

GRACE and loop integrals

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5. Summary

J.Fujimoto

KEK

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1. Introduction

GRACE is the generator of event generators

- Feynman rules based on SM, MSSM.
- Any orders of Feynman diagrams can be generated automatically.
- 1-loop diagrams of SM and MSSM can be evaluated.
- 1-loop integrals up to 4-point functions are equipped.
- 1-loop integrals up to 6-point functions are evaluated with the reduction method.
- 2-loop integrals up to 4-point functions are evaluated numerically.

GRACE Members

- KEK : Y. Kurihara, T. Kaneko, T. Ishikawa,
F. Yuasa, N. Watanabe, P.H. Khiem,
Y. Shimizu, N. Hamaguchi, J. F.
- M.G.U. : M. Kuroda,
- Kogakuin U. : K. Kato, N. Nakazawa, K. Tobimatsu
- Chiba U. C. : M. Jimbo,
- Seikei U. : T. Kon
- T. M. C. : Y. Yasui

2. Direct Computation Method(DCM)

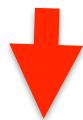
E. De Doncker, Y. Shimizu, J. Fujimoto, F. Yuasa, Comput. Phys. Comm. 159 ('04) 145.

$$I = \lim_{\epsilon \rightarrow 0} \int d(x) \frac{1}{(D(x) + i\epsilon)^n}$$

1st step

Let ϵ be finite as $\epsilon_l = \frac{\epsilon_0}{(A_c)^l}$, $A_c > 1$
with $l=0,1,2,\dots$

ϵ_0 and acceleration constant A_c are
positive numbers and given by hand.



2nd step

Evaluate the integral I numerically
and get the sequence of $I(\epsilon_l)$.



3rd step

Extrapolate the sequence $I(\epsilon_l)$ to the
limit ($\epsilon \rightarrow 0$) and determine I . ⁴

Example of the DCM application

Two-loop 4-point by DCM

$$I = (-1)^7 \times \Gamma(3)$$

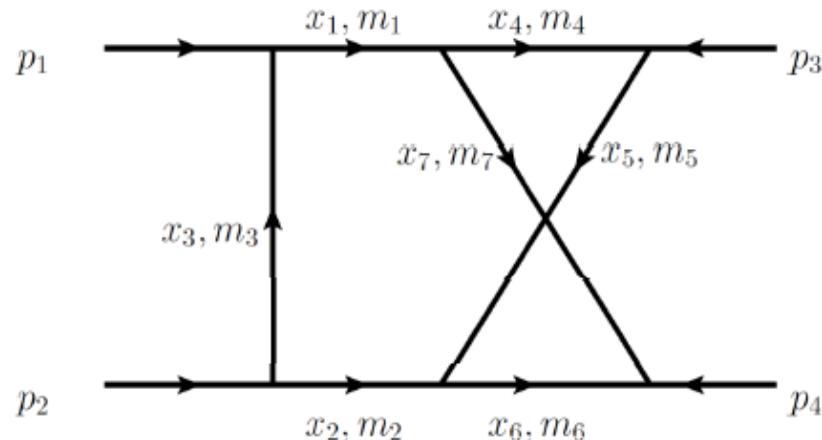
$$\times \int_0^1 \prod_{i=1}^7 dx_i \delta(1 - \sum_{i=1}^7 x_i) \frac{C}{(D_4 - i\epsilon C)^3}$$

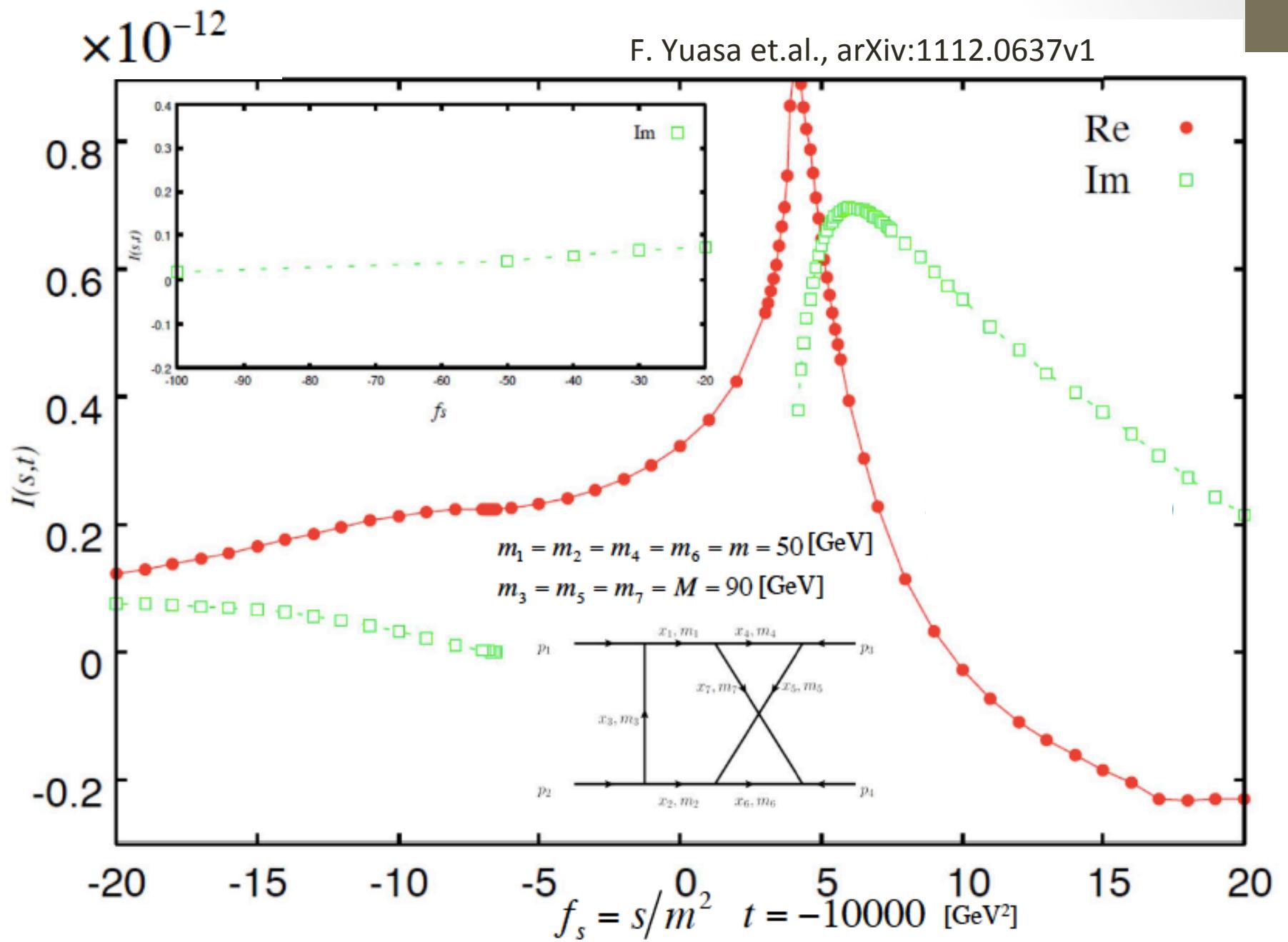
The 6-dimensional integration

$$\begin{aligned}
 D_4 &= C \sum x_\ell m_\ell^2 \\
 &- \{s(x_1x_2x_4 + x_1x_2x_5 + x_1x_2x_6 + x_1x_2x_7 + x_1x_5x_6 + x_2x_4x_7 - x_3x_4x_6) \\
 &+ t(x_3(-x_4x_6 + x_5x_7)) \\
 &+ p_1^2(x_3(x_1x_4 + x_1x_5 + x_1x_6 + x_1x_7 + x_4x_6 + x_4x_7)) \\
 &+ p_2^2(x_3(x_2x_4 + x_2x_5 + x_2x_6 + x_2x_7 + x_4x_6 + x_5x_6)) \\
 &+ p_3^2(x_1x_4x_5 + x_1x_5x_7 + x_2x_4x_5 + x_2x_4x_6 + x_3x_4x_5 + x_3x_4x_6 + x_4x_5x_6 + x_4x_5x_7) \\
 &+ p_4^2(x_1x_4x_6 + x_1x_6x_7 + x_2x_5x_7 + x_2x_6x_7 + x_3x_4x_6 + x_3x_6x_7 + x_4x_6x_7 + x_5x_6x_7)\}
 \end{aligned}$$

$$C = (x_1 + x_2 + x_3 + x_4 + x_5)(x_1 + x_2 + x_3 + x_6 + x_7) - (x_1 + x_2 + x_3)^2$$

Non-planar diagram





3. Introduction of b-functions and P-operators

$$P(s, x, \partial) D(x)^{s+1} = b(s) D(x)^s$$

- The application of this equation to Feynman integrals was introduced by F. Tkachov in 1996.
- b-functions were introduced with a- and c-functions in the theory of prehomogeneous vector spaces by M. Sato in early 1970s.
- Simultaneously and independently, the existence of $b(s)$ was proved by J. Bernstein in 1971 .
- The background of introduction of b-functions related to the analytical continuation of the hyperfunctions $D(x)_+^s$.

Application of the equation on b-functions to the 1-loop integrals

F.V. Tkachov, Nucl. Instr. Meth. Phys. Res. A389 (1996) 309 ,

G. Passarino, Nucl.Phys.B619:257-312,2001,

A. Ferroglia, G. Passarino, M. Passera, S. Uccirati, Nucl.Phys.B650:162-228,2003

$$\begin{aligned} I &= \int dx \frac{1}{D(x)^s} \\ &= \int dx \frac{1}{b(s)} P(s, x, \partial) \frac{1}{D(x)^{s-1}} \end{aligned}$$

where, $D(x) = x^T V x + 2R^T T x + Z$

$$P = 1 - \frac{(x + A)\partial_x}{2(1 + s)}, \quad A = R^T V^{-1}, \quad b(s) = Z - R^T V^{-1} R.$$

Application of the equation on b-functions to the 2-loop integrals

- As pointed out in
G. Passarino, Nucl.Phys.B619:257-312,2001,
 V of the 2-loop integrals is a highly incomplete polynomial
with few non-zero coefficients. Then, P has higher derivatives on x .
- The paper claimed to use the equation on one-loop integrals
extensively even for 2-loop integrals and more. It is called
“minimal B-T(Bernstein-Tkachov) approach”.

See also,

- G.Passarino, S. Uccirati, Nucl.Phys. B629 (2002) 97-187,
A. Ferroglia, G. Passarino, M. Passera, S. Uccirati, Nucl.Phys. B680 (2004) 199-270
S. Actis, A. Ferroglia, G. Passarino, M. Passera, S. Uccirati, Nucl.Phys. B703 (2004) 3-126,
G. Passarino, C. Sturm, S. Uccirati, Phys.Lett.B655:298-306,2007

- Minimal B-T approach works well for the numerical Integration of the 2-loop integrals.
- b-functions are still interesting, because all roots of the b-functions are negative rational numbers
(M. Kashiwara 1976).
- There exists an algorithm for computing b-functions and constructing P-operators by T.Oaku, using Groebner bases for rings of differential operators.

Oaku, T., An algorithm of computing b -functions. Duke Math. J. **87** (1997)

Outline of the algorithm by Oaku

1. Find annihilator ideal on f^s for rings with differential operators:

$$\text{Ann}_{D_n[s]} f^s = \{ A \in D_n[s] \mid Af^s = 0 \}$$

2. Because of $b(s) - P(s)f^s \in \text{Ann}_{D_n[s]} f^s$, through the left ideal C , $b(s)$ should be an element of the intersection of C and $K[s]$:

$$C = \text{Ann}_{D_n[s]} f^s + D_n[s]f,$$

$$b(s) \in B_f = C \cap K[s].$$

2. During getting B_f , Q_i and P can be obtained with generators P_i of the annihilator ideal as:

$$Q = Q_1P_1 + Q_2P_2 + \cdots + Q_kP_k + Pf$$

3. Because of $P_j f^s = 0$, then $Qf^s = Pf^{s+1}$.

4. This P satisfies $P(s, x, \partial)f(x)^{s+1} = b(s)f(x)^s$

Symbolic systems

- Groebner Bases Engine should be equipped.
- Wyle algebra can be treated:

$$[\partial_i, x_j] = \delta_{ij}$$

like Maple, Singular, Macaulay2, etc.

We have used

- Risa/Asir

<http://www.math.kobe-u.ac.jp/Asir/> by M. Noro

- OpenXM(Kan/sm1)

<http://www.math.kobe-u.ac.jp/OpenXM/> by N. Takayama

4. Examples of P-operators

(4-1) one-loop 5-point case

Kinematical variables kept:

s1,s2,s3,s4,s5,s12,s23,s34,s45,s51,xm1,xm2,xm3,xm4,xm5

f:=

```
-x1*x2*s1-x2*x3*s2-x3*x4*s3-x1*x3*s12-x2*x4*s23+x1*x3*s34+x2*x3*s34
+x3^2*s34+x3*x4*s34+x1*x4*s4+x2*x4*s4+x3*x4*s4+x4^2*s4-x1*x4*s45
+x1^2*s5+x1*x2*s5+x1*x3*s5+x1*x4*s5+x1*x2*s51+x2^2*s51+x2*x3*s51
+x2*x4*s51-x3*s34-x4*s4-x1*s5-x2*s51+x1*xm1+x2*xm2+x3*xm3+x4*xm4
-x1*xm5-x2*xm5-x3*xm5-x4*xm5+xm5$
```

$$\mathbf{b}(s) = (s + 1)$$

```
pfs1 := 2*dx1*(s1*s2*s3*s4*x4 - 2*s1*s2*s34*s4*x3 - s1*s2*s34*s4*x4 + s1*s2*s34*s4
- s1*s2*s4**2*x4 - s1*s2*s4*xm3 + s1*s2*s4*xm5 + s1*s23**2*s34*x2 +
s1*s23*s3**2*x4 - 2*s1*s23*s3*s34*x4 + s1*s23*s3*s34 - 2*s1*s23*s3*s4*x4 -
s1*s23*s3*xm3 + s1*s23*s3*xm5 + s1*s23*s34**2*x4 - s1*s23*s34**2 -
2*s1*s23*s34*s4*x2 - 2*s1*s23*s34*s4*x4 + s1*s23*s34*s4 - 2*s1*s23*s34*s51*x2 +
s1*s23*s34*xm3 - 2*s1*s23*s34*xm4 + s1*s23*s34*xm5 + s1*s23*s4**2*x4 +
s1*s23*s4*xm3 - s1*s23*s4*xm5 - s1*s3**2*s4*x4 - s1*s3**2*s51*x2 -
s1*s3**2*s51*x4 + ... (cont.)
```

(4-2) two-loop 2-point case :

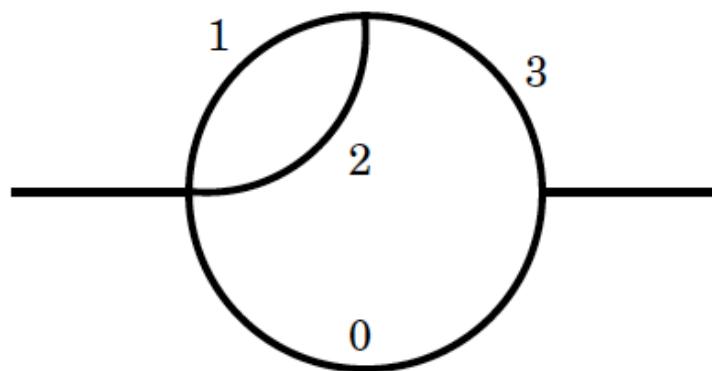
$$f = E - CM,$$

$$E = p^2[x_0x_1x_2 + (x_1 + x_2)x_0x_3],$$

$$C = x_1x_2 + (x_1 + x_2)(x_0 + x_3),$$

$$M = x_0m_0^2 + x_1m_1^2 + x_2m_2^2 + x_3m_3^2.$$

$$x_0 = 1 - x_1 - x_2 - x_3$$



[2.2.3]

4-2-1:

$$p_s^2/m_j^2 = 2, m_j^2 = m_0^2 = m_1^2 = m_2^2 = m_3^2$$

```
f:=-2*x1^2*x2-2*x1*x2^2-2*x1^2*x3-6*x1*x2*x3-2*x2^2*x3  
-2*x1*x3^2-2*x2*x3^2+x1^2+3*x1*x2+x2^2+2*x1*x3+2*x2*x3-x1-x2  
;
```

$$b(s) = (s + 1)(2s + 3)$$

```
pfs1:=  
+20407334400*s*x2*x3*Dx1-5101833600*s*x3^2*Dx1-5101833600*s*x1*x2*Dx2-15305500800*s*x2^2*Dx2  
-10203667200*s*x1*x3*Dx2-20407334400*s*x2*x3*Dx2+5101833600*s*x3^2*Dx2-10203667200*s*x1*x2*Dx3  
-5101833600*s*x1*x3*Dx3-30611001600*s*x2*x3*Dx3-15305500800*s*x3^2*Dx3-10203667200*s*x2*Dx1  
+5101833600*s*x3*Dx1+5101833600*s*x1*Dx2+10203667200*s*x2*Dx2-5101833600*s*x3*Dx2+2550916800*s*x1*Dx3  
+15305500800*s*x2*Dx3+15305500800*s*x3*Dx3  
-1417176000*s*Dx1+3684657600*s*Dx2-9069926400*s*Dx3  
+34012224000*x2*x3*Dx1-8503056000*x3^2*Dx1+283435200*x1*Dx11+566870400*x3*Dx11-8503056000*x1*x2*Dx2  
-25509168000*x2^2*Dx2-17006112000*x1*x3*Dx2-34012224000*x2*x3*Dx2+8503056000*x3^2*Dx2-283435200*x1*Dx12  
-283435200*x2*Dx12-1133740800*x3*Dx12+283435200*x2*Dx22+566870400*x3*Dx22-17006112000*x1*x2*Dx3  
-8503056000*x1*x3*Dx3-51018336000*x2*x3-25509168000*x3^2*Dx3-425152800*x1*Dx13-283435200*x2*Dx13  
-566870400*x3*Dx13-283435200*x1*Dx23-425152800*x2*Dx23-566870400*x3*Dx23+283435200*x1*Dx33+283435200*x2*Dx33  
+708588000*x3*Dx33-17006112000*x2*Dx1+8503056000*x3*Dx1-283435200*Dx11+8503056000*x1*Dx2+17006112000*x2*Dx2  
-8503056000*x3*Dx2+566870400*Dx12-283435200*Dx22+425152800*x1*Dx3+25509168000*x2*Dx3+25509168000*x3*Dx3  
+283435200*Dx13+283435200*Dx23-354294000*Dx33  
-2550916800*Dx1+5952139200*Dx2-14880348000*Dx3  
+(15305500800*s^2*x1+40814668800*s^2*x2+35712835200*s^2*x3  
-17856417600*s^2  
+40814668800*s*x1+108839116800*s*x2+95234227200*s*x3  
-47617113600*s  
+25509168000*x1+68024448000*x2+59521392000*x3  
-29760696000)*Dx0$
```

4-2-2:

$$p_s^2/m_j^2=4, m_j^2 = m_0^2 = m_1^2 = m_2^2 = m_3^2$$

$$\begin{aligned}f &:= -4*x1^2*x2 - 4*x1*x2^2 - 4*x1^2*x3 - 12*x1*x2*x3 - 4*x2^2*x3 \\&\quad - 4*x1*x3^2 - 4*x2*x3^2 + x1^2 + 5*x1*x2 + x2^2 + 4*x1*x3 + 4*x2*x3 - x1 - x2\end{aligned}$$

$$b(s) = (s+1)(2s+3)(4s+5)(4s+7)$$

```
23 pfs1:=  
24 +7739670528000*s^3*x2*Dx1+11609505792000*s^3*x3*Dx1+5804752896000*s^3*x1*Dx2+3869835264000*s^3*x2*Dx2+9674588160000*s^3*x3*Dx2  
25 -15479341056000*s^3*x3*Dx3-2794881024000*s^3*Dx1-3762339840000*s^3*Dx2-859963392000*s^3*Dx3  
26 +3869835264000*s^2*x3^2*Dx1^2-9674588160000*s^2*x3^2*Dx2-1934917632000*s^2*x1*x2*Dx2^2-1934917632000*s^2*x2*x3*Dx2^2  
27 -3869835264000*s^2*x1*x3*Dx2^2+5804752896000*s^2*x3^2*Dx2^2-3869835264000*s^2*x3^2*Dx1*Dx3-3869835264000*s^2*x1*x2*Dx2*Dx3  
28 -3869835264000*s^2*x2*x3^2*Dx2*Dx3-1934917632000*s^2*x1*x3*Dx2*Dx3-11609505792000*s^2*x2*x3*Dx2*Dx3-5804752896000*s^2*x3^2*Dx2*Dx3  
29 +3869835264000*s^2*x3^2*Dx3^2-680804352000*s^2*x1*Dx1^2-2364899328000*s^2*x2*Dx1^2-1863254016000*s^2*x3*Dx1^2  
30 +89579520000*s^2*x1*Dx2-1791590400000*s^2*x2*Dx1*Dx2+2364899328000*s^2*x3*Dx1*Dx2+465813504000*s^2*x1*Dx2^2  
31 +483729408000*s^2*x2*Dx2^2-3439853568000*s^2*x3*Dx2^2+1289945088000*s^2*x1*Dx3+3798171648000*s^2*x2*Dx1*Dx3  
32 +6055575552000*s^2*x3*Dx1*Dx3+1092870144000*s^2*x1*Dx2*Dx3+6163070976000*s^2*x2*Dx2*Dx3+6790127616000*s^2*x3*Dx2*Dx3  
33 -429981696000*s^2*x1*Dx3^2-429981696000*s^2*x2*Dx3^2-3726508032000*s^2*x3*Dx3^2+3869835264000*s^2*x1*Dx1  
34 +3869835264000*s^2*x2*Dx1+54177693696000*s^2*x3*Dx1+633028608000*s^2*Dx1^2+21284093952000*s^2*x1*Dx2  
35 +7739670528000*s^2*x2*Dx2+38698352640000*s^2*x3*Dx2+424009728000*s^2*Dx1*Dx2+1295917056000*s^2*Dx2^2  
36 -3869835264000*s^2*x1*Dx3-7739670528000*s^2*x2*Dx3-7739670528000*s^2*x3*Dx3-1797562368000*s^2*Dx1*Dx3  
37 -1645277184000*s^2*Dx2*Dx3-65691648000*s^2*Dx3^2-13257768960000*s^2*Dx1-12666544128000*s^2*Dx2-2615721984000*s^2*Dx3  
38 -286654464000*s*x3^2*Dx1^3+429981696000*s*x3^2*Dx1^2*Dx2+214990848000*s*x3^2*Dx1*Dx2^2+501645312000*s*x1*x2*Dx2^3  
39 +214990848000*s*x2^2*Dx2^3+429981696000*s*x1*x3*Dx2^3-358318080000*s*x3^2*Dx2^3+1074954240000*s*x3^2*Dx1^2*Dx3  
40 -967458816000*s*x3^2*Dx1*Dx2*Dx3+35831808000*s*x1*x2*Dx2^2*Dx3+609140736000*s*x2*Dx2^2*Dx3-286654464000*s*x1*x3*Dx2*Dx3  
41 +1289945088000*s*x2*x3*Dx2^2*Dx3+1254113280000*s*x3^2*Dx2^2*Dx3-1074954240000*s*x3^2*Dx1*Dx3^2-214990848000*s*x1*x2*Dx2*Dx3^2  
42 -501645312000*s*x2^2*Dx2*Dx3^2+179159040000*s*x1*x3*Dx2*Dx3^2-644972544000*s*x2*x3*Dx2*Dx3^2-752467968000*s*x3^2*Dx2*Dx3^2  
43 +788299776000*s*x3^2*Dx3^3+11609505792000*s*x3^2*Dx1^2+71663616000*s*x1*Dx1^3+47775744000*s*x2*Dx1^3+286654464000*s*x3*Dx1^  
44 3-29023764480000*s*x3^2*Dx1*Dx2+11943936000*s*x1*Dx1^2*Dx2+95551488000*s*x2*Dx1^2*Dx2-429981696000*s*x3*Dx1^2*Dx2  
45 -5804752896000*s*x1*x2*Dx2^2-5804752896000*s*x2^2*Dx2^2-11609505792000*s*x1*x3*Dx2^2+17414258688000*s*x3^2*Dx2^2  
46 -209018880000*s*x1*Dx1*Dx2^2-23887872000*s*x2*Dx1*Dx2^2-214990848000*s*x3*Dx1*Dx2^2-203046912000*s*x1*Dx2^3  
47 -125411328000*s*x2*Dx2^3+358318080000*s*x3*Dx2^3-11609505792000*s*x3^2*Dx1*Dx3-316514304000*s*x1*Dx1^2*Dx3  
48 -274710528000*s*x2*Dx1^2*Dx3-1003290624000*s*x3*Dx1^2*Dx3-11609505792000*s*x1*x2*Dx2*Dx3-11609505792000*s*x2*x3*Dx2*Dx3  
49 -5804752896000*s*x1*x3*Dx2*Dx3-34828517376000*s*x2*x3*Dx2*Dx3-17414258688000*s*x3^2*Dx2*Dx3-206032896000*s*x1*Dx1*Dx2*Dx3  
50 -394149888000*s*x2*Dx1*Dx2*Dx3+501645312000*s*x3*Dx1*Dx2*Dx3+119439360000*s*x1*Dx2^2*Dx3-916697088000*s*x2*Dx2^2*Dx3
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51 -1272029184000*s*x3*Dx2^2*Dx3+11609505792000*s*x3^2*Dx3+513589248000*s*x1*Dx1^2+471785472000*s*x2*Dx1*Dx3^2
52 +1209323520000*s*x3*Dx1^2*Dx3^2+234399744000*s*x1*Dx2*Dx3^2+934612992000*s*x2*Dx2*Dx3^2+864442368000*s*x3*Dx2*Dx3^2
53 -191102976000*s*x1*Dx3^3-191102976000*s*x2*Dx3^3-716636160000*s*x3*Dx3^3-2400731136000*s*x1*Dx1^2-7668006912000*s*x2*Dx1^2
54 -3726508032000*s*x3*Dx1^2-71663616000*s*x1*Dx1^3-17915904000*s*x1*Dx1^2-6306398208000*s*x2*Dx1^2+3081535488000*s*x3*Dx1^2
55 +107495424000*s*x1*Dx1^2*Dx2+1934917632000*s*x1*Dx2^2+3242778624000*s*x2*Dx2^2-7560511488000*s*x3*Dx2^2+53747712000*s*x1*Dx2^2
56 -89579520000*s*x2*Dx2^3+4550639616000*s*x1*Dx3+12397805568000*s*x2*Dx1*Dx3+16410968064000*s*x3*Dx1*Dx3
57 +232906752000*s*x1*Dx1^2*Dx3+3690676224000*s*x1*Dx2*Dx3+19779158016000*s*x2*Dx2*Dx3+20818280448000*s*x3*Dx2*Dx3
58 -8957952000*s*x1*Dx2*Dx3+322486272000*s*x2*Dx3-1433272320000*s*x1*Dx3^2-2221572096000*s*x2*Dx3^2-8993783808000*s*x3*Dx3^2
59 -291133440000*s*x1*Dx1^2*Dx3^2-212751360000*s*x2*Dx3^2+71663616000*s*x3*Dx3^2+11609505792000*s*x1*Dx1+63637291008000*s*x2*Dx1
60 +83846430720000*s*x3*Dx1^2+1516879872000*s*x1*Dx1^2+24508956672000*s*x1*Dx2-3009871872000*s*x2*Dx2+50522849280000*s*x3*Dx2
61 +2657525760000*s*x1*Dx2+2341011456000*s*x2*Dx2^2-11609505792000*s*x1*Dx3-23219011584000*s*x2*Dx3-127274582016000*s*x3*Dx3
62 -5100060672000*s*x1*Dx3-5485252608000*s*x2*Dx3-471785472000*s*x3*Dx3^2-22036561920000*s*x1*Dx1-11072028672000*s*x2*Dx2-286654464000*s*Dx3
63 +26873856000*x3*Dx1^2*Dx3^2-67184640000*x3*Dx2*Dx3^2-13436928000*x1*x2*Dx2^2*Dx3^2-13436928000*x2*Dx2^2*Dx3^2
64 -26873856000*x1*x3*Dx2^2*Dx3^2+40310784000*x3*Dx2^2*Dx3^2-26873856000*x3*Dx1*Dx3^3-26873856000*x1*x2*Dx2*Dx3^3
65 -26873856000*x2*Dx2*Dx3^3-13436928000*x1*x3*Dx2*Dx3^3-80621568000*x2*x3*Dx2*Dx3^3-40310784000*x3*Dx2*Dx3^3
66 +26873856000*x3*Dx3^4-429981696000*x3*Dx1^3+644972544000*x3*Dx1^2*Dx2+322486272000*x3*Dx1*Dx2^2
67 +752467968000*x1*x2*Dx2^3+322486272000*x2*Dx2^3+644972544000*x1*x3*Dx2^3-537477120000*x3*Dx2^3
68 +1612431360000*x3*Dx1^2*Dx3-1451188224000*x3*Dx1^2*Dx2*Dx3+53747712000*x1*x2*Dx2^2*Dx3+913711104000*x2*Dx2^2*Dx3
69 -429981696000*x1*x3*Dx2^2*Dx3+1934917632000*x2*x3*Dx2^2*Dx3+1881169920000*x3*Dx2^2*Dx3-1612431360000*x3*Dx1*Dx3^2
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71 -752467968000*x2*Dx2^2*Dx3^2+26873856000*x1*x3*Dx2*Dx3^2-967458816000*x2*x3*Dx2*Dx3^2-1128701952000*x3*Dx2*Dx3^2
72 +5598720000*x1*Dx1^2*Dx2*Dx3^2-4478976000*x2*Dx1^2*Dx3^2+6718464000*x3*Dx1*Dx2*Dx3^2+15676416000*x1*Dx2^2*Dx3^2
73 +3359232000*x2*Dx2^2*Dx3^2-40310784000*x3*Dx2^2*Dx3^2+1182449664000*x3*Dx3^3+17915904000*x1*Dx1*Dx3^3
74 +13436928000*x2*Dx1*Dx3^3+33592320000*x3*Dx1*Dx3^3+3359232000*x1*Dx2*Dx3^3+58226688000*x2*Dx2*Dx3^3
75 +5038848000*x3*Dx2*Dx3^3-8957952000*x1*Dx3^4-8957952000*x2*Dx3^4-26873856000*x3*Dx3^4+859963392000*x3*Dx1^2
76 +107495424000*x1*Dx1^3+71663616000*x2*Dx1^3+429981696000*x3*Dx1^3-214990848000*x3*Dx1^2*Dx2+17915904000*x1*Dx1^2*Dx2
77 +143327232000*x2*Dx1^2*Dx2-644972544000*x3*Dx1^2*Dx2-429981696000*x1*x2*Dx2^2-429981696000*x2*Dx2^2
78 -859963392000*x1*x3*Dx2^2+1289945088000*x3*Dx2^2-31352832000*x1*Dx1*Dx2^2-35831808000*x2*Dx1*Dx2^2
79 -322486272000*x3*Dx1*Dx2^2-304570368000*x1*Dx2^3-188116992000*x2*Dx2^3+53747712000*x3*Dx2^3-859963392000*x3*Dx1*Dx3
80 -456855552000*x1*Dx1^2*Dx3-412065792000*x2*Dx1^2*Dx3-13616088704000*x3*Dx1^2*Dx3-859963392000*x1*x2*Dx2*Dx3
81 -859963392000*x2*Dx2*Dx3-429981696000*x1*x3*Dx2*Dx3-2579890176000*x2*x3*Dx2*Dx3-1289945088000*x3*Dx2*Dx3
82 -326965248000*x1*Dx1*Dx2*Dx3-609140736000*x2*Dx1*Dx2*Dx3+412065792000*x3*Dx1*Dx2*Dx3+125411328000*x1*Dx2^2*Dx3
83 -1357129728000*x2*Dx2^2*Dx3-1710968832000*x3*Dx2^2*Dx3+859963392000*x3*Dx3^2+770383872000*x1*Dx1*Dx3^2
84 +71663616000*x2*Dx1*Dx3^2+1684094976000*x3*Dx1*Dx3^2+6718464000*Dx1^2*Dx3^2+273217536000*x1*Dx2*Dx3^2
85 +1146617856000*x2*Dx2*Dx3^2+1065996288000*x3*Dx2*Dx3^2-16796160000*Dx1*Dx2*Dx3^2+10077696000*Dx2*Dx3^2
86 -295612416000*x1*Dx3^3-322486272000*x2*Dx3^3-976416768000*x3*Dx3^3-10077696000*Dx1*Dx3^3-15116544000*Dx2*Dx3^3
87 +5038848000*Dx3^4-2078244864000*x1*Dx1^2-6163070976000*x2*Dx1^2-13616088704000*x3*Dx1^2-107495424000*Dx1^3
88 -214990848000*x1*Dx1*Dx2-53747712000*x2*Dx1*Dx2-824131584000*x3*Dx1*Dx2+161243136000*Dx1*Dx2
89 +17915904000*x1*Dx2^2+3726508032000*x2*Dx2^2-3511517184000*x3*Dx2^2+80621568000*Dx1*Dx2^2-13436928000*Dx2^3
90 +3905667072000*x1*Dx1*Dx3+1003290624000*x2*Dx1*Dx3+1092870144000*x3*Dx1*Dx3+277696512000*Dx1*Dx2
91 +3099451392000*x1*Dx2*Dx3+15622668288000*x2*Dx2*Dx3+15819743232000*x3*Dx2*Dx3+156764160000*Dx1*Dx2*Dx3
92 +385191936000*Dx2^2*Dx3-1182449664000*x1*Dx3^2-2364899328000*x2*Dx3^2-5088116736000*x3*Dx3^2-362797056000*Dx1*Dx3^2
93 -164602368000*Dx2*Dx3^2+73903104000*Dx3^3+859963392000*x1*Dx1+3439853568000*x2*Dx1+429981696000*x3*Dx1
94 +949542912000*Dx1^2+859963392000*x1*Dx2-859963392000*x2*Dx2+214990848000*x3*Dx2+282175488000*Dx1*Dx2
95 +707678208000*Dx2^2-859963392000*x1*Dx3-1719926784000*x2*Dx3-6879707136000*x3*Dx3-3681718272000*Dx1*Dx3
96 -470292480000*Dx2*Dx3-555393024000*Dx3^2-12576964608000*Dx1-913711104000*Dx2+2364899328000*Dx3
97 +(11609505792000*s^4+69657034752000*s^3+153503465472000*s^2+147053740032000*s+5159780352000)*Dx0$
```

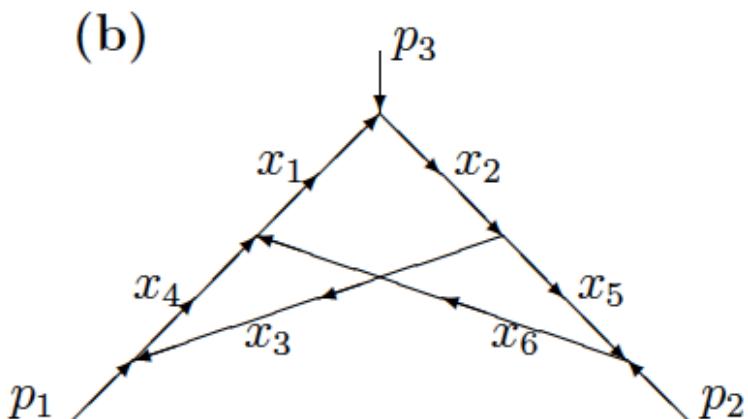
(4-3) two-loop non-planar vertex case

$$J = \int_0^1 dx_1 \cdots dx_6 \delta(1 - \sum x_j) \frac{1}{(\mathcal{D} + i\epsilon)^2}$$

$$\mathcal{D} = \sum_S W_S p_S^2 - U \sum_j x_j m_j^2$$

$$\sum S W_S p_S^2 = f_1 s_1 + f_2 s_2 + f_3 s_3 .$$

$$U = x_{12}x_{3456} + x_{34}x_{56}$$



$$f_1 = x_{1256}x_3x_4 + x_1x_3x_6 + x_2x_4x_5$$

$$f_2 = x_{1234}x_5x_6 + x_2x_3x_6 + x_1x_4x_5$$

$$f_3 = x_{3456}x_1x_2 + x_2x_4x_6 + x_1x_3x_5$$

$$p_1^2 = p_3^2 = 10, \quad p_2^2 = 0,$$

$$m_1^2 = 1, \quad m_2^2 = 2, \quad m_3^2 = 3, \quad m_4^2 = 4, \quad m_5^2 = 5, \quad m_6^2 = 6.$$

```
f := - 5*x1**3 - 24*x1**2*x2 - 18*x1**2*x3 - 7*x1**2*x4 - x1**2*x5 + 11*x1**2 -
23*x1*x2**2 - 25*x1*x2*x3 - 23*x1*x2*x4 - 2*x1*x2*x5 + 31*x1*x2 - 18*x1*x3**2 -
25*x1*x3*x4 - x1*x3*x5 + 24*x1*x3 - 7*x1*x4**2 - x1*x4*x5 + 13*x1*x4 + x1*x5 -
6*x1 - 4*x2**3 - 7*x2**2*x3 - 16*x2**2*x4 - x2**2*x5 + 10*x2**2 - 7*x2*x3**2 -
23*x2*x3*x4 - x2*x3*x5 + 13*x2*x3 - 16*x2*x4**2 - x2*x4*x5 + 22*x2*x4 + x2*x5 -
6*x2 - 3*x3**3 - 18*x3**2*x4 - x3**2*x5 + 9*x3**2 - 17*x3*x4**2 - 2*x3*x4*x5 +
27*x3*x4 + x3*x5 - 6*x3 - 2*x4**3 - x4**2*x5 + 8*x4**2 + x4*x5 - 6*x4$
```

$$b(s) = (s+1)(2s+3)$$

```
Pfs1:=
-8457445293636043504800000*s*Dx5+8457445293636043504800000*x1*Dx25
+8457445293636043504800000*x2*Dx25+8457445293636043504800000*x3*Dx35
+8457445293636043504800000*x4*Dx35-160691460579084826591200000*x1*Dx55
-67659562349088348038400000*x2*Dx55-50744671761816261028800000*x3*Dx55
-126861679404540652572000000*x4*Dx55-8457445293636043504800000*x5*Dx55
+140957421560600725080000*Dx11-281914843121201450160000*Dx12
+140957421560600725080000*Dx22+140957421560600725080000*Dx13
-140957421560600725080000*Dx23+140957421560600725080000*Dx33
-140957421560600725080000*Dx14+140957421560600725080000*Dx24
-281914843121201450160000*Dx34+140957421560600725080000*Dx44
-1832446480287809426040000*Dx15-986701950924205075560000*Dx25
-1832446480287809426040000*Dx35-986701950924205075560000*Dx45
+84997325201042237223240000*Dx55-16914890587272087009600000*Dx5$
```

5. Summary

- Direct Computation Method(DCM) works well to treat the Feynman loop-integrals in GRACE.
- b-functions may be useful to analyze the structures of the Feynman loop-integrals.
- There exists an algorithm to calculate b-functions and P-operators by T. Oaku.
- P-operators are constructed in some cases of two-loop diagrams using symbolic systems.
- Applications of P-operators to the numerical integration of the multi-loop integrals are still open question.

Back up

1st example: 1-loop box integral with complex masses reported in PoS(ACAT08)122

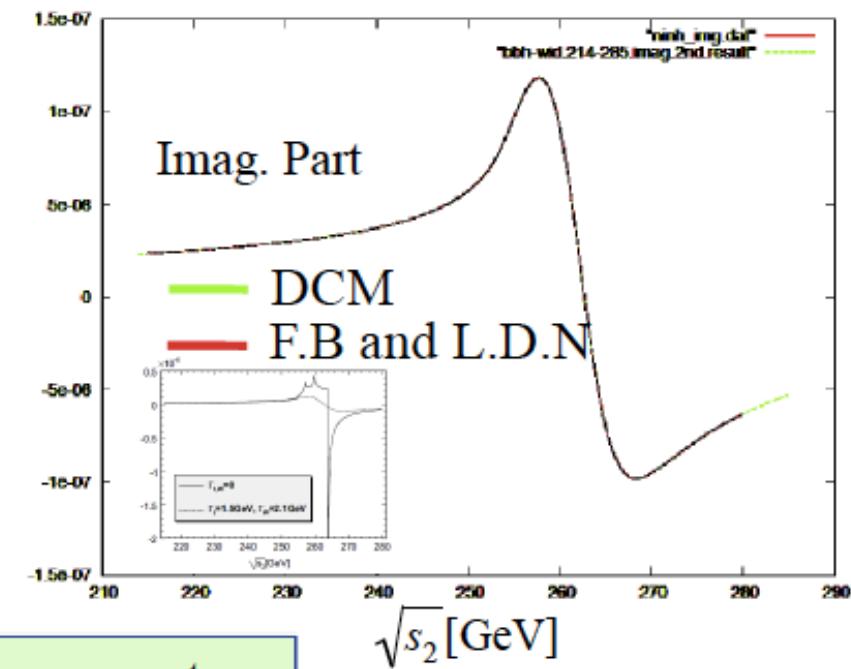
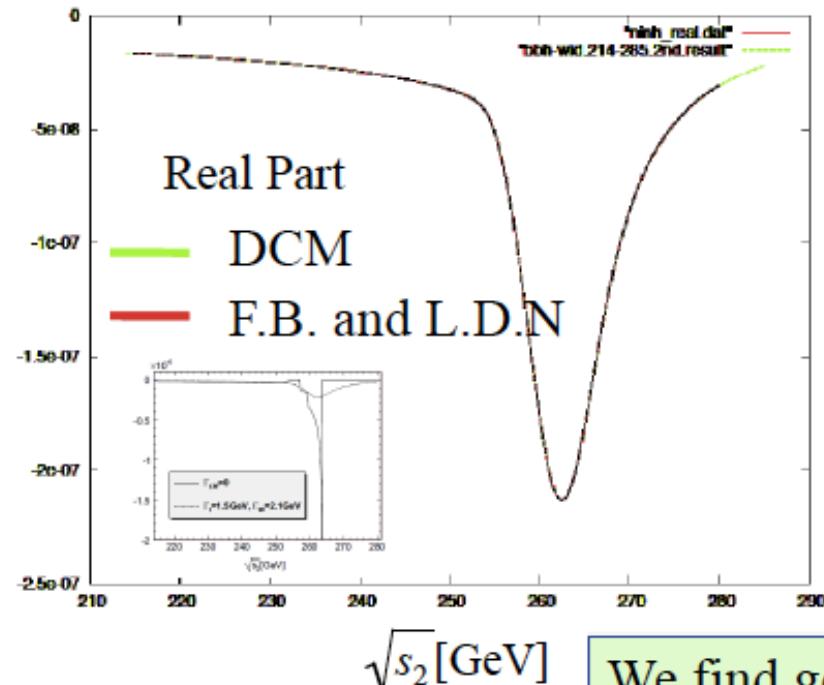
F.Boudjema and LE Duc Ninh Phys.Rev.D78:093005,2008

A box diagram contributing to $gg \rightarrow bb H$ that can develop a Landau singularity for $M_H \geq 2M_W$ and $\sqrt{s} \geq 2m_t$.

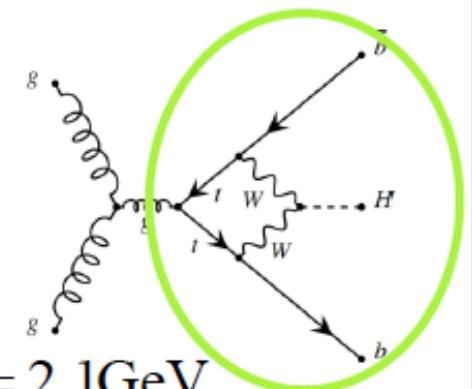
$m_t = 174$ GeV, $M_W = 80.3766$ GeV, $\sqrt{s} = 353$ GeV and $M_H = 165$ GeV.

$$m_t^2 \rightarrow m_t^2 - im_t\Gamma_t, \Gamma_t = 1.5 \text{ GeV} \quad M_W^2 \rightarrow M_W^2 - iM_W\Gamma_w, \Gamma_w = 2.1 \text{ GeV}$$

$$\times 10^{-6}$$

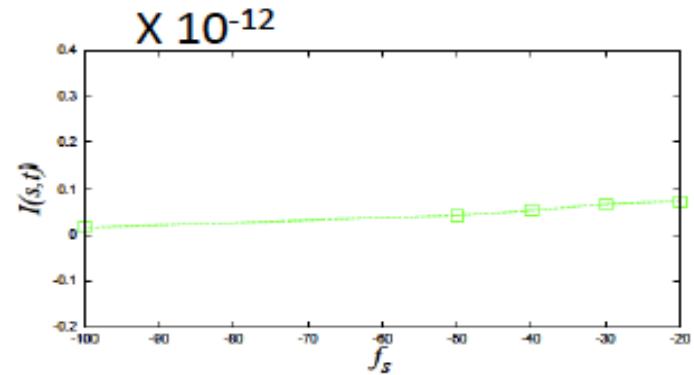
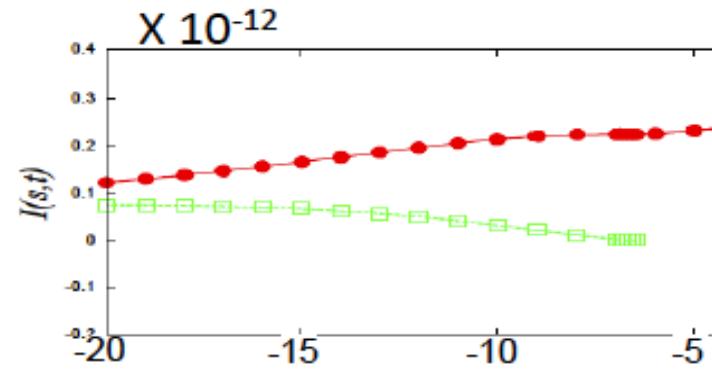


We find good agreement.



Numerical results of Two-loop **non-planar** box with masses

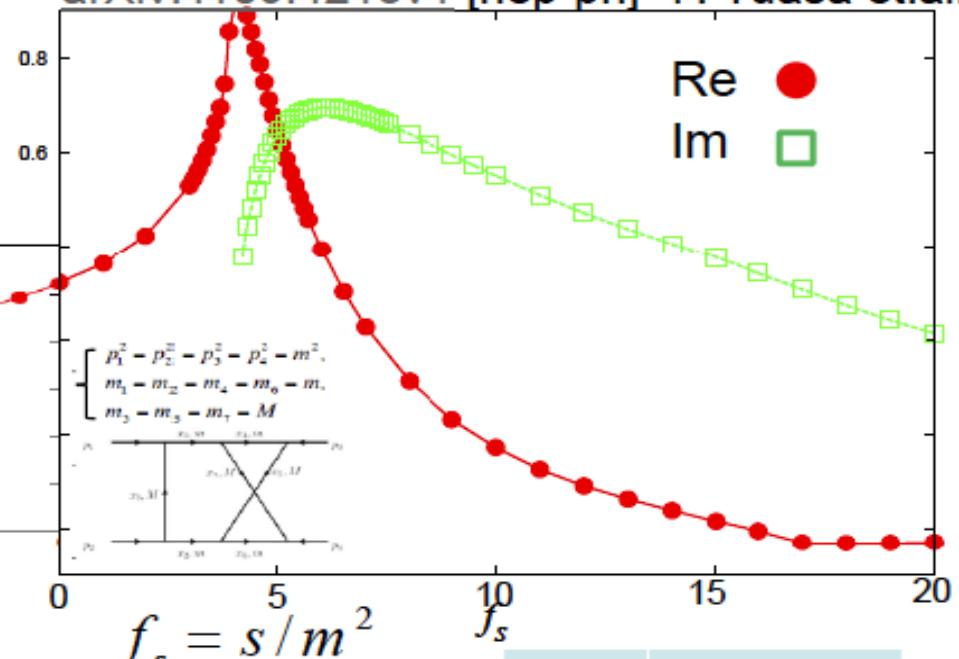
$m=50 \text{ GeV}$, $M=90 \text{ GeV}$, $t=-100^2 \text{ GeV}^2$



Dispersion Relation

$$\Re(I(s)) = \frac{1}{\pi} \left(P \int_{-\infty}^{s_0'} \frac{\Im(I(s'))}{s - s'} ds' + P \int_{s_0}^{\infty} \frac{\Im(I(s'))}{s - s'} ds' \right),$$

where s_0 and s_0' are the threshold in s -channel and that in u -channel,



	Re. fs	CPU time
Intel(R) Xeon(R) X5460 @ 3.16GHz	6.0	16 hours
	7.0	2 days
	10.0	1 week

