

# TOP QUARK PAIRS AT TWO LOOPS AND REDUZE 2

Andreas v. Manteuffel



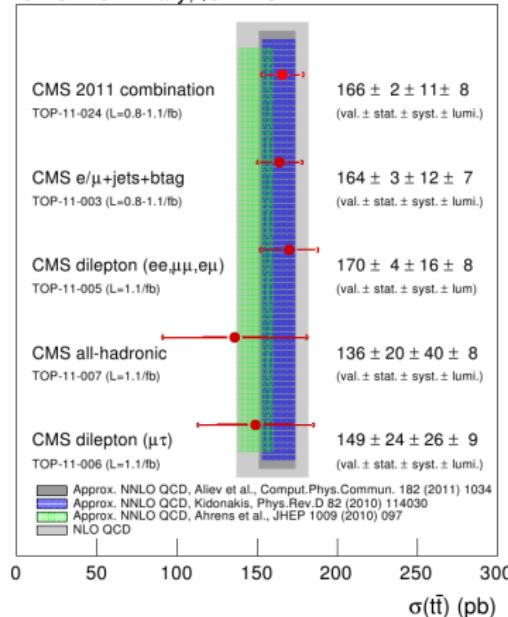
Loops and Legs in Quantum Field Theory



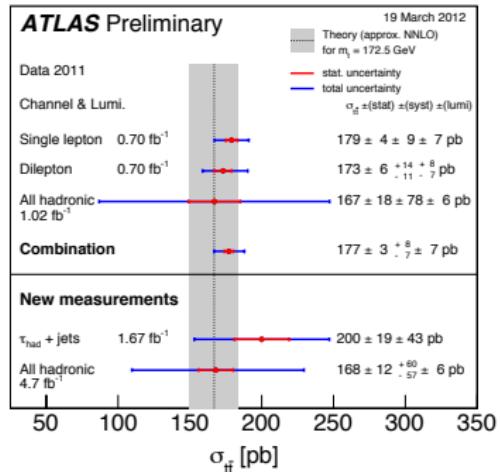
*11th Workshop organized by Theory Group, DESY, Zeuthen*  
15-20 April 2012, Wernigerode

# TOP PAIR PRODUCTION AT HADRON COLLIDERS

CMS Preliminary,  $\sqrt{s}=7$  TeV



ATLAS Preliminary



- **LHC precision** below NLO accuracy already: see talk by **G. Dissertori**
- **full  $WWbb$**  at NLO: see talks by **S. Dittmaier, A. van Hameren**
- **threshold/resummations**: see talks by **M. Neubert, A. Mitov** and refs. therein

# INGREDIENTS FOR FULL NNLO CALCULATION

- **VV**: two-loop ME for  $q\bar{q} \rightarrow t\bar{t}$  and  $gg \rightarrow t\bar{t}$

- **RV**: one-loop ME for  $t\bar{t} + 1$  parton

Dittmaier, Uwer, Weinzierl '07

- **RR**: tree level ME for  $t\bar{t} + 2$  partons

- **subtraction terms**: up to 2 unresolved partons needed

Gehrman-De Ridder, Ritzmann '09; Daleo et al. '09; Boughezal et al. '10; Glover, Pires '10; Czakon '10, '11; Anastasiou, Herzog, Lazopoulos '10; Abelof, Gehrman-De Ridder '11; Gehrman, Monni '11; Bierenbaum, Czakon, Mitov '11

→ see talks by I. Bierenbaum, G. Abelov, A. Mitov, J. Pires

consider **VV** ( $2 \rightarrow 2$  ingredients) :

$$\sum_{\text{spin, colour}} |\mathcal{M}|^2 = 16\pi^2 \alpha_s^2 \left[ \mathcal{A}_0 + \left( \frac{\alpha_s}{\pi} \right) \mathcal{A}_1 + \left( \frac{\alpha_s}{\pi} \right)^2 \mathcal{A}_2 + \mathcal{O}(\alpha_s^3) \right]$$

with

$$\mathcal{A}_0 = \langle \mathcal{M}^{(0)} | \mathcal{M}^{(0)} \rangle$$

$$\mathcal{A}_1 = 2 \operatorname{Re} \langle \mathcal{M}^{(0)} | \mathcal{M}^{(1)} \rangle$$

$$\mathcal{A}_2 = \langle \mathcal{M}^{(1)} | \mathcal{M}^{(1)} \rangle + 2 \operatorname{Re} \langle \mathcal{M}^{(0)} | \mathcal{M}^{(2)} \rangle$$

$\langle \mathcal{M}^{(1)} | \mathcal{M}^{(1)} \rangle$ : Kniehl, Körner, Merebashvili, Rogal '05-'08, Anastasiou, Aybat '08

# GAUGE INVARIANT SUBSETS IN TWO-LOOP CONTRIBUTIONS

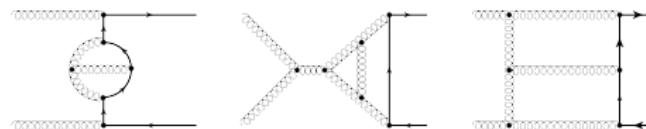
**gg channel:** 789 two-loop diagrams (+ ghost init.) contrib. to 16 coeff.:

$$\begin{aligned} 2 \operatorname{Re} \left\langle \mathcal{M}^{(0)} | \mathcal{M}^{(2)} \right\rangle = & 2 C_F N_c (N_c^3 \mathbf{A} + N_c \mathbf{B} + \frac{1}{N_c} \mathbf{C} + \frac{1}{N_c^3} \mathbf{D} \\ & + N_c^2 n_I \mathbf{E}_I + n_I \mathbf{F}_I + \frac{n_I}{N_c^2} \mathbf{G}_I + N_c n_I^2 \mathbf{H}_I + \frac{n_I^2}{N_c} \mathbf{I}_I \\ & + N_c^2 n_h \mathbf{E}_h + n_h \mathbf{F}_h + \frac{n_h}{N_c^2} \mathbf{G}_h + N_c n_h^2 \mathbf{H}_h + \frac{n_h^2}{N_c} \mathbf{I}_h \\ & + N_c n_I n_h \mathbf{H}_{Ih} + \frac{n_I n_h}{N_c} \mathbf{I}_{Ih}) \end{aligned}$$

**$q\bar{q}$  channel:** 218 two-loop diagrams, 10 coefficients

example: for **leading  $N_c$  coefficient  $A$**  we need:

- 300 two-loop diagrams (+ ghost initiated), e.g.:



- two independent ratios of scales
- up to: 4-point, 7 propagators, 4 loop momenta in numerator

# STEPS TOWARD COMPLETE NNLO CALCULATION for $2 \operatorname{Re} \langle \mathcal{M}^{(0)} | \mathcal{M}^{(2)} \rangle$ :

## small mass expansion :

- Czakon, Mitov, Moch (2006) for  $q\bar{q}$ ,  $gg$

## IR poles :

- Ferroglia, Neubert, Pecjak, Yang (2009) for  $q\bar{q}$ ,  $gg$

## $q\bar{q}$ with full dependence on $s$ , $t$ , $m_t$ , $\mu$ :

- numerical result for all contributions:  
Czakon (2008)
- analytical result for fermionic:  
Bonciani, Ferroglia, Gehrmann, Maitre, Studerus (2008)
- analytical result for leading  $N_c$ :  
Bonciani, Ferroglia, Gehrmann, Studerus (2009)

## $gg$ with full dependence on $s$ , $t$ , $m_t$ , $\mu$ :

- analytical result for leading  $N_c$ :  
Bonciani, Ferroglia, Gehrmann, A.v.M., Studerus (2010)
- analytical result for light fermionic:  
Bonciani, Ferroglia, Gehrmann, A.v.M., Studerus (in preparation)
- numerical result for all contributions:  
Czakon, Bärnreuther (in preparation)

# ORGANISATION OF OUR CALCULATION

## RECIPE

- ① generate **Feynman diagrams** with QGRAF by Nogueira
- ② build **interference** terms
- ③ **reduce** scalar integrals to masters with **parallel Laporta**
- ④ **solve masters** with differential equations
- ⑤ **renormalize**:  $\overline{MS}$ , pole mass

⇒ **analytical result** in terms of generalized polylogarithms,  
allows fast **numerical evaluation, expansions**, ...

various tasks automatized in computer program **Reduze 2**

## LAPORTA ALGORITHM

### index integrals:

- define **integral family**: set of propagators  $\{1/D_1, \dots, 1/D_N\}$  such that:  
all scalar products with loop momenta are linear combinations of  $D_i$
- **sector** defined by subset of propagators (denominators)
- counting propagator exponents indexes integrals:

Feynman integrals of some topologies  $\rightarrow \mathbb{Z}^N$

$$\int d^d k_1 \cdots d^d k_L \frac{1}{D_1^{n_1} \cdots D_N^{n_N}} \mapsto \{n_1, \dots, n_N\} \quad \text{with } n_i \in \mathbb{Z}$$

## LAPORTA ALGORITHM

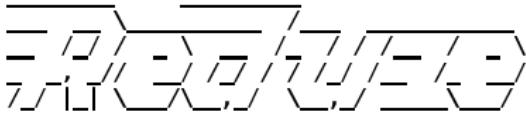
- ① define **ordering** for integrals  $I(n_1, \dots, n_N)$
- ② generate **integration by parts identities** (IBPs): **sparse system** of equations
- ③ **solve linear system** of equations

### based on:

- Chetyrkin, Tkachov '81

### public implementations:

- Anastasiou: AIR, Smirnov: FIRE, Studerus: Reduze 1



## Reduze 2 - Distributed Feynman Integral Reduction

A.v.M., Studerus

arXiv:1201.4330

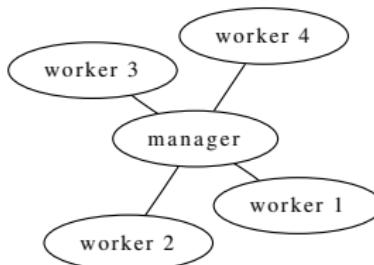
<http://projects.hepforge.org/reduze>

## REDUZE 2: PARALLELIZATION OF LAPORTA-ALGORITHM

- generate the system of equations
- sort equations in **blocks** with the **same leading integral**

$$\begin{array}{lcl} \mathbf{l}_5 + c_{14}\mathbf{l}_4 + c_{13}\mathbf{l}_3 & = 0 \\ \mathbf{l}_5 + c_{24}\mathbf{l}_4 & + c_{22}\mathbf{l}_2 & = 0 \\ \mathbf{l}_5 & + c_{33}\mathbf{l}_3 + c_{32}\mathbf{l}_2 & = 0 \\ \hline \mathbf{l}_3 + c_{42}\mathbf{l}_2 & = 0 \\ \mathbf{l}_3 & + c_{51}\mathbf{l}_1 = 0 \\ \hline \mathbf{l}_2 + c_{61}\mathbf{l}_1 & = 0 \end{array}$$

- send blocks to workers



# REDUZE 2 AND EXTERNAL SOFTWARE

- **key features**

- ▶ fully parallelized reductions, resumable
- ▶ topological analysis of diagrams and sectors

- **open source C++**

- **libraries / programs** used (no proprietary mandatory):

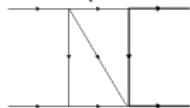
- ▶ GiNaC by Bauer, Frink, Kreckel
- ▶ yaml-cpp
- ▶ optional: MPI
- ▶ optional: Berkeley DB
- ▶ optional: Fermat CAS by Lewis (closed source)

- **interfaces**

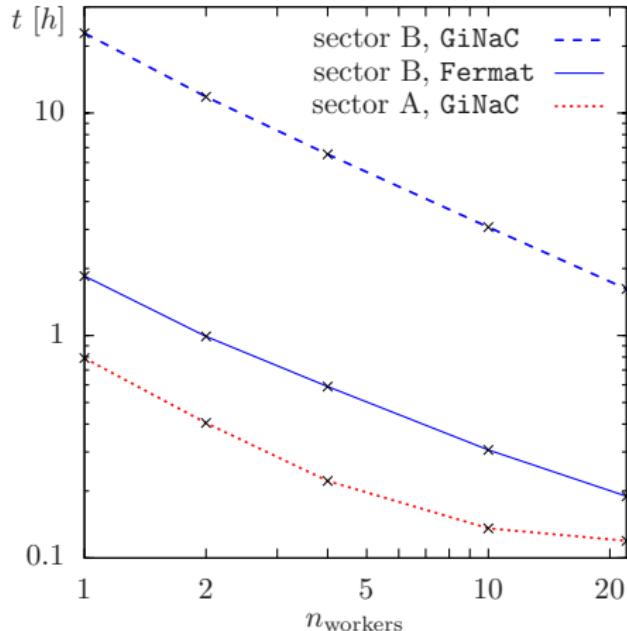
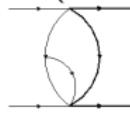
- ▶ input: QGRAF, YAML
- ▶ output: FORM, Mathematica, Maple

## REDUZE 2: PERFORMANCE FOR SINGLE SECTORS

sector B (for  $s \leq 4$ ):



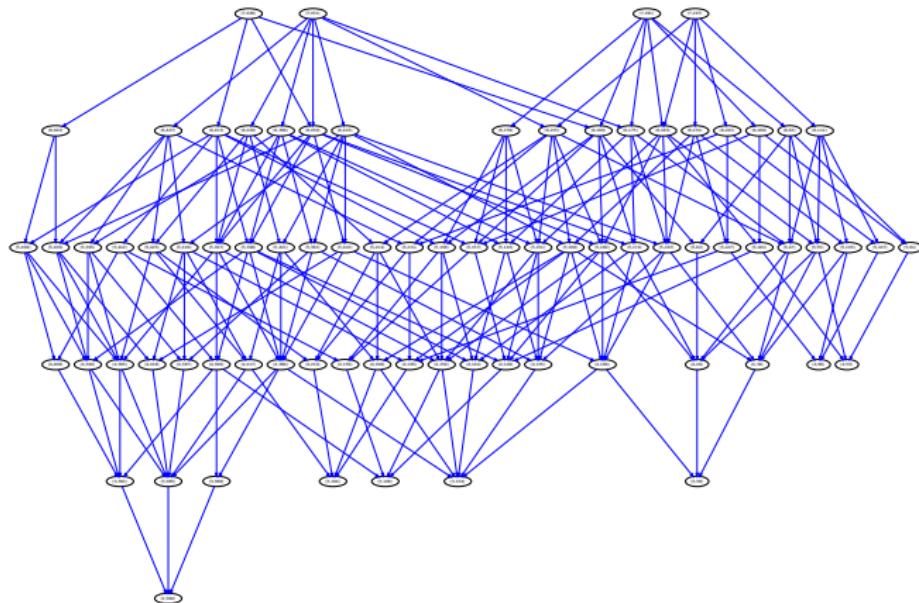
sector A (for  $s \leq 3$ ):



- Fermat very fast GCD (speedup often  $> 10\times$ )
- scaling problem specific
- single sector:  $t\bar{t}$  typically benefits from up to 10-20 cores (speedup 5x-10x)

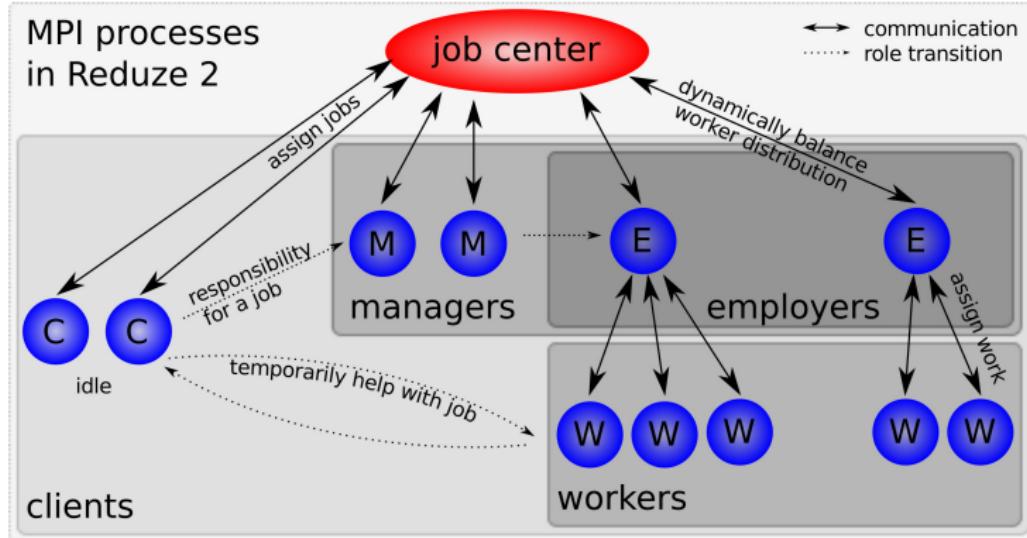
## REDUZE 2: PARALLEL EXECUTION OF JOBS

consider reduction of multiple sectors:



⇒ parallel reduction of several sectors ("jobs"), balance workers between them

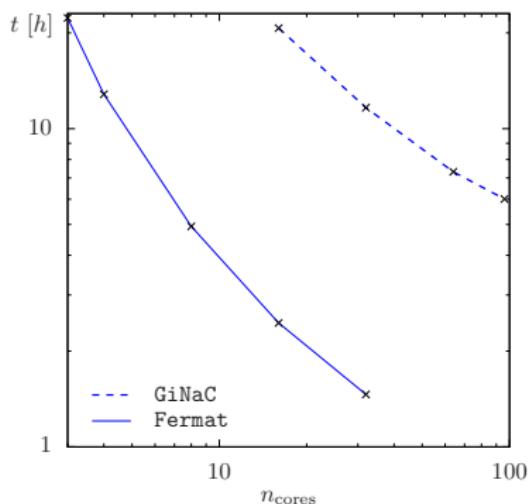
# MPI PROCESSES IN REDUZE 2



## REDUZE 2: PERFORMANCE FOR MULTIPLE SECTORS

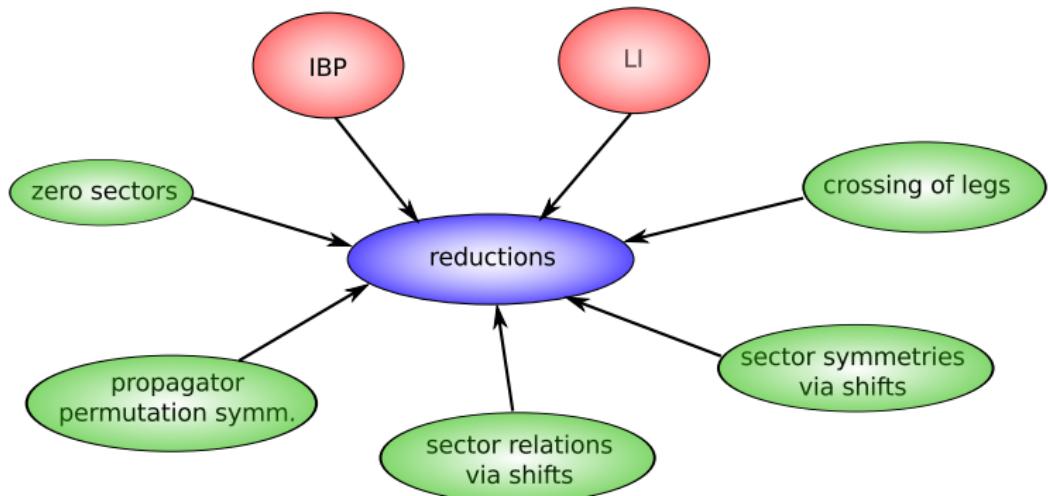


reductions incl. all subsectors (for  $s \leq 4$ ):

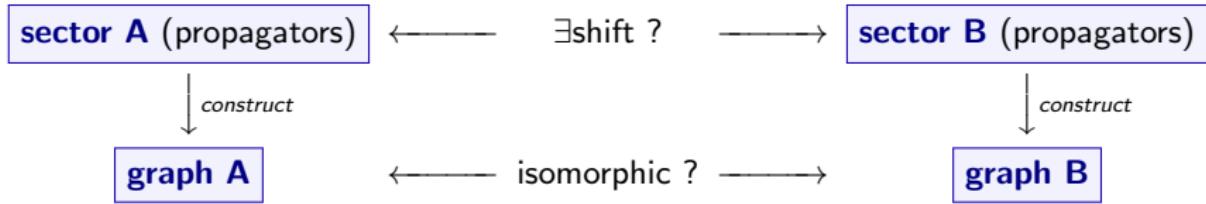


- scaling problem specific
- multiple sectors: may benefit from up to 100 cores (speedup up to  $\mathcal{O}(50\times)$ )

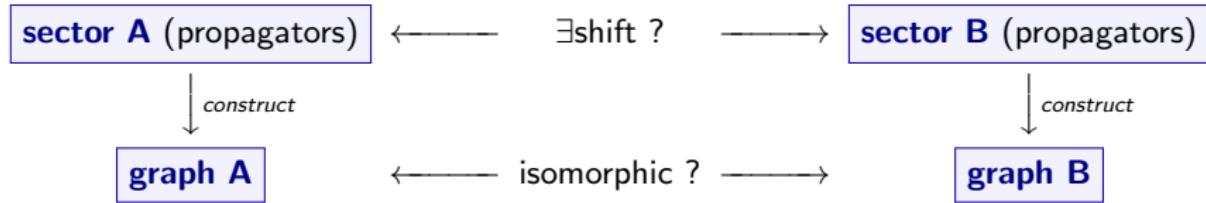
## REDUZE 2: REMOVING AMBIGUITIES FOR INTEGRALS



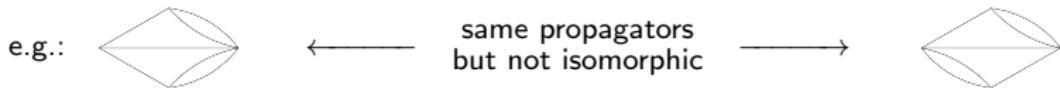
## REDUZE 2: SECTORS, GRAPHS AND MATROIDS



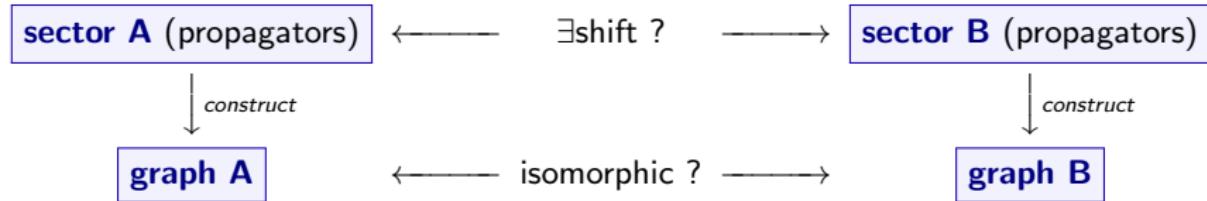
## REDUZE 2: SECTORS, GRAPHS AND MATROIDS



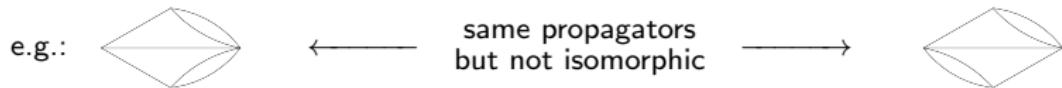
- **problem:** graphs not unique !



## REDUZE 2: SECTORS, GRAPHS AND MATROIDS



- **problem:** graphs not unique !



- **solution:** select unique representative by allowing for **twists**

idea based on **theorem** by Bogner, Weinzierl (2010):

**first Symanzik polynomials  $\mathcal{U}$  isomorphic  $\Leftrightarrow$  graphs isomorphic up to twists**

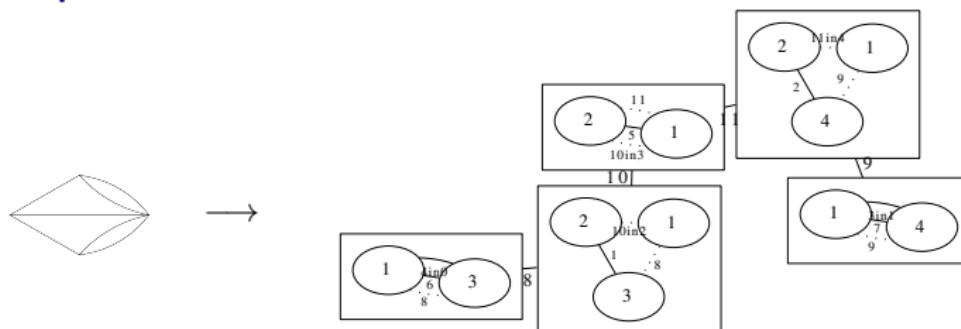
proof based on **Whitney's theorem** for isomorphisms of cycle matroids

# REDUZE 2: SHIFT FINDING ALGORITHM

## SHIFT FINDER

- ① generate graph for sector
- ② colour edges according to masses
- ③ connect external legs with a new vertex
- ④ decompose into triconnected components (Hopcroft, Tarjan '73; Gutwenger, Mutzel '01)
- ⑤ minimize graph by twists
- ⑥ check for graph isomorphism (McKay '81)

example:



tree of triconnected components  
(dashed “virtual edges” mark positions for **Tutte** twists)

# APPLICATION EXAMPLE

auto-generated shifts for non-planar double box family:

The screenshot shows a Windows command-line window titled "figures : less". The window contains a large amount of text representing auto-generated shift code for a non-planar double box family. The code is organized into several sections: "sector\_mappings:", "sector\_relations:", "crossed\_sector\_relations:", and "sector\_symmetries:". Each section contains multiple entries, each consisting of a sector index (e.g., 3, 6, 7, 12, etc.) followed by a list of two-sector pairs. The sectors are represented by lists of indices, such as [k1, k2]. The code also includes comments indicating truncated lists and specific indices like 207, 335, 463, and 20.

```
...  
sector_mappings:  
  name: box2n  
  zero_sectors:  
    t=0: [0]  
    t=1: [1, 4, 16, 64, 256]  
    t=2: [3, 5, 6, 12, 17, 18, 20, 24, 48, 65, 66, 68, 72, 80, 96, 192, 257, 260, 272, 320]  
    [...] truncated  
  sectors_without_graph:  
    t=3: [22, 28, 52, 82, 84, 88, 104, 112, 276]  
    t=4: [23, 29, 30, 53, 54, 60, 83, 85, 86, 89, 90, 92, 105, 106, 108, 113, 114, 116, 120, 21  
    t=5: [31, 55, 61, 87, 91, 93, 94, 107, 109, 110, 115, 117, 118, 121, 122, 124, 211, 213, 21  
    [...] truncated  
  sector_relations:  
    3: [[box2p, 3], [[k1, k1], [k2, k2]]]  
    6: [[box2p, 3], [[k1, -p1+k2], [k2, k1]]]  
    7: [[box2p, 11], [[k1, k1-p1], [k2, k2]]]  
    12: [[box2p, 3], [[k1, k1-p1], [k2, k2-p2]]]  
    13: [[box2p, 11], [[k1, k1-p1], [k2, k2-p2]]]  
    15: [[box2p, 58], [[k1, -p1+k2], [k2, k1-p1+p2]]]  
    18: [[box2p, 3], [[k1, k2+p3], [k2, k1]]]  
    19: [[box2px13x24, 11], [[k1, k1+p3], [k2, k2]]]  
    24: [[box2p, 3], [[k1, k2+p3], [k2, k1-p2]]]  
    25: [[box2px13x24, 11], [[k1, k1+p3], [k2, k2-p2]]]  
    26: [[box2px12x34, 11], [[k1, k2+p3], [k2, k1-p2]]]  
    27: [[box2px13, 58], [[k1, k2+p3], [k2, k1-p2+p3]]]  
    [...] truncated  
  crossed_sector_relations:  
    x12:  
      207: [[box2n, 207], [[k1, -k2-p2], [k2, -k1-p1]]]  
      335: [[box2n, 335], [[k1, -k2-p2], [k2, -k1-p1]]]  
      463: [[box2n, 463], [[k1, -k2-p2], [k2, -k1-p1]]]  
    x123:  
      335: [[box2n, 335], [[k1, -k2-p2], [k2, k1+p1-k2]]]  
    x124:  
      335: [[box2nx14x23, 335], [[k1, -k1+p1+k2-p3], [k2, -k1+p1+p2-p3]]]  
    x1243:  
      335: [[box2nx12x34, 335], [[k2, k1-k2+p2]]]  
  sector_symmetries:  
    207:  
      - [[k1, -k1-p1], [k2, -k2-p2]]  
      - [[k1, -k2-p2], [p1, p2], [k2, -k1-p1], [p2, p1], [p3, p1+p2-p3]]  
      - [[k1, k2], [p1, p2], [k2, k1], [p2, p1], [p3, p1+p2-p3]]  
    463:  
      - [[k1, k2], [p1, p2], [k2, k1], [p2, p1], [p3, p1+p2-p3]]  
(END)
```

# AVAILABLE JOBS IN REDUZE 2

```
andreas : bash
Datei Bearbeiten Ansicht Verlauf Lesezeichen Einstellungen Hilfe
andreas@chili:~$ reduze -h jobs

List of available job types:

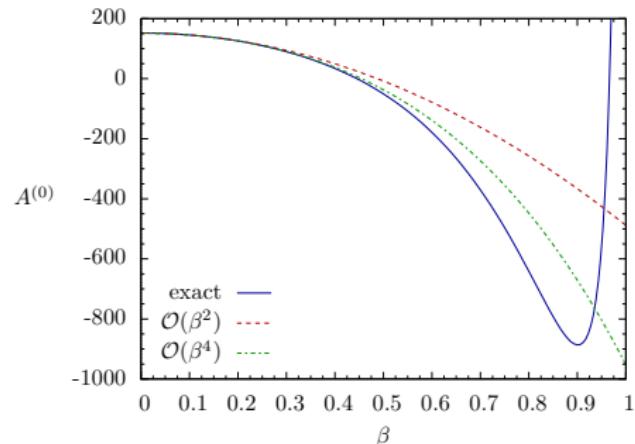
apply_crossings:           Generates reduction results for crossed sectors.
cat_files:                 Concatenates files.
collect_integrals:          Collects all integrals appearing in the input file.
compute_diagram_interferences: Computes interferences of diagrams.
compute_differential_equations: Computes derivatives of integrals wrt invariants.
export:                     Exports to FORM, Mathematica or Maple format.
find_diagram_shifts:        Matches diagrams to sectors via graphs.
find_diagram_shifts_alt:    Matches diagrams to sectors via combinatorics.
generate_identities:        Generates identities like IBPs for given seeds.
generate_seeds:             Generates integrals from a sector.
insert_reductions:          Inserts reductions in expressions.
normalize:                  Simplifies linear combinations and equations.
print_reduction_info_file: Analyzes reductions in a file.
print_reduction_info_sectors: Analyzes reductions available for sectors.
print_sector_info:          Prints diagrams and other information for sectors.
reduce_files:               Reduces identities in given files.
reduce_sectors:             Reduces integrals from a selection of sectors.
run_reduction:              Low-level job to run a reduction.
select_reductions:          Selects reductions for integrals.
setup_sector_mappings:      Finds shifts between sectors via graphs.
setup_sector_mappings_alt:  Finds shifts between sectors via combinatorics.
sum_terms:                  Sums terms.
test:                      Empty job template.
verify_same_terms:          Verifies two files contain the same terms.

andreas@chili:~$
```

## Top quark pair production at two loops: gluon channel results

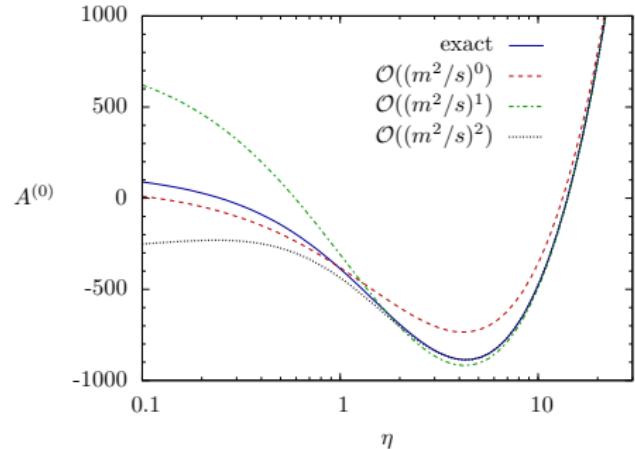
# RESULTS FOR $gg$ LEADING $N_c$ : EXPANSIONS

threshold expansion:



$$\beta = \sqrt{1 - 4m_t^2/s}, \\ \text{c.m. scatt. angle} = \pi/2$$

small mass expansion:



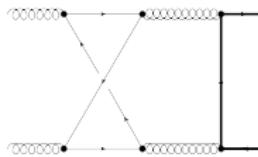
$$\eta = \frac{s}{4m_t^2} - 1, \\ \text{c.m. scatt. angle} = \pi/2$$

(careful with phase space for small  $m$  expansion:  
don't introduce forw.-backw. asymmetry,  $\phi = -(t - m_t^2)/s = \text{const}$  )

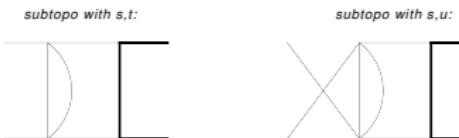
Bonciani, Ferroglio, Gehrmann, A.v.M., Studerus (2010)

# LIGHT FERMION LOOPS IN $gg \rightarrow t\bar{t}$

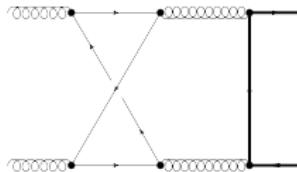
- analytical result for light fermion loops in  $gg \rightarrow t\bar{t}$ :  
Bonciani, Ferroglio, Gehrmann, A.v.M., Studerus (in preparation)
- required calculation of 11 new master integrals
  - ▶ differential equations + Mellin-Barnes for const.
  - ▶ used: MB.m by Czakon '05, planar: AMBRE by Gluza, Kajda, Riemann '07 + Yundin '10
- most difficult integrals: massive non-planar double box



- ▶ 3 master integrals (+ subtopo MIs)
- ▶ analytical solution for massless case: Tausk 1999
- ▶ 3 + 1 scale problem:  $s$ ,  $t$ ,  $u$  and  $m$  appear naturally



# ANALYTICAL SOLUTION FOR MASSIVE NON-PLANAR DOUBLE BOX



solved with differential equations:

A.v.M, Studerus (in preparation)

- all masters up to and including finite parts
- determined integration constants from
  - ▶ regularity conditions
  - ▶ symmetry conditions
  - ▶ Mellin-Barnes (cmp. with SecDec by Carter, Heinrich '10)  
in kinematical limits (finite const. up to 3-fold scaleless M.B.)

found **result** in terms of  $(x, y)$ :

- linear combinations of GPLs with rational prefactors
- transcendality up to 4
- 805 GPLs total, 166 two-dim.
- GPLs with argument  $y$ : weights  $\{0, -1, -x, -1/x, -1/x - x, -1/x - x + 1\}$
- GPLs with argument  $x$ : weights  $\{0, \pm 1, \pm i, (1 \pm i\sqrt{3})/2\}$ 
  - ▶ for univariate cmp. Ablinger, Blümlein, Schneider (2011) & talk by C. Schneider

## SIMPLIFICATION VIA SYMBOLS

**result** for  $1/\epsilon$  of scalar master:

- using GPLs with variables  $(x, y)$ : **62 different GPLs** (23 two-dim.)
- choosing different arguments via **symbols**, this **simplifies to**:

$$\begin{aligned} M_1|_{1/\epsilon} = & \frac{1}{16m^2y_1z_1(y_1 + z_1)} \times \\ & \times \left( \text{Li}_3\left(\frac{y_1z_1}{y_1 + z_1}\right) \right. \\ & + \text{Li}_2\left(\frac{y_1z_1}{y_1 + z_1}\right) (\log(-y_1 - z_1) - \log y_1 - \log z_1) \\ & \left. + \text{polynomial in } \log(y_1), \log(z_1), \log(-y_1 - z_1), \log\left(1 - \frac{y_1z_1}{y_1 + z_1}\right) \right) \end{aligned}$$

$$\begin{aligned} \text{with } y_1 &:= -t/m^2 + 1, \\ z_1 &:= -u/m^2 + 1 \end{aligned}$$

- recent progress with symbols: **Duhr, Gangl, Rhodes '11, Duhr '12** & talk by **C. Duhr**

## CONCLUSIONS

- **Reduze 2** is open source tool for
  - ▶ **parallelized reductions** of Feynman integrals
  - ▶ **finding shifts** between sectors or diagrams
- $gg \rightarrow t\bar{t}$ : analytical two-loop corrections for **leading  $N_c$  + light fermionic**
- **massive non-planar double box**: analytical solution
- **choice of polylogs via symbols**: simplifies QCD integrals with mass