Symbolic integration and multiple polylogarithms

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Wernigerode, 19.04.2012

European Research Council

Contents:

- Iterated integrals in several variables
- The 'symbol map' ψ
- Integration over a universal set of multiple polylogarithms
- Applications in Feynman integral computations

We are familiar with iterated integrals like:

$$\begin{aligned} I\,(a_w,\, a_{w-1},\, ...,\, a_2,\, a_1;\, x) &= \int_0^x \underbrace{f_{a_w}(x^{(w)})dx^{(w)}}_{\omega_w} ... \int_0^{x'''} \underbrace{f_{a_2}(x'')dx''}_{\omega_2} \int_0^{x''} \underbrace{f_{a_1}(x')dx'}_{\omega_1} \\ &\equiv \int_0^x \omega_w ... \omega_2 \omega_1 \equiv [\omega_w|...|\omega_2|\omega_1] \quad \text{(short-hand notation)} \end{aligned}$$

The differential one-forms ω_i belong to a chosen set Ω.

Examples:

- $\Omega_{\text{Polylogs}} = \left\{ \frac{dx}{x}, \frac{dx}{1-x} \right\}$
  - **classical polylogarithms**: $\mathrm{Li}_w(x) = \underbrace{\left[\frac{dx}{x} | ... | \frac{dx}{x} | \frac{dx}{1-x} \right]}_{w \text{ times}}$,
  - **multiple polylogarithms in one variable**: $\mathrm{Li}_{n_1 n_2 ...}(x) = [...|\underbrace{\frac{dx}{x}|...|\frac{dx}{x}|\frac{dx}{1-x}}_{n_2 \text{ times}}|\underbrace{\frac{dx}{x}|...|\frac{dx}{x}|\frac{dx}{1-x}}_{n_1 \text{ times}}]$

Examples:

- $\Omega_{\text{HPL}} = \left\{ \frac{dx}{x}, \frac{dx}{1-x}, \frac{dx}{1+x} \right\}$

  Harmonic Polylogarithms (Remiddi, Vermaseren 1999),
  (implementation Maitre '05, '07)

- $\Omega_{\text{Cyclotomic}} = \left\{ \frac{dx}{x}, \frac{x^l\,dx}{\phi_k(x)} | k \in \mathbb{N}_+, 0 \le l \le \varphi(k), \phi_k(x) : \text{ cyclotomic polyn.} \right\}$

  Cyclotomic Harmonic Polylogarithms (Ablinger, Blümlein, Schneider '11),
  (implementation Ablinger)

- $\Omega_{\text{2dHPL}} = \left\{ \frac{dx}{x}, \frac{dx}{1-x}, \frac{dx}{x+z}, \frac{dx}{x+z-1} \right\}$

  Two-dimensional Harmonic Polylogarithms (Gehrmann, Remiddi '01):
  x variable, z fixed parameter

- $\Omega_{\text{Hyperlog}} = \left\{ \frac{dx}{x}, \frac{dx}{x-z_i} \right\}, \ i = 1, \ldots, n$

  Hyperlogarithms (Poincare, Kummer 1840, Lappo-Danilevsky 1911):
  x variable, $z_1, \ldots, z_n$ fixed parameters

- $\Omega_{n\text{VarPoly}} = \left\{ \frac{dx_i}{x_i}, \frac{dx_i}{1-x_i}, \frac{dx_i - dx_j}{x_i - x_j} \right\}, \ i, j = 1, \ldots, n$

  (includes) **Multiple Polylogarithms in n variables** (Goncharov '01)
  (implementation as nested sums: Weinzierl '02)

For iterated integrals involving n parameters x_1, ..., x_n, we distinguish:

- functions I_F defined on the **fiber**:

  x_1, ..., x_{n-1} are **arbitrary but fixed**, x_n is **variable**

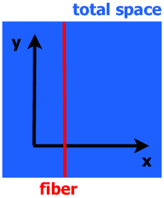
  $\Rightarrow$ only differentiate/integrate w.r.t. x_n.
  (Hyperlogarithms, 2dim. HPLs)

- functions I_T defined on the **total space**:

  all x_1, ..., x_n are **variable**

  $\Rightarrow$ differentiate/integrate w.r.t. all x_1, ..., x_n.

  (Multiple Polylogarithms of several variables)



If the total space has a fibration (i.e. Ω has good properties) then we can obtain one from the other:

$$I_T \text{ on total space} \quad \underset{\longleftarrow}{\overset{dx_1 = .. = dx_{n-1} = 0}{\longrightarrow}} \quad I_F \text{ on fiber}$$

$$\text{the 'symbol map' } \psi$$

First Example: $\ln(1-xy)$

On the **fiber** where x **is fixed** and y **is variable**: $\int_0^y \frac{x\,dy'}{1-xy'} \equiv \left[\frac{x\,dy}{1-xy}\right]$.

What is the corresponding integral $\psi\left(\left[\frac{x\,dy}{1-xy}\right]\right)$ on the total space?

- Differentiation w.r.t. y: $\frac{\partial}{\partial y}\psi\left(\left[\frac{x\,dy}{1-xy}\right]\right) = \psi\left(\frac{\partial}{\partial y}\left[\frac{x\,dy}{1-xy}\right]\right) = \frac{x}{1-xy}$
- Define an auxiliary operator ∂_x with the properties:
  a) $\frac{\partial}{\partial y}\partial_x f = \partial_x \frac{\partial}{\partial y} f$ and b) $\partial_x f(x) = \frac{\partial}{\partial x} f(x)$ for rational functions $f(x)$.
- We reconstruct $\frac{\partial}{\partial x}\psi\left(\left[\frac{x\,dy}{1-xy}\right]\right)$ from $\partial_x\left[\frac{x\,dy}{1-xy}\right] = ?$

$$\frac{\partial}{\partial y}\partial_x\left[\frac{x\,dy}{1-xy}\right] \overset{a)}{=} \partial_x \frac{\partial}{\partial y}\left[\frac{x\,dy}{1-xy}\right] = \partial_x \frac{x}{1-xy} \overset{b)}{=} \frac{1}{(1-xy)^2}$$

$$\Rightarrow \partial_x\left[\frac{x\,dy}{1-xy}\right] = \int_0^y \frac{dy'}{(1-xy')^2} = \frac{y}{1-xy} \Rightarrow \frac{\partial}{\partial x}\psi\left(\left[\frac{x\,dy}{1-xy}\right]\right) = \frac{y}{1-xy}$$

$\Rightarrow$ The total differential is $d\psi\left(\left[\frac{x\,dy}{1-xy}\right]\right) = \frac{x\,dy}{1-xy} + \frac{y\,dx}{1-xy}$

$\Rightarrow \psi\left(\left[\frac{x\,dy}{1-xy}\right]\right) = \int_0^y \frac{x\,dy'}{1-xy'} + \int_0^x \frac{y\,dx'}{1-x'y} \equiv \left[\frac{x\,dy + y\,dx}{1-xy}\right]$, with x **and** y **variable**.

Second Example: $\mathrm{Li}_{1,1}(x, y)$

On the fiber where x **is fixed** and y **is variable**: $\int_0^y \frac{dy'}{1-y'} \int_0^{y'} \frac{x\,dy''}{1-xy''} \equiv \left[\frac{dy}{1-y} \middle| \frac{x\,dy}{1-xy}\right]$.

What is the corresponding integral $\psi\left(\left[\frac{dy}{1-y} \middle| \frac{x\,dy}{1-xy}\right]\right)$ on the total space?

- $\frac{\partial}{\partial y}\psi\left(\left[\frac{dy}{1-y} \middle| \frac{x\,dy}{1-xy}\right]\right) = \frac{1}{1-y} \underbrace{\psi\left(\left[\frac{x\,dy}{1-xy}\right]\right)}_{\text{see previous example}} = \frac{1}{1-y}\left[\frac{x\,dy+y\,dx}{1-xy}\right]$
- Reconstruct $\frac{\partial}{\partial x}\psi([\ldots])$ from $\partial_x[\ldots] = ?$

$$\frac{\partial}{\partial y}\partial_x\left[\frac{dy}{1-y} \middle| \frac{x\,dy}{1-xy}\right] = \partial_x \frac{\partial}{\partial y}[\ldots] = \frac{1}{1-y} \underbrace{\partial_x\left[\frac{x\,dy}{1-xy}\right]}_{\text{see previous example}} = \frac{1}{1-y}\frac{y}{1-xy}$$

$$\Rightarrow \partial_x\left[\frac{dy}{1-y} \middle| \frac{x\,dy}{1-xy}\right] = \int_0^y \frac{y'\,dy'}{(1-y')(1-xy')} = \ldots = \frac{1}{1-x}\left[\frac{dy}{1-y}\right] - \left(\frac{1}{1-x}+\frac{1}{x}\right)\left[\frac{x\,dy}{1-xy}\right]$$

$$\Rightarrow \frac{\partial}{\partial x}\psi\left(\left[\frac{dy}{1-y} \middle| \frac{x\,dy}{1-xy}\right]\right) = \frac{1}{1-x}\psi\left(\left[\frac{dy}{1-y}\right]\right) - \left(\frac{1}{1-x}+\frac{1}{x}\right)\psi\left(\left[\frac{x\,dy}{1-xy}\right]\right)$$

$$d\psi\left(\left[\frac{dy}{1-y} \middle| \frac{x\,dy}{1-xy}\right]\right) = \frac{dy}{1-y}\psi\left(\left[\frac{x\,dy}{1-xy}\right]\right) + \frac{dx}{1-x}\psi\left(\left[\frac{dy}{1-y}\right]\right) - \left(\frac{dx}{1-x}+\frac{dx}{x}\right)\psi\left(\left[\frac{x\,dy}{1-xy}\right]\right)$$

$$\Rightarrow \psi\left(\left[\frac{dy}{1-y} \middle| \frac{x\,dy}{1-xy}\right]\right) = \left[\frac{dy}{1-y} \middle| \frac{x\,dy+y\,dx}{1-xy}\right] + \left[\frac{dx}{1-x} \middle| \frac{dy}{1-y}\right] - \left[\frac{dx}{1-x}+\frac{dx}{x} \middle| \frac{x\,dy+y\,dx}{1-xy}\right]$$

Recursive algorithm for the 'symbol map' $\psi : I_F \mapsto I_T$:

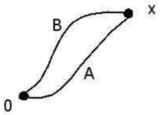
- Let I_F be an iterated integral on the fiber, where x is fixed and y is variable.
- For rational functions f we have $\psi(f) = f$.
- Define the operator ∂_x : $\partial_x I_F = \int \partial_x \left(\frac{\partial}{\partial y} I_F \right) dy$
- Compute $\frac{\partial}{\partial y} \psi(I_F) = \psi\left(\frac{\partial}{\partial y} I_F \right)$ and $\frac{\partial}{\partial x} \psi(I_F) = \psi(\partial_x I_F)$.
- $\psi(I_F) = \int d\psi(I_F) = \int dy\, \psi\left(\frac{\partial}{\partial y} I_F \right) + \int dx\, \psi(\partial_x I_F)$

Extend accordingly to several variables x_1, x_2, ...

For an equivalent construction using polygons see Duhr, Gangl, Rhodes 2011.
A (slightly different) notion of Goncharov (Goncharov 2009) found applications in N=4 SYM theory: Goncharov, Spradlin, Vergu, Volovich 2010, ...
Also see the talk by Blümlein and recent work by Duhr et al.

Remark:

Consider homotopic paths A and B with common end-point x and ω a smooth one-form.
We have $\int_A \omega = \int_B \omega$ if and only if $d\omega = 0$ (Stokes).
$\Rightarrow \int_A \omega$ is a well-defined (homotopy invariant) function of x.



Chen (1977): On iterated integrals $I = \sum_{I=(i_1,\ldots,i_l)} c_I \left[\omega_{i_1}|\omega_{i_2}|\ldots|\omega_{i_l}\right]$, with $c_I \in \mathbb{Q}$ and ω_i closed, the **analogous condition** for being a well-defined function on the total space is

$$\sum_{I=(i_1,\ldots,i_l)} c_I \omega_{i_1} \otimes \ldots \otimes \omega_{i_k} \wedge \omega_{i_{k+1}} \otimes \ldots \otimes \omega_{i_l} = 0 \text{ for each } 1 \leq k \leq l-1.$$

Integrals obtained from the 'symbol map' ψ satisfy this condition.

We obtain well-defined functions of n variables.

We define an alphabet Ω_{UPL} of differential forms on the fiber:

- 1 variable: $\Omega^{(1)}_{\mathrm{UPL}} = \{\frac{dy}{y}, \frac{dy}{1-y}\}$,

  ($=\Omega_{\mathrm{Polylogs}}$ multiple polylogarithms in one variable)

- 2 variables: $\Omega^{(2)}_{\mathrm{UPL}} = \{\frac{dy}{y}, \frac{dy}{1-y}, \frac{x_1\, dy}{1-x_1 y}\}$ (see examples above)

  (contains harmonic polylogarithms: $\Omega^{(2)}_{\mathrm{UPL}}\Big|_{x_1=-1} = \{\frac{dy}{y}, \frac{dy}{1-y}, -\frac{dy}{1+y}\} = \pm\Omega_{\mathrm{HPL}}$)

- 3 variables: $\Omega^{(3)}_{\mathrm{UPL}} = \{\frac{dy}{y}, \frac{dy}{1-y}, \frac{x_1\, dy}{1-x_1 y}, \frac{x_1 x_2\, dy}{1-x_1 x_2 y}\}$

  (contains two-dimensional harmonic polylogarithms: $\Omega^{(3)}_{\mathrm{UPL}}\Big|_{x_1=-\frac{1}{z},\, x_2=-\frac{z}{1-z}} = \pm\Omega_{\mathrm{2dHPL}}$)

- ... n variables:$\Omega^{(n)}_{\mathrm{UPL}} = \dots$

Definition:

- Let $\mathcal{B}_F$ be the iterated integrals on the fiber, obtained from $\Omega^{(n)}_{\mathrm{UPL}}$.

- $\mathcal{B}_T$ are the corresponding integrals on the total space,
  i.e. the **image of $\mathcal{B}_F$ under** ψ. (Contains multiple polylogarithms in n variables.)

Which **properties** make $\mathcal{B}_T$ a good class of iterated integrals?

- **Well defined on the total space** (homotopy invariance) Chen '77
- ψ maps complicated **functional relations** to trivial identities.
- **Closed** under taking **primitives**:

  The primitive $\int \sum dx_i f_i I_T$ (where $I_T \in \mathcal{B}_T$ and where f_i have the same denominators as $\omega \in \Omega^{(n)}_{\mathrm{UPL}}$) belongs again to $\mathcal{B}_T$.
  There is an algorithm for taking primitives. Brown '05
- **Limits at zero and one** are under control:

  One obtains combinations of $I_T \in \mathcal{B}_T$ and multiple zeta values.

  There is an algorithm for taking limits at zero and one. Brown '05

We can integrate from zero to one and the result stays in this class of functions.

Our **Maple-program**

- contains an implementation of the map ψ and the functions $\mathcal{B}_T$.
- allows to differentiate, take primitives and limits (at 0 and 1) of functions in $\mathcal{B}_T$.
- **computes integrals** of the form

$$\int_0^1 \sum_i dx_k\, f_i(x_1, \ldots, x_n)\, l_i(x_1, \ldots, x_n), \text{ with } x_k \in \{x_1, \ldots, x_n\}$$

where $l_i \in \mathcal{B}_T$ and where the f_i have the same denominators as $\omega_i \in \Omega^{(n)}_{\mathrm{UPL}}$.
The result lands in the same class of functions (with MZV prefactors)

Remark: This integration step was trivial for one-variable iterated integrals,

$$\text{e.g. } \int_0^1 \underbrace{dx\, f_i(x)}_{\in \Omega_{\mathrm{HPL}}} \mathrm{HPL}(\ldots; x) = \mathrm{HPL}(a_i, \ldots; 1) \text{ by definition.}$$

For n-variable iterated integrals, the step is not trivial (but algorithmically solved).

(Possible) Application 1: Differential Equations Method (Kotikov '91, Remiddi '97, Gehrmann, Remiddi 2000)

For each I_j of a set of IBP-master-integrals I_1, ..., I_m one derives a first order differential equation:

$$\underbrace{\frac{\partial}{\partial z} I_j + a_j I_j}_{\text{the MI we want to solve}} = \underbrace{\sum_{i \neq j} a_i I_i}_{\text{the other MIs}}$$

Usually the I_i are obtained in terms of HPLs.

$\Rightarrow$ Solving for I_j implies integrating over HPLs as functions of z.

Assume some I_i are known as iterated integrals with **several parameters** z_1, ..., z_n.

It may be advantageous to consider differential equations in several variables:

$$\begin{aligned}
\frac{\partial}{\partial z_1} I_j + a_j I_j &= \sum_{i \neq j} a_i I_i, \\
\frac{\partial}{\partial z_2} I_k + b_k I_k &= \sum_{i \neq k} b_i I_i, \\
&\dots
\end{aligned}$$

Then the $I_i(z_1, z_2, \dots)$ have to be **functions in n variables** and the (non-trivial) integration step can be done with our program, provided the a_i and b_i satisfy the mentioned conditions.

Application 2: Systematic Integration over Feynman Parameters using the universal class of multiple polylogarithms $\mathcal{B}_T$ (Brown '08)

- Express a Feynman integral with the help of **finite integrals** $\int_0^1 d^n x \ldots$.
  where x_1, ..., x_n are **Feynman parameters** and the integrand is given by the Symanzik polynomials.
  (Brown, Kreimer '11 extends the class of graphs for which this can be done)
- Use manipulations on **Symanzik polynomials** to obtain integrands whose denominator is **linear** in a Feynman parameter x_k.
- Then the program does the **integration step** over x_k from 0 to 1.
- **Iterate** for all Feynman parameters, if possible. (Conditions were studied in Brown '08, '09)

Using the above class of functions is essential for this approach:

- The iterated integrals are **functions of n variables.**
  $\Rightarrow$ We can integrate w.r.t. each Feynman parameter.
- After an integration we **stay in the same class of functions**.
  $\Rightarrow$ The iteration can be automated.

Conclusions:

- From iterated integrals on the **fiber** ($n-1$ fixed parameters, 1 variable) we obtain **well-defined** integrals on the **total space** (n variables) via the **'symbol map'** ψ.
- Using ψ and an appropriate set of differential forms $\Omega^{(n)}_{\mathrm{UPL}}$ we obtain a **universal class of polylogarithm functions** $\mathcal{B}_T$.
- $\mathcal{B}_T$ is closed under taking **primitives**. **Limits** at zero vanish, limits at one are multiple zeta values.
- Our **program computes definite integrals** from zero to one over linear combinations in $\mathcal{B}_T$.
- We plan to apply this integration step to systematic Feynman integral computations.

A well known **functional equation** is the five-term-relation:

$$-\mathrm{Li}_2\left(\frac{1-y}{1-\frac{1}{x}}\right)-\mathrm{Li}_2\left(\frac{1-x}{1-\frac{1}{y}}\right)+\mathrm{Li}_2(xy)-\mathrm{Li}_2(x)-\mathrm{Li}_2(y)=\frac{1}{2}\ln^2(1-x)+\frac{1}{2}\ln^2(1-y)$$

Writing each function as iterated integral on the total space (using ψ), the relation becomes obvious:

$$\mathrm{Li}_2\left(\frac{1-y}{1-\frac{1}{x}}\right)=\left[\frac{dx}{x}+\frac{dx}{1-x}-\frac{dy}{1-y}\middle|\frac{xdy+ydx}{1-xy}\right]-\left[\frac{dx}{1-x}\middle|\frac{dy}{1-y}\right]-\left[\frac{dx}{x}+\frac{dx}{1-x}\middle|\frac{dx}{1-x}\right]$$

$$\mathrm{Li}_2\left(\frac{1-x}{1-\frac{1}{y}}\right)=\left[\frac{dy}{y}+\frac{dy}{1-y}-\frac{dx}{1-x}\middle|\frac{xdy+ydx}{1-xy}\right]+\left[\frac{dx}{1-x}\middle|\frac{dy}{1-y}\right]-\left[\frac{dy}{y}+\frac{dy}{1-y}\middle|\frac{dy}{1-y}\right]$$

$$\mathrm{Li}_2(xy)=\left[\frac{dx}{x}+\frac{dy}{y}\middle|\frac{xdy+ydx}{1-xy}\right],\ \mathrm{Li}_2(x)=\left[\frac{dx}{x}\middle|\frac{dx}{1-x}\right],\ \mathrm{Li}_2(y)=\left[\frac{dy}{y}\middle|\frac{dy}{1-y}\right]$$