

Efficient contraction of 1-loop N-point tensor integrals

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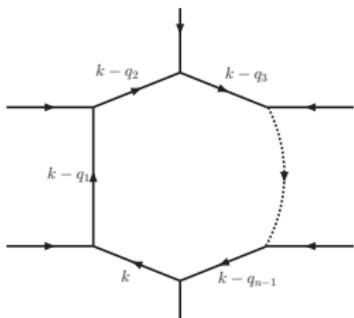
Introduction

n -point tensor integrals of rank R : (n, R)-integrals

$$I_n^{\mu_1 \dots \mu_R} = \int \frac{d^d k}{i\pi^{d/2}} \frac{\prod_{r=1}^R k^{\mu_r}}{\prod_{j=1}^n c_j^{\nu_j}},$$

$d = 4 - 2\epsilon$ and denominators c_j have *indices* ν_j and *chords* q_j

$$c_j = (k - q_j)^2 - m_j^2 + i\epsilon$$



tensor integrals due to, e.g.:

- fermion propagators
- three-gauge boson couplings

History

- [D.B.Melrose, Nuovo Cim.**40** (1965), Reduction of Feynman diagrams, Cayley determinants]
- [Davydychev:Phys.Lett. **B263** (1991), integrals in diff. space-time dim. see also Z.Bern et al. Phys. Lett. **B302** (1993).]
- [Tarasov:Phys.Rev. **D54** (1996), dimensional recurrence relations.]
- [JF,Jegerlehner,Tarasov: Nucl. Phys. **B566** (2000), 1-loop and “signed minors”.]
- [T. Binoth, J. Guillet, G. Heinrich, E. Pilon, C. Schubert, An algebraic / numerical formalism for one-loop multi-leg amplitudes, JHEP 10 (2005) 015].
- [JF and T.Riemann: Phys. Rev. **D83** (2011), Complete reduction of 1-loop tensors.]
- [V.Yundin C++ package PJFry. Available at <https://github.com/Vayu/PJFry>]
- [JF and T.Riemann: Phys. Lett. **B707** (2012), A solution for tensor reduction of one-loop n -point functions with $n \geq 6$.]

Tensors expressed in terms of integrals in higher dimension

Following [Davydychev:1991], also [J.F. et al.:2000] express tensors by means of scalar integrals in higher dimensions ($n_{ij} = \nu_{ij} = 1 + \delta_{ij}$, $n_{ijk} = \nu_{ij}\nu_{jk}$, $\nu_{ijk} = 1 + \delta_{ik} + \delta_{jk}$ etc.):

$$I_n^\mu = \int^d k^\mu \prod_{r=1}^n c_r^{-1} = - \sum_{i=1}^n q_i^\mu I_{n,i}^{[d+]}$$

$$I_n^{\mu\nu} = \int^d k^\mu k^\nu \prod_{r=1}^n c_r^{-1} = \sum_{i,j=1}^n q_i^\mu q_j^\nu n_{ij} I_{n,ij}^{[d+]^2} - \frac{1}{2} g^{\mu\nu} I_n^{[d+]}$$

$$I_n^{\mu\nu\lambda} = \int^d k^\mu k^\nu k^\lambda \prod_{r=1}^n c_r^{-1} = - \sum_{i,j,k=1}^n q_i^\mu q_j^\nu q_k^\lambda n_{ijk} I_{n,ijk}^{[d+]^3} + \frac{1}{2} \sum_{i=1}^n g^{\mu\nu} q_i^\lambda I_{n,i}^{[d+]^2}$$

$$I_n^{\mu\nu\lambda\rho} = \int^d k^\mu k^\nu k^\lambda k^\rho \prod_{r=1}^n c_r^{-1} = \sum_{i,j,k,l=1}^n q_i^\mu q_j^\nu q_k^\lambda q_l^\rho n_{ijkl} I_{n,ijkl}^{[d+]^4}$$

$$-\frac{1}{2} \sum_{i,j=1}^n g^{\mu\nu} q_i^\lambda q_j^\rho n_{ij} I_{n,ij}^{[d+]^3} + \frac{1}{4} g^{\mu\nu} g^{\lambda\rho} I_n^{[d+]^2}$$

The integrals

$$I_{p,ijk\cdots}^{[d+]^l,stu\cdots} = \int^{[d+]^l} \prod_{r=1}^n \frac{1}{C_r^{1+\delta_{ri}+\delta_{rj}+\delta_{rk}+\cdots-\delta_{rs}-\delta_{rt}-\delta_{ru}-\cdots}}, \quad \int^d \equiv \int \frac{d^d k}{\pi^{d/2}},$$

where $[d+]^l = 4 + 2l - 2\varepsilon$.

$$I_{n-1,ab}^{\{\mu_1, \dots\},s}, \quad a, b \neq s$$

is obtained from

$$I_n^{\{\mu_1, \dots\}}$$

by

- shrinking line s
- raising the powers of inverse propagators a, b .

Comments

- External momenta and $g^{\mu\nu}$ in 4 dimensions.
- Integration in higher dimension (generic: $d = 4 - 2\varepsilon$).
- No contraction of $g^{\mu\nu}$ with an integration momentum.
- $g^{\mu\nu}$ cancels for $n = 5$. Will be exemplified.
- Considering an N -point tensor, $q_N = 0$ is chosen.
Needed also for

$$(q_i \cdot q_j) = \frac{1}{2} [Y_{ij} - Y_{iN} - Y_{Nj} + Y_{NN}].$$

- Cancellation of (“reducible”) scalar propagators is avoided.
If needed, the integration rules also hold for reduced tensors of rank $N - 1$ with $q_{N-1} \neq 0$. No shift of integration momentum.

Notations: modified Cayley determinant

Modified Cayley determinant $(\cdot)_N$ of a diagram with N internal lines and chords q_j :

$$(\cdot)_N \equiv \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & Y_{11} & Y_{12} & \dots & Y_{1N} \\ 1 & Y_{12} & Y_{22} & \dots & Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1N} & Y_{2N} & \dots & Y_{NN} \end{vmatrix},$$

with matrix elements

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2, \quad (i, j = 1 \dots N)$$

Gram determinant G_n : $G_n = |2q_i q_j|$, $i, j = 1, \dots, n$

For a choice $q_n = 0$, both determinants are related: $(\cdot)_N = -G_{N-1}$

⇒ The determinant $(\cdot)_N$ does not depend on the masses.

Notations: signed minors

We also need **signed minors** of $(\cdot)_N$, constructed by deleting m rows and m columns from $(\cdot)_N$, and multiplying with a sign factor:

$$\begin{aligned} & \left(\begin{array}{cccc} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{array} \right)_N = \\ & \equiv (-1)^{\sum_i (j_i + k_i)} \operatorname{sgn}_{\{j\}} \operatorname{sgn}_{\{k\}} \left| \begin{array}{c|c} \text{rows } j_1 \cdots j_m \text{ deleted} \\ \hline \text{columns } k_1 \cdots k_m \text{ deleted} \end{array} \right| \end{aligned}$$

where $\operatorname{sgn}_{\{j\}}$ and $\operatorname{sgn}_{\{k\}}$ are the signs of permutations that sort the deleted rows $j_1 \cdots j_m$ and columns $k_1 \cdots k_m$ into ascending order.

The signed minors are **antisymmetric** under exchange of neighboring indices j and k .

Some algebraic relations [D.B.Melrose, Nuovo Cim.**40** (1965)]

$$\sum_{i=1}^n \binom{0}{i}_n = ()_n$$

and

$$\sum_{i=1}^n \binom{j}{i}_n = 0, \quad (j \neq 0).$$

$$()_n \binom{il}{jk}_n = \binom{i}{j}_n \binom{l}{k}_n - \binom{i}{k}_n \binom{l}{j}_n; \quad i, j, k, l = 0, \dots, n.$$

"master formula" ((A.13) of D.B. Melrose)

$$\binom{s}{i}_n \binom{\tau s}{0s}_n = \binom{s}{0}_n \binom{\tau s}{is}_n + \binom{s}{s}_n \binom{\tau s}{0i}_n, \quad \tau = 0, 1, \dots, 5,$$

Recursion for integrals

Following [Tarasov:1996, JF:2000]: apply recurrence relations relating scalar integrals of different dimensions in order to get rid of the high dimensions.

$$\begin{aligned}\nu_j j^+ I_n^{(d+2)} &= \frac{1}{\binom{0}{n}} \left[-\binom{j}{0}_5 + \sum_{k=1}^n \binom{j}{k}_n \mathbf{k}^- \right] I_n^d \\ (d - \sum_{i=1}^n \nu_i + 1) I_n^{(d+2)} &= \frac{1}{\binom{0}{n}} \left[\binom{0}{0}_n - \sum_{k=1}^n \binom{0}{k}_n \mathbf{k}^- \right] I_n^d,\end{aligned}$$

where the operators i^\pm, j^\pm, k^\pm act by shifting the indices ν_i, ν_j, ν_k by ± 1 . Example for a “scratched” integral ($\nu_{ij} = 1 + \delta_{ij}$):

$$\nu_{ij} I_{4,ij}^{[d+]^2,s} = -\frac{\binom{0s}{js}_5}{\binom{s}{s}_5} I_{4,i}^{[d+],s} + \frac{\binom{is}{js}_5}{\binom{s}{s}_5} I_4^{[d+],s} + \sum_{t=1}^5 \frac{\binom{ts}{js}_5}{\binom{s}{s}_5} I_{3,i}^{[d+],st}.$$

Recursion for tensors

5-point tensor recursion (5-PTR): Express any $(5, R)$ pentagon by a $(5, R - 1)$ pentagon plus $(4, R - 1)$ boxes

[T.Diakonidis,JF.T.Riemann,J.B.Tausk: Phys.Lett. **B683** (2010)]

$$I_5^{\mu_1 \dots \mu_{R-1} \mu} = I_5^{\mu_1 \dots \mu_{R-1}} Q_0^\mu - \sum_{s=1}^5 I_4^{\mu_1 \dots \mu_{R-1}, s} Q_s^\mu,$$

auxiliary vectors with inverse Gram determinants

$$Q_s^\mu = \sum_{i=1}^5 q_i^\mu \frac{(_i)_5}{(_5)}, \quad s = 0, \dots, 5$$

For e.g. $R = 3$, again $[1/(_5)]^3$ will occur.

Contractions I

$$q_{i_1\mu_1} \cdots q_{i_R\mu_R} \quad I_5^{\mu_1 \cdots \mu_R} = \int \frac{d^d k}{i\pi^{d/2}} \frac{\prod_{r=1}^R (q_{i_r} \cdot k)}{\prod_{j=1}^5 c_j},$$

$$g_{\mu_1, \mu_2} q_{i_1\mu_3} \cdots q_{i_R\mu_R} \quad I_5^{\mu_1 \cdots \mu_R} \neq \int \frac{k^2 d^d k}{i\pi^{d/2}} \frac{\prod_{r=3}^R (q_{i_r} \cdot k)}{\prod_{j=1}^5 c_j} \quad ?$$

etc., are obtained by constructing **projection operators**
or

by calculating scalar differential cross sections
(Born \times 1-loop)

For $q_n = 0$, $a = 1, \dots, n-1$, $s = 1, \dots, n$

$$(q_a \cdot Q_0) = \sum_{j=1}^{n-1} (q_a \cdot q_j) \frac{\binom{0}{j}_n}{\binom{0}{n}} = -\frac{1}{2} (Y_{an} - Y_{nn}),$$

$$(q_a \cdot Q_s) = \sum_{j=1}^{n-1} (q_a \cdot q_j) \frac{\binom{s}{j}_n}{\binom{0}{n}} = \frac{1}{2} (\delta_{as} - \delta_{ns}),$$

Writing the 5-PTR as $(q_5 = 0, i = 1 \dots 4)$:

$$\int \frac{k^{\mu_1} \cdots k^{\mu_{R-1}} (kq_i)}{c_1 c_2 c_3 c_4 c_5} = \int \frac{k^{\mu_1} \cdots k^{\mu_{R-1}}}{c_1 c_2 c_3 c_4 c_5} (Q_0 q_i) - \sum_{s=1}^5 \int \frac{k^{\mu_1} \cdots k^{\mu_{R-1}} \cdot c_s}{c_1 c_2 c_3 c_4 c_5} (Q_s q_i),$$

with $kq_i = -\frac{1}{2} [c_i - c_5 + Y_{i5} - Y_{55}]$, follows its proof.

Cancellation of $g^{\mu\nu}$

$$I_5^{\mu\nu} = \sum_{i,j=1}^4 q_i^\mu q_j^\nu \nu_{ij} I_{5,ij}^{[d+]^2} - \frac{1}{2} g^{\mu\nu} I_5^{[d+]},$$

$$\frac{1}{2} g^{\mu\nu} = \sum_{i,j=1}^5 \frac{\binom{i}{j}_5}{\binom{0}{0}_5} q_i^\mu q_j^\nu.$$

$$\nu_{ij} I_{5,ij}^{[d+]^2} = - \frac{\binom{0}{j}_5}{\binom{0}{0}_5} I_{5,i}^{[d+]} + \sum_{s=1, s \neq i}^5 \frac{\binom{s}{j}_5}{\binom{0}{0}_5} I_{4,i}^{[d+],s} + \frac{\binom{i}{j}_5}{\binom{0}{0}_5} I_5^{[d+]},$$

“Scratched” vector:

$$Q_\tau^{s,\mu} = \sum_{i=1}^5 q_i^\mu \frac{\binom{\tau s}{i s}_5}{\binom{s}{s}_5}, \quad \tau = 0 \dots 5$$

Scalar Expressions: Contracting after integration

$$I_5^{\mu\nu} = I_5^\mu Q_0^\nu - \sum_{s=1}^5 \left\{ Q_0^{s,\mu} I_4^s - \sum_{t=1}^5 Q_t^{s,\mu} I_3^{st} \right\} Q_s^\nu,$$

Contracting with vectors $q_{a\mu} q_{b\nu}$

$$\begin{aligned} q_{a\mu} q_{b\nu} I_5^{\mu\nu} &= (q_a \cdot I_5)(q_b \cdot Q_0) \\ &\quad - \sum_{s=1}^5 \left\{ (q_a \cdot Q_0^s) I_4^s - \sum_{t=1}^5 (q_a \cdot Q_t^s) I_3^{st} \right\} (q_b \cdot Q_s), \end{aligned}$$

with

$$(q_a \cdot I_5) = E(q_a \cdot Q_0) - \sum_{s=1}^5 I_4^s (q_a \cdot Q_s).$$

Recall

$$(q_b \cdot Q_s) = \frac{1}{2} (\delta_{bs} - \delta_{5s}).$$

Further

$$(q_a \cdot Q_0^s) = \frac{1}{\binom{s}{s}_5} \Sigma_a^{2,s}, \quad (q_a \cdot Q_t^s) = \frac{1}{\binom{s}{s}_5} \Sigma_a^{1,st},$$

and with $Y_a = Y_{a5} - Y_{55}$ and $D_a^s = \delta_{as} - \delta_{5s}$.

$$\Sigma_a^{2,s} \equiv \sum_{i=1}^4 (q_a \cdot q_i) \binom{0s}{is}_5 = -\frac{1}{2} \left\{ \binom{s}{s}_5 Y_a + \binom{s}{0}_5 D_a^s \right\},$$

$$\Sigma_a^{1,st} \equiv \sum_{i=1}^4 (q_a \cdot q_i) \binom{ts}{is}_5 = \frac{1}{2} \left\{ \binom{s}{s}_5 D_a^t - \binom{s}{t}_5 D_a^s \right\}.$$

For all such sums see JF and T.Riemann, Phys.Lett. **B701** (2011) and Phys.Lett. **B707** (2012).

So far we assumed the 4-point sub-Gram $\binom{s}{s}_5$ to be "large"
 - but some cancellations occur already !

Another important point is that the sum over the vector indices i, j, \dots is performed in closed form.

$g_{\mu\nu} I_5^{\mu\nu}$:

We take the following scalar products

$$(Q_0 \cdot Q_0) = \frac{1}{2} \left[\frac{\binom{0}{0}_5}{\binom{0}{0}_5} + Y_{55} \right], \quad (Q_0 \cdot Q_s) = \frac{1}{2} \left[\frac{\binom{s}{0}_5}{\binom{0}{0}_5} - \delta_{s5} \right],$$

$$(Q_0^s \cdot Q_s) = -\frac{1}{2} \delta_{s5}, \quad (Q_t^s \cdot Q_s) = 0, \quad \text{and}$$

$$\begin{aligned} (I_5 \cdot Q_0) &= E(Q_0 \cdot Q_0) - \sum_{s=1}^5 I_4^s (Q_0 \cdot Q_s) \\ &= \frac{1}{2} \left\{ \frac{1}{\binom{0}{0}_5} \sum_{s=1}^5 \binom{s}{0}_5 I_4^s \cdot \left[\frac{\binom{0}{0}_5}{\binom{0}{0}_5} + Y_{55} \right] - \sum_{s=1}^5 I_4^s \left[\frac{\binom{s}{0}_5}{\binom{0}{0}_5} - \delta_{s5} \right] \right\} \end{aligned}$$

- cancellation of $\frac{1}{\binom{0}{0}_5}$ - and finally (not surprisingly)

$$g_{\mu\nu} I_5^{\mu\nu} = \frac{Y_{55}}{2} E + I_4^5.$$

The principle

We obtained a simple formula for the reduction of 5-point tensors to 4-point tensors, **5-PTR**.

One of its properties is the **the cancellation of $g^{\mu\nu}$** .

Of course it was necessary to **reintroduce $g^{\mu\nu}$** in order to provide a "**standard**" version of the tensor decomposition - and this resulted in the publicly available program **PJFry** for the numerical evaluation of the 5-point tensors.

Contracting higher N -point tensors one also needs this version in order to avoid the inverse Gram determinant of the 5-point function.

Nevertheless one may ask why to reintroduce **$g^{\mu\nu}$** instead of **systematically eliminating** it and thus simplifying the 5-point tensor.

Moreover: Separating the indices from the integrals, we saw that the summation over the indices may be performed analytically,

which saves the numerical iteration to calculate tensor coefficients of the 4-point functions.

As a more complicated example, we reproduce the following 4-point part

$$\begin{aligned}
 n_{ijkl} l_{4,ijkl}^{[d+]} &= \frac{\binom{0}{i} \binom{0}{j} \binom{0}{k} \binom{0}{l}}{\binom{0}{0} \binom{0}{0} \binom{0}{0} \binom{0}{0}} d(d+1)(d+2)(d+3) l_4^{[d+]} \\
 &+ \frac{\binom{0i}{0j} \binom{0}{k} \binom{0}{l} + \binom{0i}{0k} \binom{0}{j} \binom{0}{l} + \binom{0j}{0k} \binom{0}{i} \binom{0}{l} + \binom{0i}{0l} \binom{0}{j} \binom{0}{k} + \binom{0j}{0l} \binom{0}{i} \binom{0}{k} + \binom{0k}{0l} \binom{0}{i} \binom{0}{j}}{\binom{0}{0}^3} d(d+1) l_4^{[d+]} \\
 &+ \frac{\binom{0i}{0l} \binom{0j}{0k} + \binom{0j}{0l} \binom{0i}{0k} + \binom{0k}{0l} \binom{0i}{0j}}{\binom{0}{0}^2} l_4^{[d+]} + \dots
 \end{aligned}$$

Tensor of rank 3: remains to contract the 4-point tensor of rank 2

Special: $I_5^{\mu\nu\lambda} = I_5^{\mu\nu} \cdot Q_0^\lambda - \sum_{s=1}^5 I_4^{\mu\nu,s} \cdot Q_s^\lambda$.

The corresponding 4-point function reads ($q_5 = 0$):

$$I_4^{\mu\nu,s} = \sum_{i,j=1}^4 q_i^\mu q_j^\nu \nu_{ij} I_{4,ij}^{[d+]^2,s} - \frac{1}{2} g^{\mu\nu} I_4^{[d+],s},$$

$$\nu_{ij} I_{4,ij}^{[d+]^2,s} = - \frac{\binom{0s}{js}}{\binom{s}{s} 5} I_{4,i}^{[d+],s} + \frac{\binom{is}{js}}{\binom{s}{s} 5} I_4^{[d+],s} + \sum_{t=1}^5 \frac{\binom{ts}{js}}{\binom{s}{s} 5} I_{3,i}^{[d+],st}.$$

For $i, j = s$ the integrals $I_{4,i}^{[d+],s}$ and $I_{4,ij}^{[d+]^2,s}$ vanish (due to vanishing signed minors) and summation over all values of i, j is possible. The following identity allows to eliminate $g^{\mu\nu}$

$$\frac{\binom{is}{js}}{\binom{s}{s} 5} = \frac{\binom{j}{j}}{\binom{0}{0} 5} - \frac{\binom{s}{i} \binom{s}{j}}{\binom{0}{5} \binom{s}{s} 5}$$

$q_{a\mu} q_{b\nu} I_5^{\mu\nu}$:

$$\begin{aligned} q_{a\mu} q_{b\nu} I_4^{\mu\nu,s} &= (q_a \cdot I_4^s) (q_b \cdot Q_0^s) - \sum_{t=1}^5 (q_a \cdot I_3^{st}) (q_b \cdot Q_t^s) \\ &\quad - \frac{\binom{()}{5}}{\binom{s}{s} 5} (q_a \cdot Q_s) (q_b \cdot Q_s) I_4^{[d+],s} \end{aligned}$$

We need the additional sums

$$(q_a \cdot Q_0^{st}) = \frac{1}{\binom{st}{st} 5} \Sigma_a^{3,st}, \quad (q_a \cdot Q_u^{st}) = \frac{1}{\binom{st}{st} 5} \Sigma_a^{1,stu}$$

with

$$\begin{aligned} \Sigma_a^{3,st} &= -\frac{1}{2} \left\{ \binom{st}{st} 5 Y_a + \binom{st}{s0} 5 D_a^t + \binom{st}{0t} 5 D_a^s \right\}, \\ \Sigma_a^{1,stu} &= \frac{1}{2} \left\{ \binom{st}{st} 5 D_a^u - \binom{st}{su} 5 D_a^t - \binom{ts}{tu} 5 D_a^s \right\}. \end{aligned}$$

Result for $q_{a\mu} q_{b\nu} I_4^{\mu\nu,s}$

$$q_{a\mu} q_{b\nu} I_4^{\mu\nu,s} = q_{a\mu} q_{b\nu} \bar{I}_4^{\mu\nu,s} - \frac{1}{4} (\delta_{as} - \delta_{5s})(\delta_{bs} - \delta_{5s}) \frac{\binom{0}{5}}{\binom{s}{s} 5} I_4^{[d+],s} \quad \text{with}$$

$$\begin{aligned} q_{a\mu} q_{b\nu} \bar{I}_4^{\mu\nu,s} = & \frac{1}{4} (Y_{a5} - Y_{55})(Y_{b5} - Y_{55}) \cdot I_4^s \\ & + \frac{1}{4} (\delta_{as} - \delta_{5s})(\delta_{bs} - \delta_{5s}) \frac{1}{\binom{s}{s} 5} \left[\binom{s}{0}_5 R^s - \sum_{t=1}^5 \binom{s}{t}_5 R^{st} \right] \\ & + \frac{1}{4} [(\delta_{as} - \delta_{5s})(Y_{b5} - Y_{55}) + (Y_{a5} - Y_{55})(\delta_{bs} - \delta_{5s})] R^s \\ & + \frac{1}{4} \sum_{t=1}^5 [(\delta_{as} - \delta_{5s})(\delta_{bt} - \delta_{5t}) + (\delta_{bs} - \delta_{5s})(\delta_{at} - \delta_{5t})] R^{st} \\ & + \frac{1}{4} \sum_{t=1}^5 [(\delta_{at} - \delta_{5t})(Y_{b5} - Y_{55}) + (Y_{a5} - Y_{55})(\delta_{bt} - \delta_{5t})] I_3^{st} \\ & + \frac{1}{4} \sum_{t=1}^5 (\delta_{at} - \delta_{5t})(\delta_{bt} - \delta_{5t}) R^{ts} + \frac{1}{4} \sum_{t,u=1}^5 (\delta_{bt} - \delta_{5t})(\delta_{au} - \delta_{5u}) I_2^{stu}. \end{aligned}$$

$$\begin{aligned}
 R^s &\equiv \frac{1}{\binom{s}{s}_5} \left[\binom{s}{0}_5 I_4^s - \sum_{t=1}^5 \binom{s}{t}_5 I_3^{st} \right] = \frac{1}{\binom{0s}{0s}_5} \left[\binom{s}{0}_5 I_4^{[d+],s} - \sum_{t=1}^5 \binom{0s}{0t}_5 I_3^{st} \right] \\
 R^{st} &\equiv \frac{1}{\binom{st}{st}_5} \left[\binom{st}{0t}_5 I_3^{st} - \sum_{u=1}^5 \binom{st}{ut}_5 I_2^{stu} \right] \\
 &= \frac{1}{\binom{0st}{0st}_5} \left[\binom{st}{0t}_5 (d-2) I_3^{[d+],st} - \sum_{u=1}^5 \binom{0st}{0ut}_5 I_2^{stu} \right].
 \end{aligned}$$

and

$$\begin{aligned}
 &\frac{1}{\binom{s}{s}_5} \left\{ -()_5 I_4^{[d+],s} + \binom{s}{0}_5 R^s - \sum_{t=1}^5 \binom{s}{t}_5 R^{st} \right\} \\
 &= \frac{-1}{\binom{0s}{0s}_5} \left\{ ()_5 (d-2)(d-1) I_4^{[d+]^2,s} - \binom{0}{0}_5 I_4^{[d+],s} + \sum_{t=1}^5 \binom{t}{0}_5 (d-2) I_3^{[d+],st} \right. \\
 &\quad \left. + \sum_{t=1}^5 \binom{0s}{0t}_5 R^{st} \right\}
 \end{aligned}$$

Elimination of $\frac{1}{\binom{s}{s}_5}$ solved - $a = b = c = s_0 = 5$ only!

Results for $g_{\mu\nu} I_5^{\mu\nu\lambda}$

Contracting $g_{\mu\nu} I_4^{\mu\nu}$, we take the following scalar products

$$(Q_0^s \cdot Q_0^s) = \frac{1}{2 \binom{s}{s}_5} \left[\binom{0s}{0s}_5 + 2 \binom{s}{0}_5 \delta_{s5} \right] + \frac{1}{2} Y_{55},$$

$$(Q_s \cdot Q_s) = \frac{1}{2} \frac{\binom{s}{s}_5}{\binom{}{5}},$$

$$(Q_t^s \cdot Q_0^s) = \frac{1}{2 \binom{s}{s}_5} \left[\binom{ts}{0s}_5 - \binom{s}{s}_5 \delta_{t5} + \binom{s}{t}_5 \delta_{s5} \right],$$

$$(Q_t^s \cdot Q_0^{st}) = \frac{1}{2 \binom{s}{s}_5} \left[\quad - \binom{s}{s}_5 \delta_{t5} + \binom{s}{t}_5 \delta_{s5} \right],$$

$$(Q_t^s \cdot Q_u^{st}) = 0,$$

which yields

$$g_{\mu\nu} I_4^{\mu\nu,s} = \frac{Y_{55}}{2} I_4^s + I_3^{s5} + \delta_{s5} R^s$$

General Approach for N -point tensors

Following ideas presented in

Z. Bern, L. J. Dixon, D. A. Kosower, Nucl. Phys. B412 (1994) 751–816,

an iterative approach has been systematically worked out in

T. Binoth, J. Guillet, G. Heinrich, E. Pilon, C. Schubert, JHEP 10 (2005) 015.

$$I_N^{\mu_1 \mu_2 \dots \mu_R} = - \sum_{r=1}^N C_r^{\mu_1} I_{N-1}^{\mu_2 \dots \mu_R, r} \quad \text{with the conditions}$$

$$\sum_{j=1}^N C_j^\mu q_j^\nu = \frac{1}{2} g_{[4]}^{\mu\nu}, \quad \sum_{j=1}^N C_j^\mu = 0.$$

The solution of this set is not unique. Assume

$$\frac{1}{2} g^{\mu\nu} = \sum_{i,j=1}^{N-1} G_{ij} q_i^\mu q_j^\nu, \quad \text{then}$$

$$C_j^\mu = \sum_{i=1}^{N-1} G_{ij} q_i^\mu.$$

$N = 6$

We have to find a **proper representation of the metric tensor**. E.g. we have

$$\sum_{i,j=1}^{n-1} (q_a \cdot q_i)(q_b \cdot q_j) \begin{pmatrix} si \\ sj \end{pmatrix}_n = \frac{1}{2} (q_a \cdot q_b) \begin{pmatrix} s \\ s \end{pmatrix}_n - \frac{1}{4} ()_n (\delta_{as} - \delta_{ns}) (\delta_{bs} - \delta_{ns}).$$

Observation: all our sums are valid for arbitrary n and have to be considered as identities in terms of arbitrary (symmetric) Y_{ij} .

From $()_6 = 0$ for the "physical" ones, we obtain

$$\frac{1}{2} g^{\mu\nu} = \sum_{i,j=1}^5 \frac{1}{\binom{s}{s}_6} \begin{pmatrix} si \\ sj \end{pmatrix}_6 q_i^\mu q_j^\nu, \quad \text{or}$$

$$C_r^{s,\mu} = \sum_{i=1}^5 q_i^\mu \frac{\binom{sr}{si}_6}{\binom{s}{s}_6} = Q_r^{s,\mu}, \quad s = 1 \dots 6.$$

$$\sum_{r=1}^6 \begin{pmatrix} sr \\ si \end{pmatrix}_6 = 0, \quad s = 1, \dots 6$$

yields the second condition. Further : $Q_r^{0,\mu}$ or $\frac{\binom{sr}{si}_6}{\binom{s}{s}_6} \Rightarrow \frac{\binom{0r}{si}_6}{\binom{0}{s}_6}$.

$N = 7, 8 \dots$

Starting with a sum, again, to find the proper metrik tensor:

$$\begin{aligned} \sum_{i,j=1}^{n-1} (q_a \cdot q_i)(q_b \cdot q_j) \begin{pmatrix} sti \\ stj \end{pmatrix}_n &= \frac{1}{2} (q_a \cdot q_b) \begin{pmatrix} st \\ st \end{pmatrix}_n \\ -\frac{1}{4} \left\{ \left[\begin{pmatrix} s \\ s \end{pmatrix}_n (\delta_{at} - \delta_{nt}) - \begin{pmatrix} s \\ t \end{pmatrix}_n (\delta_{as} - \delta_{ns}) \right] (\delta_{bt} - \delta_{nt}) \right. \\ \left. + \left[\begin{pmatrix} t \\ t \end{pmatrix}_n (\delta_{as} - \delta_{ns}) - \begin{pmatrix} s \\ t \end{pmatrix}_n (\delta_{at} - \delta_{ns}) \right] (\delta_{bs} - \delta_{ns}) \right\}, \end{aligned}$$

With dimension 4 of the chords, all $(\)_n$, $n \geq 7$, have rank 6. (Melrose, 1965).
The $(\)_7$ is of order 8 and thus

$$\begin{pmatrix} s \\ t \end{pmatrix}_7 = 0 \text{ and } \begin{pmatrix} s \\ s \end{pmatrix}_7 = 0,$$

and therefore the whole curly bracket vanishes with the result

$$C_r^{st,\mu} = \sum_{i=1}^6 \frac{1}{\begin{pmatrix} st \\ st \end{pmatrix}_7} \begin{pmatrix} sti \\ str \end{pmatrix}_7 q_i^\mu = Q_r^{st,\mu}.$$

Similar for the 8-point function, etc..

Complete contraction

$$I_7^{\mu_1 \mu_2 \dots \mu_R} = - \sum_{r_1=1}^7 Q_{r_1}^{s,t,\mu_1} I_6^{\mu_2 \dots \mu_R, r_1} \quad s, t : \text{redundancy indices},$$

$$I_6^{\mu_2 \dots \mu_R, r_1} = - \sum_{r_2=1}^7 Q_{r_2}^{r_1, u, \mu_2} I_5^{\mu_3 \dots \mu_R, r_1, r_2} \quad \text{with}$$

$$Q_{r_1}^{s,t,\mu_1} = \sum_{i=1}^6 \frac{1}{\binom{st}{st}_7} \binom{st}{st r_1}_7 q_i^\mu$$

$$Q_{r_2}^{r_1, u, \mu_2} = \sum_{i=1}^6 \frac{1}{\binom{r_1 u}{r_1 u}_7} \binom{r_1 u i}{r_1 u r_2}_7 q_i^\mu$$

$$\begin{aligned} (q_a \cdot Q_r^{st}) &= \frac{1}{2 \binom{st}{st}_7} \left[\binom{st}{st}_7 (\delta_{ar} - \delta_{7r}) - \binom{st}{sr}_7 (\delta_{at} - \delta_{7t}) - \binom{ts}{tr}_7 (\delta_{as} - \delta_{7s}) \right] \\ &= \frac{1}{2} (\delta_{ar} - \delta_{7r}) \quad \text{for } s, t \neq a, 7 \end{aligned}$$

The 5-point vector: no redundancy index

Two versions

$$a) \quad I_5^{\mu, r_1 r_2} = E^{r_1 r_2} Q_0^{r_1 r_2, \mu} - \sum_{r_3=1}^7 I_4^{r_1 r_2 r_3} Q_{r_3}^{r_1 r_2, \mu} \quad \text{with}$$

$$Q_0^{r_1 r_2, \mu} = \sum_{i=1}^6 q_i^\mu \frac{\binom{r_1 r_2 0}{r_1 r_2 i}}{\binom{r_1 r_2}{r_1 r_2} 7}, \quad Q_{r_3}^{r_1 r_2, \mu} = \sum_{i=1}^6 q_i^\mu \frac{\binom{r_1 r_2 r_3}{r_1 r_2 i}}{\binom{r_1 r_2}{r_1 r_2} 7}$$

This version contains the inverse Gram of the 5-point function! It can be avoided by using the version

$$b) \quad I_5^{\mu, r_1 r_2} = - \sum_{r_3=1}^7 Q_{r_3}^{0 r_1 r_2, \mu} I_4^{r_1 r_2 r_3}, \quad Q_{r_3}^{0 r_1 r_2, \mu} = \sum_{i=1}^6 q_i^\mu \frac{\binom{0 r_1 r_2 r_3}{0 r_1 r_2 i}}{\binom{0 r_1 r_2}{0 r_1 r_2} 7}$$

$$\begin{aligned} \sum_{i=1}^6 (q_a \cdot q_i) \binom{0stu}{0sti} 7 &= \frac{1}{2} \left\{ \binom{stu}{st0} 7 (Y_{a7} - Y_{77}) \right. \\ &+ \left. \binom{0st}{0st} 7 (\delta_{au} - \delta_{7u}) - \binom{0st}{0su} 7 (\delta_{at} - \delta_{7t}) - \binom{0ts}{0tu} 7 (\delta_{as} - \delta_{7s}) \right\}. \end{aligned}$$

The 5-point tensor of rank 2 : avoiding $\frac{1}{\binom{r_1 r_2}{r_1 r_2}_7}$ and cancel $g^{\mu\nu}$

$$I_5^{\mu\nu} = \sum_{i,j=1}^4 q_i^\mu q_j^\nu E_{ij} + g^{\mu\nu} E_{00} \quad \text{with}$$

$$\begin{aligned} E_{00} &= - \sum_{s=1}^5 \frac{1}{2} \frac{1}{\binom{0}{0}_5} \binom{s}{0}_5 I_4^{[d+],s}, \\ E_{ij} &= \sum_{s=1}^5 \frac{1}{\binom{0}{0}_5} \left[\binom{0i}{sj}_5 I_4^{[d+],s} + \binom{0s}{0j}_5 I_{4,i}^{[d+],s} \right]. \end{aligned}$$

Contractions with chords yields:

$$q_{a\mu} q_{b\nu} I_5^{\mu\nu} = \sum_{i,j=1}^4 (q_a \cdot q_i)(q_b \cdot q_j) E_{ij} + (q_a \cdot q_b) E_{00}.$$

With

$$\sum_{i,j=1}^4 (q_a \cdot q_i)(q_b \cdot q_j) \binom{0i}{sj}_5 = \frac{1}{2} (q_a \cdot q_b) \binom{s}{0}_5 + \frac{1}{4} ()_5 Y_b D_a^s.$$

The scratched sum for the 7-point function

$$\begin{aligned}
 & \sum_{i,j=1}^6 (q_a \cdot q_i)(q_b \cdot q_j) \begin{pmatrix} 0itu \\ sjtu \end{pmatrix}_7 = \frac{1}{2} (q_a \cdot q_b) \begin{pmatrix} stu \\ 0tu \end{pmatrix}_7 \\
 & + \frac{1}{4} \left[\begin{pmatrix} tu \\ tu \end{pmatrix}_7 (\delta_{as} - \delta_{7s}) - \begin{pmatrix} su \\ tu \end{pmatrix}_7 (\delta_{at} - \delta_{7t}) - \begin{pmatrix} st \\ ut \end{pmatrix}_7 (\delta_{au} - \delta_{7u}) \right] (Y_{b7} - Y_{77}) \\
 & + \frac{1}{4} \left[\begin{pmatrix} ut \\ 0t \end{pmatrix}_7 (\delta_{bu} - \delta_{7u}) + \begin{pmatrix} tu \\ 0u \end{pmatrix}_7 (\delta_{bt} - \delta_{7t}) \right] (\delta_{as} - \delta_{7s}) \\
 & - \frac{1}{4} \left[\begin{pmatrix} su \\ 0u \end{pmatrix}_7 (\delta_{at} - \delta_{7t})(\delta_{bt} - \delta_{7t}) + \begin{pmatrix} st \\ 0t \end{pmatrix}_7 (\delta_{au} - \delta_{7u})(\delta_{bu} - \delta_{7u}) \right] \\
 & + \frac{1}{4} \left[\begin{pmatrix} st \\ 0u \end{pmatrix}_7 (\delta_{au} - \delta_{7u})(\delta_{bt} - \delta_{7t}) + \begin{pmatrix} su \\ 0t \end{pmatrix}_7 (\delta_{at} - \delta_{7t})(\delta_{bu} - \delta_{7u}) \right].
 \end{aligned}$$

Conclusion

- Starting with the representation given by **Davydychev**,
- a systematic application of the algebra of signed minors according to **Melrose**
- was first presented in **JF et al.** in connection with the recursion relations of **Tarasov**.
- **Cancellation of the $g^{\mu\nu}$** for the 5-point function was observed.
- After separation of the indices from the integrals
- for the **contraction** a method for the **summation over the indices** was developed, yielding analytic expressions, **JF & T.Riemann** .
- An almost "trivial" **cancellation of $(\cdot)_5$** is observed.
- Avoidance of the inverse 4-point Gram $\binom{s}{s}_5$ is achieved.
- Applicability for any **higher N -point functions** is demonstrated.
- **Package in preparation** in collaboration with J.Gluza.

Small $\binom{s}{s}_5$

JF and T.Riemann: Phys. Rev. D83 (2011):

$$Z_4^{[d+]^l} = \frac{1}{\binom{0}{0}} \sum_{t=1}^4 \binom{t}{0} I_3^{[d+]^l, t}, \quad l = 1, \dots$$

$$I_4^{[d+]^L} = \sum_{j=0}^{\infty} a_j^L r^j \left[Z_4^{(L+j)} - b_j^L Z_{4D}^{(L+j)} \right], \quad L = 0, \dots, 4.$$

with $r = \frac{()}{\binom{0}{0}}$ and

$$a_j^L = 2^j \frac{\Gamma(L+j+\frac{1}{2})}{\Gamma(L+\frac{1}{2})}, \quad b_j^L = \psi(L+j+\frac{1}{2}) - \psi(L+\frac{1}{2}),$$

The $\psi(z)$ is the logarithmic derivative of the Gamma function.

The 4-point $g^{\mu\nu}$

Using the **5-PTR**, takes into account already some cancellation of the $g_{\mu\nu}$,
 $g_{\mu\nu}$ -contributions from 4-points simpler to handle!.

A remark concerning the **4-point function**: here only **3** of the 4 vectors $q_i, i = 1, \dots, 4$ are **independent**:

$$g_{\mu\nu} \rightarrow G^{\mu\nu} = g^{\mu\nu} - 2 \sum_{i,j=1}^4 q_i^\mu q_j^\nu \frac{\binom{i}{j}}{\binom{4}{2}} = \frac{8v^\mu v^\nu}{\binom{4}{2}},$$

with

$$v^\mu = \varepsilon^{\mu\lambda\rho\sigma} (q_1 - q_4)_\lambda (q_2 - q_4)_\rho (q_3 - q_4)_\sigma.$$

Assume $q_4 = 0 \rightarrow (q_i \cdot v) = 0$, i.e. $q_{i\mu} G^{\mu\nu} = 0$:
effectively vanishing !