structure function  $g_1$ :

$$\Gamma_1^N \equiv \int_0^1 \frac{dx}{2} (g_1^p + g_1^n)(x, Q^2 = 3 \text{ GeV}^2) = 0.050 \pm 0.007$$

Theory gives (without higher twists!):

$$\Gamma_1^N = \frac{1}{36} C^{NS}(Q^2 = 3 \text{ GeV}^2) a_8 + \frac{1}{9} \hat{C}^S(Q^2 = 3 \text{ GeV}^2) \hat{a}_0$$

Assuming  $a_8 = 0.585 \pm 0.025$  /from hyperon  $\beta$  decay/, we find:

$$\text{LO : } \hat{a}_0 = 0.332 \pm 0.03(\text{stat.}) \pm 0.02(\text{syst})$$

$$\text{NLO : } \hat{a}_0 = 0.340 \pm 0.03(\text{stat.}) \pm 0.02(\text{syst})$$

$$\text{NNLO : } \hat{a}_0 = 0.345 \pm 0.03(\text{stat.}) \pm 0.02(\text{syst})$$

$$\text{N}^3\text{LO : } \hat{a}_0 = 0.350 \pm 0.03(\text{stat.}) \pm 0.02(\text{syst})$$

which could be compared with the very recent lattice result  
(QCDSF collaboration, arXiv:1112.3354 , December 2011):

$$\hat{a}_0 = 0.376 \pm 0.08$$

# Conclusions I

- The “10 years +” project of computing  $R(s)$  and  $\Gamma(Z \rightarrow \text{hadrons})$  at order  $\alpha_s^4$  is finished!
- The last missing ingredients — singlet contributions — to the (massless) VV and AA as well as (massive)  $\langle A_t A_t \rangle$  and  $\langle A_t A_b \rangle$  correlators at  $\mathcal{O}(\alpha_s^4)$  are now available
- singlet  $\mathcal{O}(\alpha_s^4)$  contributions are numerically tiny
- the net effects of  $\mathcal{O}(\alpha_s^4)$  term in  $\Gamma_Z^h$  are: an increase of  $\delta\alpha_s(M_Z) = \mathbf{0.0004}$

$$\mathcal{O}(\alpha_s^3) : \quad \alpha_s(M_Z)^{NNLO} = \mathbf{0.1185} \pm \mathbf{0.0026}^{\text{exp}} \pm \mathbf{0.002}^{\text{th}}$$

$$\mathcal{O}(\alpha_s^4) : \quad \alpha_s(M_Z)^{NNNLO} = \mathbf{0.1189} \pm \mathbf{0.0026}^{\text{exp}} \pm \mathbf{0.0005}^{\text{th}}$$

and *four-fold* decrease of the theory error!

/ K.Ch, Baikov and Kühn, PLR 101 (2008) 012002/

## Conclusions II

- the **5-loop** QED  $\beta$ -function is computed  $\Leftarrow$  the first example of 5-loop RG function in a (normal) 4-D gauge theory: almost exactly thirty years after similar result in a  $\phi^4$  model /K.Ch, Gorishnii, Larin, and Tkachov, Phys.Lett. B132 (1983) 351/; Kleinert, Neu, Schulte-Frohlinde , K.Ch., and Larin, Phys.Lett. B272 (1991) 39-44/
- All our methods and tools are equally well applicable to evaluation of the **5-loop**  $\beta$ -functions and anomalous dimensions in general **non-Abelian** gauge theories. An interesting example could be: QCD  $\beta$ -function (important for a better understanding of the  $\tau$ -lepton decay rate within the so-called contour-improved method)

## Conclusions III

- All three classical DIS sum-rules — Bjorken, Gross-Llewellyn Smith and Ellis-Jaffe — have been extended to  $\mathcal{O}(\alpha_s^4)$  during last three years
- Exper. data for Bjorken sum-rule seem to favour the  $\mathcal{O}(\alpha_s^4)$  input
- Newly computed corrections to the (anomalous part of) Ellis-Jaffe sum-rule are quite small (unlike the situation with the the Bjorken one)
- the 5-loop anomalous dimension of the singlet axial vector current ( needed for meaningful treatment of Ellis-Jaffe sum-rule) has been computed

Last but not the least:

- all previous  $\alpha_s^3$  results for the DIS sum rules<sup>1,2,3</sup> as well as the  $\alpha_s^4$  4-loop anomalous dimension of the singlet axial vector current<sup>2</sup> have been confirmed, *without any use of tools like MINCER from the Amsterdam's group /not counting the very FORM!/-*

---

<sup>1</sup>S.A. Larin, F.V. Tkachov and J.A.M. Vermaseren , Phys.Rev.Lett. 66: 862-863, 1991;

<sup>2</sup>S.A. Larin and J.A.M. Vermaseren, Phys. Lett. B259: 345-352, 1991;

<sup>3</sup> .A. Larin, T.van Ritbergen and J.A.M. Vermaseren, Phys. Lett. B404: 153-160, 1997