

# Simultaneous decoupling of $b$ and $c$ quarks

A. G. Grozin, M. Höschele, J. Hoff, M. Steinhauser

Institut für Theoretische Teilchenphysik  
Karlsruher Institut für Technologie

# Introduction

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Ordinary QCD Lagrangian  
(with re-defined fields and parameters)  
plus  $1/m_Q^n$  corrections

Traditionally — 1 flavour at a time,  
a tower of effective theories

# History

- ▶ 2 loops

W. Bernreuther, W. Wetzel (1982); Erratum (1998)

S. A. Larin, T. van Ritbergen, J. A. M. Vermaseren  
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K. G. Chetyrkin, B. A. Kniehl, M. Steinhauser (1998)

- ▶ 4 loops

K. G. Chetyrkin, J. H. Kühn, C. Sturm (2006)

Y. Schröder, M. Steinhauser (2006)

# 2- and 1-step decoupling

$$n_f = n_l + n_c + n_b$$

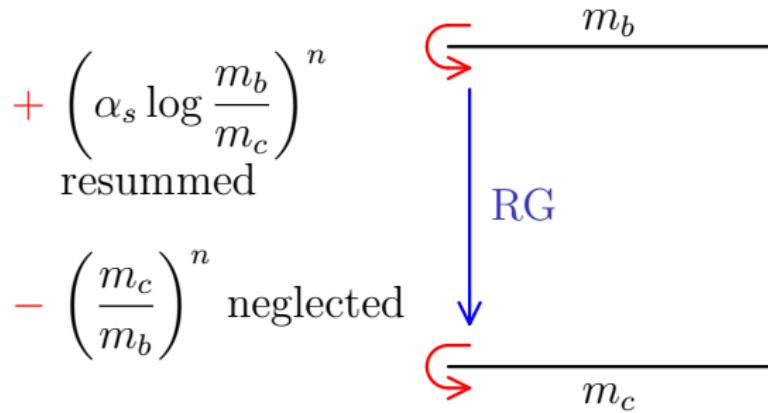
$$\overline{m_b}$$

$$\overline{m_c}$$

$$n_l$$

## 2- and 1-step decoupling

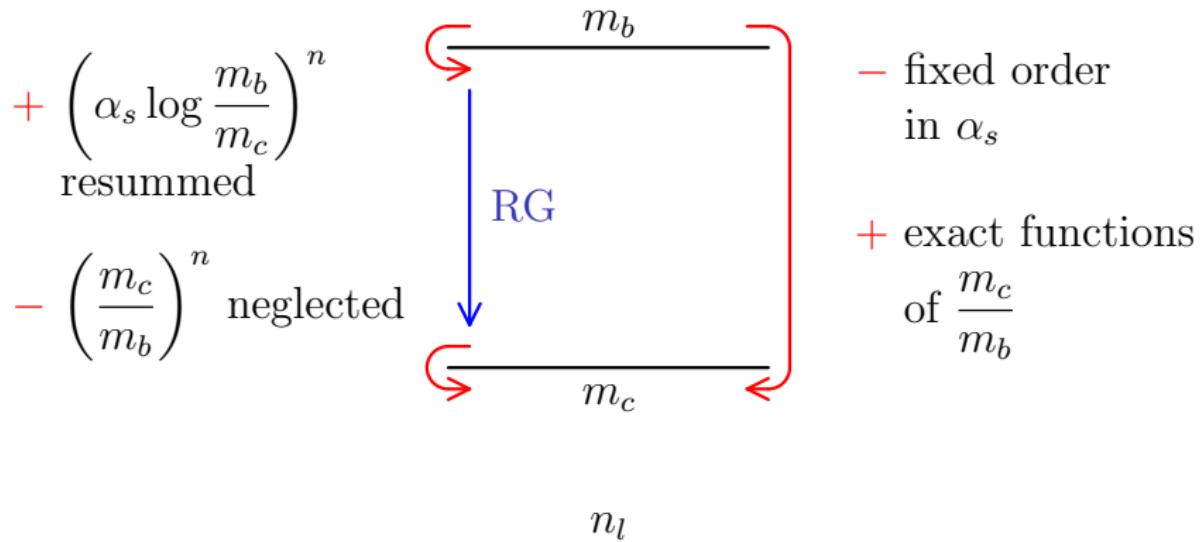
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$n_l$

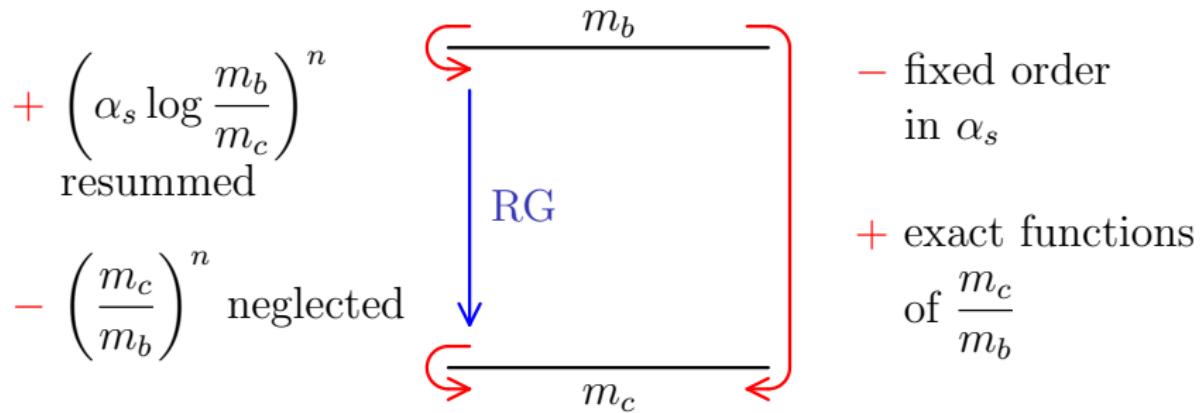
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## 2- and 1-step decoupling

$$n_f = n_l + n_c + n_b$$



$n_l$

$$\left( \frac{\alpha_s}{\pi} \right)^3 \frac{m_c^2}{m_b^2} \sim \left( \frac{\alpha_s}{\pi} \right)^4$$

# Bare decoupling

## Fields

$$A_0^{(n_l)} = (\zeta_A^0)^{1/2} A_0^{(n_f)}$$

$$c_0^{(n_l)} = (\zeta_c^0)^{1/2} c_0^{(n_f)}$$

$$q_0^{(n_l)} = (\zeta_q^0)^{1/2} q_0^{(n_f)}$$

# Bare decoupling

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## Parameters of the Lagrangian

$$\begin{aligned}\alpha_{s0}^{(n_l)} &= \zeta_\alpha^0 \alpha_{s0}^{(n_f)} \\a_0^{(n_l)} &= \zeta_A^0 a_0^{(n_f)} \\m_{q0}^{(n_l)} &= \zeta_m^0 m_{q0}^{(n_f)}\end{aligned}$$

# Renormalized decoupling

## Fields

$$A^{(n_l)}(\mu') = \zeta_A^{1/2}(\mu', \mu) A^{(n_f)}(\mu)$$

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## Parameters of the Lagrangian

$$\alpha_s^{(n_l)}(\mu') = \zeta_\alpha(\mu', \mu) \alpha_s^{(n_f)}(\mu)$$

$$a^{(n_l)}(\mu') = \zeta_A(\mu', \mu) a^{(n_f)}(\mu)$$

$$m_q^{(n_l)}(\mu') = \zeta_m(\mu', \mu) m_q^{(n_f)}(\mu)$$

# Renormalization

$$\zeta_\alpha(\mu', \mu) = \frac{Z_\alpha^{(n_f)}\left(\alpha_s^{(n_f)}(\mu)\right)}{Z_\alpha^{(n_l)}\left(\alpha_s^{(n_l)}(\mu')\right)} \zeta_\alpha^0\left(\alpha_{s0}^{(n_f)}\right)$$
$$\zeta_m(\mu', \mu) = \frac{Z_m^{(n_f)}\left(\alpha_s^{(n_f)}(\mu)\right)}{Z_m^{(n_l)}\left(\alpha_s^{(n_l)}(\mu')\right)} \zeta_m^0\left(\alpha_{s0}^{(n_f)}\right)$$

# Renormalization

$$\zeta_A(\mu', \mu) = \frac{Z_A^{(n_f)} \left( \alpha_s^{(n_f)}(\mu), a^{(n_f)}(\mu) \right)}{Z_A^{(n_l)} \left( \alpha_s^{(n_l)}(\mu'), a^{(n_l)}(\mu') \right)} \zeta_A^0 \left( \alpha_{s0}^{(n_f)}, a_0^{(n_f)} \right)$$
$$\zeta_q(\mu', \mu) = \frac{Z_q^{(n_f)} \left( \alpha_s^{(n_f)}(\mu), a^{(n_f)}(\mu) \right)}{Z_q^{(n_l)} \left( \alpha_s^{(n_l)}(\mu'), a^{(n_l)}(\mu') \right)} \zeta_q^0 \left( \alpha_{s0}^{(n_f)}, a_0^{(n_f)} \right)$$
$$\zeta_c(\mu', \mu) = \frac{Z_c^{(n_f)} \left( \alpha_s^{(n_f)}(\mu), a^{(n_f)}(\mu) \right)}{Z_c^{(n_l)} \left( \alpha_s^{(n_l)}(\mu'), a^{(n_l)}(\mu') \right)} \zeta_c^0 \left( \alpha_{s0}^{(n_f)}, a_0^{(n_f)} \right)$$

# Decoupling: gluon field

Effective theory ( $n_l$ )

$$A_0^{(n_l)} = Z_A^{\text{os}(n_l)} A_{\text{os}}^{(n_l)}$$

Full QCD ( $n_f$ )

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$$A_0^{(n_l)} = \zeta_A^0 A_0^{(n_f)} \quad \zeta_A^0 = \frac{Z_A^{\text{os}(n_l)}}{Z_A^{\text{os}(n_f)}}$$

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$$Z_A^{\text{os}(n_l)} = \frac{1}{1 + \Pi_A^{(n_l)}(0)} = 1$$
$$\Pi_A^{(n_l)}(0) = 0$$

$$Z_A^{\text{os}(n_f)} = \frac{1}{1 + \Pi_A^{(n_f)}(0)}$$

at least 1 heavy-quark loop

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$$\zeta_A^0(\alpha_{s0}^{(n_f)}, a_0^{(n_f)}) = 1 + \Pi_A^{(n_f)}(0)$$

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$$\zeta_A^0(\alpha_{s0}^{(n_f)}, a_0^{(n_f)}) = 1 + \Pi_A^{(n_f)}(0)$$

$$\zeta_A(\mu', \mu) = \frac{Z_A^{(n_f)}}{Z_A^{(n_l)}} \frac{Z_A^{\text{os}(n_l)}}{Z_A^{\text{os}(n_f)}}$$

# Bare decoupling: fields

$$\zeta_A^0(\alpha_{s0}^{(n_f)}, a_0^{(n_f)}) = 1 + \Pi_A(0) = [Z_A^{\text{os}}]^{-1}$$

$$\zeta_c^0(\alpha_{s0}^{(n_f)}, a_0^{(n_f)}) = 1 + \Pi_c(0) = [Z_c^{\text{os}}]^{-1}$$

$$\zeta_q^0(\alpha_{s0}^{(n_f)}, a_0^{(n_f)}) = 1 + \Sigma_V(0) = [Z_q^{\text{os}}]^{-1}$$

$$\alpha_{s0} = g_0^2/(4\pi)^{1-\varepsilon}, \quad \Sigma(q) = q\Sigma_V(q^2) + m_{q0}\Sigma_S(q^2)$$

## Bare decoupling: $\alpha_s$ , $m_q$

Expand vertex functions in their external momenta up to the first non-vanishing term

- ▶ Quark–gluon:  $\gamma^\mu$
- ▶ 3–gluon:  $f^{a_1 a_2 a_3} (g^{\mu_1 \mu_2} (k_1 - k_2)^{\mu_3} + \text{cycle})$ ,  
 $d^{a_1 a_2 a_3} (g^{\mu_1 \mu_2} k_3^{\mu_3} + \text{cycle})$ . Slavnov–Taylor identity  
 $\langle T\{\partial^\mu A_\mu(x), \partial^\nu A_\nu(y), \partial^\lambda A_\lambda(z)\} \rangle = 0 \Rightarrow$   
 $\Gamma_{\mu_1 \mu_2 \mu_3}^{a_1 a_2 a_3} k_1^{\mu_1} k_2^{\mu_2} k_3^{\mu_3} = 0$
- ▶ Ghost–gluon:  $p^\mu$  (outgoing ghost momentum)

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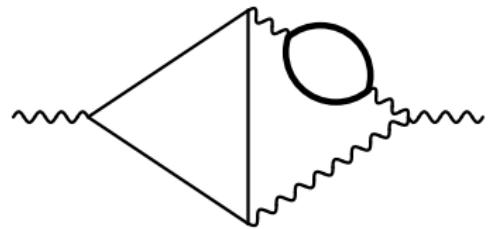
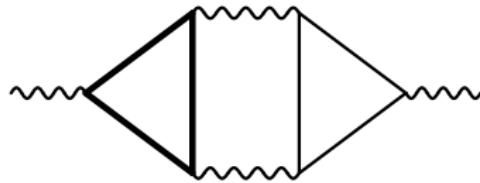
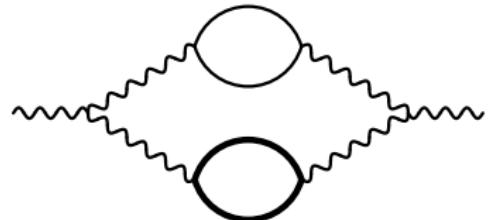
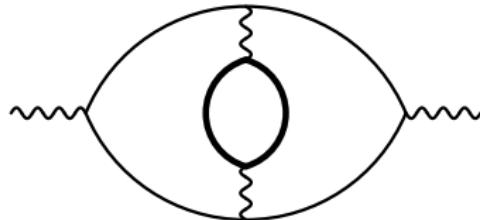
$$\begin{aligned}\zeta_\alpha^0(\alpha_{s0}^{(n_f)}) &= (1 + \Gamma_{A\bar{c}c})^2 (Z_q^{\text{os}})^2 Z_A^{\text{os}} = (1 + \Gamma_{A\bar{q}q})^2 (Z_c^{\text{os}})^2 Z_A^{\text{os}} \\ &= (1 + \Gamma_{AAA})^2 (Z_A^{\text{os}})^3\end{aligned}$$

$$\zeta_m^0(\alpha_{s0}^{(n_f)}) = Z_q^{\text{os}} [1 + \Sigma_S(0)]$$

# Gluon self-energy

$$\begin{aligned}\Pi_A(0) &= \frac{1}{3} \left( n_b m_{b0}^{-2\varepsilon} + n_c m_{c0}^{-2\varepsilon} \right) T_F \frac{\alpha_{s0}^{(n_f)}}{\pi} \Gamma(\varepsilon) \\ &+ P_h \left( n_b m_{b0}^{-4\varepsilon} + n_c m_{c0}^{-4\varepsilon} \right) T_F \left( \frac{\alpha_{s0}^{(n_f)}}{\pi} \Gamma(\varepsilon) \right)^2 \\ &+ \left[ (P_{hg} + P_{hl} T_F n_l) \left( n_b m_{b0}^{-6\varepsilon} + n_c m_{c0}^{-6\varepsilon} \right) \right. \\ &\quad \left. + P_{hh} T_F \left( n_b^2 m_{b0}^{-6\varepsilon} + n_c^2 m_{c0}^{-6\varepsilon} \right) \right. \\ &\quad \left. + P_{bc} \left( \frac{m_{c0}}{m_{b0}} \right) T_F n_b n_c (m_{b0} m_{c0})^{-3\varepsilon} \right] T_F \left( \frac{\alpha_{s0}^{(n_f)}}{\pi} \Gamma(\varepsilon) \right)^3 + \dots\end{aligned}$$

# Gluon self-energy



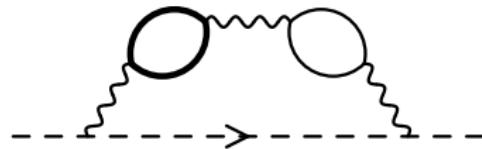
$$P_{bc}(x^{-1}) = P_{bc}(x)$$

$$P_{bc}(1) = 2P_{hh}$$

$$P_{bc}^{\text{hard}}(x \rightarrow 0) \rightarrow P_{hl} x^{3\varepsilon}$$

# Ghost self-energy

$$\begin{aligned}\Pi_c(0) = & C_h \left( n_b m_{b0}^{-4\varepsilon} + n_c m_{c0}^{-4\varepsilon} \right) C_A T_F \left( \frac{\alpha_{s0}^{(n_f)}}{\pi} \Gamma(\varepsilon) \right)^2 \\ & + \left[ (C_{hg} + C_{hl} T_F n_l) \left( n_b m_{b0}^{-6\varepsilon} + n_c m_{c0}^{-6\varepsilon} \right) \right. \\ & \quad \left. + C_{hh} T_F \left( n_b^2 m_{b0}^{-6\varepsilon} + n_c^2 m_{c0}^{-6\varepsilon} \right) \right. \\ & \quad \left. + C_{bc} \left( \frac{m_{c0}}{m_{b0}} \right) T_F n_b n_c (m_{b0} m_{c0})^{-3\varepsilon} \right] C_A T_F \left( \frac{\alpha_{s0}^{(n_f)}}{\pi} \Gamma(\varepsilon) \right)^3 + \dots\end{aligned}$$



# Light-quark self-energy

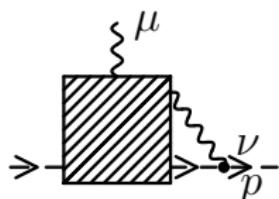
$$\begin{aligned}\Sigma_V(0) = & V_h \left( n_b m_{b0}^{-4\varepsilon} + n_c m_{c0}^{-4\varepsilon} \right) C_F T_F \left( \frac{\alpha_{s0}^{(n_f)}}{\pi} \Gamma(\varepsilon) \right)^2 \\ & + \left[ (V_{hg} + V_{hl} T_F n_l) \left( n_b m_{b0}^{-6\varepsilon} + n_c m_{c0}^{-6\varepsilon} \right) \right. \\ & \quad \left. + V_{hh} T_F \left( n_b^2 m_{b0}^{-6\varepsilon} + n_c^2 m_{c0}^{-6\varepsilon} \right) \right. \\ & \quad \left. + V_{bc} \left( \frac{m_{c0}}{m_{b0}} \right) T_F n_b n_c (m_{b0} m_{c0})^{-3\varepsilon} \right] C_F T_F \left( \frac{\alpha_{s0}^{(n_f)}}{\pi} \Gamma(\varepsilon) \right)^3 + \dots\end{aligned}$$

Similarly for  $\Sigma_S(0)$

# Ghost–gluon vertex

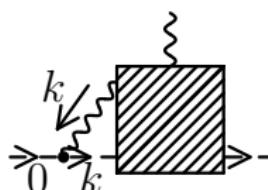
$$\begin{array}{c} \xi^\mu \\ \hbox{\scriptsize\rm hatched square} \\ \Rightarrow \end{array} = A^{\mu\nu} p_\nu \qquad A^{\mu\nu}(0) = Ag^{\mu\nu}$$

# Ghost-gluon vertex



Feynman diagram showing a ghost-gluon vertex. A horizontal dashed line with an arrow pointing right enters a shaded square from the left. From the right side of the square, two lines emerge: a wavy line labeled  $\zeta^\mu$  and a gluon line labeled  $p^\nu$ . The ghost-gluon vertex is at the top-right corner of the square.

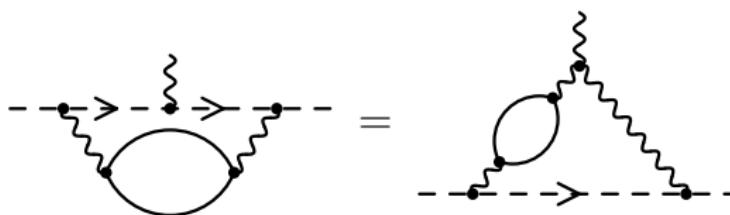
$$= A^{\mu\nu} p_\nu \quad A^{\mu\nu}(0) = Ag^{\mu\nu}$$



Feynman diagram showing a ghost-gluon vertex correction. A horizontal dashed line with an arrow pointing right enters a shaded square from the left. A wavy line labeled  $k$  enters the square from the left. From the right side of the square, a gluon line labeled  $p$  emerges. The ghost-gluon vertex is at the top-right corner of the square.

$$\sim k^\lambda$$

All corrections vanish in Landau gauge



Feynman diagram showing a ghost-gluon vertex correction vanishes in Landau gauge. On the left, a ghost-gluon vertex correction is shown with a ghost line (dashed with arrow), a gluon line (solid with arrow), and a ghost-gluon vertex loop. This is followed by an equals sign. On the right, the result is shown as a ghost line (dashed with arrow) and a gluon line (solid with arrow) meeting at a ghost-gluon vertex, with a value of zero indicated.

$$= 0$$

# Ghost-gluon vertex

$$\begin{array}{c} \text{Diagram 1: } \text{ghost} \rightarrow \text{ghost} + \text{ghost} \\ \text{Diagram 2: } \text{ghost} \rightarrow \text{ghost} + \text{ghost} \end{array} = a_0 \left[ \begin{array}{c} \text{Diagram 3: } \text{ghost} \rightarrow \text{ghost} + \text{ghost} \\ \text{Diagram 4: } \text{ghost} \rightarrow \text{ghost} + \text{ghost} \end{array} - \begin{array}{c} \text{Diagram 5: } \text{ghost} \rightarrow \text{ghost} + \text{ghost} \\ \text{Diagram 6: } \text{ghost} \rightarrow \text{ghost} + \text{ghost} \end{array} \right]$$

Diagrams:

- Diagram 1: A ghost line (dashed) enters from the left, splits into two gluon lines (wavy), which then recombine into a ghost line that exits to the right.
- Diagram 2: Similar to Diagram 1, but the ghost line exits from the right.
- Diagram 3: A ghost line (dashed) enters from the left, splits into two gluon lines (wavy), which then recombine into a ghost line that exits to the right. This diagram is enclosed in brackets with a minus sign.
- Diagram 4: Similar to Diagram 3, but the ghost line exits from the right.
- Diagram 5: A ghost line (dashed) enters from the left, splits into two gluon lines (wavy), which then recombine into a ghost line that exits to the right. This diagram is enclosed in brackets with a minus sign.
- Diagram 6: Similar to Diagram 5, but the ghost line exits from the right.

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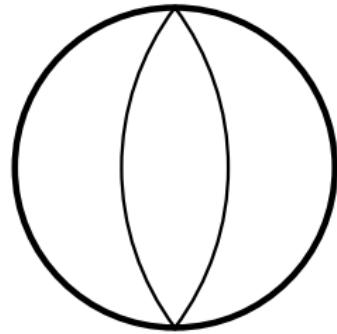
$$\begin{array}{c} \text{Diagram 7: } \text{ghost} \rightarrow \text{ghost} + \text{ghost} \\ \text{Diagram 8: } \text{ghost} \rightarrow \text{ghost} + \text{ghost} \end{array} = a_0 \left[ \begin{array}{c} \text{Diagram 9: } \text{ghost} \rightarrow \text{ghost} + \text{ghost} \\ \text{Diagram 10: } \text{ghost} \rightarrow \text{ghost} + \text{ghost} \end{array} - \begin{array}{c} \text{Diagram 11: } \text{ghost} \rightarrow \text{ghost} + \text{ghost} \\ \text{Diagram 12: } \text{ghost} \rightarrow \text{ghost} + \text{ghost} \end{array} \right]$$

$$\begin{array}{c} \text{Diagram 13: } \text{ghost} \rightarrow \text{ghost} + \text{ghost} \\ \text{Diagram 14: } \text{ghost} \rightarrow \text{ghost} + \text{ghost} \end{array} = a_0 \left[ \begin{array}{c} \text{Diagram 15: } \text{ghost} \rightarrow \text{ghost} + \text{ghost} \\ \text{Diagram 16: } \text{ghost} \rightarrow \text{ghost} + \text{ghost} \end{array} - \begin{array}{c} \text{Diagram 17: } \text{ghost} \rightarrow \text{ghost} + \text{ghost} \\ \text{Diagram 18: } \text{ghost} \rightarrow \text{ghost} + \text{ghost} \end{array} \right]$$

# Reduction

FIRE A. V. Smirnov (2008)

Master integrals



S. Bekavac, AGG, D. Seidel,

V. A. Smirnov (2009)

$$I_1(x) = I_1(x^{-1}) = \frac{(m_b m_c)^{-2+3\varepsilon}}{(i\pi^{d/2})^3 \Gamma^3(\varepsilon)} \int \frac{d^d k_1 d^d k_2 d^d k_3}{D_1 D_2 D_3 D_4}$$

$$I_2(x) = I_2(x^{-1}) = \frac{(m_b m_c)^{-3+3\varepsilon}}{(i\pi^{d/2})^3 \Gamma^3(\varepsilon)} \int \frac{N d^d k_1 d^d k_2 d^d k_3}{D_1 D_2 D_3 D_4}$$

$$D_1 = m_b^2 - k_1^2 \quad D_2 = m_b^2 - k_2^2 \quad D_3 = m_c^2 - k_3^2$$

$$D_4 = m_c^2 - (k_1 - k_2 + k_3)^2 \quad N = -(k_1 - k_2)^2$$

# Master integrals

$$I_2(1) = -\frac{4}{3} \left( I_1(1) + \frac{8}{(d-2)^3} \right)$$

D. J. Broadhurst (1991), (1996)

# Decoupling: $\alpha_s$

$$\alpha_{s0}^{(n_l)} = \zeta_\alpha^0 \alpha_{s0}^{(n_f)}$$

$$\zeta_\alpha^0 = 1 - \frac{1}{3} \left( n_b m_{b0}^{-2\varepsilon} + n_c m_{c0}^{-2\varepsilon} \right) T_F \frac{\alpha_{s0}^{(n_f)}}{\pi} \Gamma(\varepsilon) + \dots$$

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Re-expressing via renormalized quantities

$$\frac{\alpha_{s0}^{(n_f)}}{\pi} \Gamma(\varepsilon) = \frac{\alpha_s^{(n_f)}(\mu)}{\pi \varepsilon} Z_\alpha^{(n_f)} \left( \alpha_s^{(n_f)}(\mu) \right) e^{\gamma_E \varepsilon} \Gamma(1 + \varepsilon) \mu^{2\varepsilon}$$

$$m_{b0} = Z_m^{(n_f)} \left( \alpha_s^{(n_f)}(\mu) \right) m_b(\mu) \quad m_{c0} = Z_m^{(n_f)} \left( \alpha_s^{(n_f)}(\mu) \right) m_c(\mu)$$

Convenient to use  $m_b(\bar{m}_b) = \bar{m}_b$

Obtain  $\alpha_{s0}^{(n_l)}$  via  $\alpha_s^{(n_f)}(\bar{m}_b)$ ,  $\bar{m}_b$ , and  $m_c(\bar{m}_b)$

# Decoupling: $\alpha_s$

Inverting the series

$$\frac{\alpha_{s0}^{(n_l)}}{\pi} \Gamma(\varepsilon) = \frac{\alpha_s^{(n_l)}(\mu')}{\pi \varepsilon} Z_\alpha^{(n_l)} \left( \alpha_s^{(n_l)}(\mu') \right) e^{\gamma_E \varepsilon} \Gamma(1 + \varepsilon) \mu'^{2\varepsilon}$$

we obtain  $\alpha_s^{(n_l)}(\mu')$

Convenient to use  $\mu' = m_c(\bar{m}_b)$ , then via  $\alpha_s^{(n_f)}(\bar{m}_b)$  and

$$x = \frac{m_c(\bar{m}_b)}{\bar{m}_b}$$

# Decoupling: $a$

$$a_0^{(n_l)} = \zeta_A^0 a_0^{(n_f)}$$

$$\zeta_A^0 = 1 + \frac{1}{3} \left( n_b m_{b0}^{-2\varepsilon} + n_c m_{c0}^{-2\varepsilon} \right) T_F \frac{\alpha_{s0}^{(n_f)}}{\pi} \Gamma(\varepsilon) + \dots$$

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Re-expressing via renormalized quantities

$$a_0^{(n_f)} = Z_A^{(n_f)} \left( \alpha_s^{(n_f)}(\mu), a^{(n_f)}(\mu) \right) a^{(n_f)}(\mu)$$

$$\mu = \bar{m}_b$$

$a_0^{(n_l)}$  via  $a^{(n_f)}(\bar{m}_b)$ ,  $\alpha_s^{(n_f)}(\bar{m}_b)$ ,  $m_c(\bar{m}_b)$

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$$\mu = \bar{m}_b$$

$$a_0^{(n_l)} \text{ via } a^{(n_f)}(\bar{m}_b), \alpha_s^{(n_f)}(\bar{m}_b), m_c(\bar{m}_b)$$

Solving by iterations

$$a_0^{(n_l)} = Z_A^{(n_l)} \left( \alpha_s^{(n_l)}(\mu'), a^{(n_l)}(\mu') \right) a^{(n_l)}(\mu')$$

$$\mu' = m_c(\bar{m}_b)$$

$$a^{(n_l)}(m_c(\bar{m}_b)) \text{ via } a^{(n_f)}(\bar{m}_b), \alpha_s^{(n_f)}(\bar{m}_b), x$$

# Bilinear quark currents

Full QCD

$$j_n^{(n_f)}(\mu) = \left[ Z_n^{(n_f)}(\alpha_s^{(n_f)}(\mu)) \right]^{-1} j_{n0}^{(n_f)}$$
$$j_{n0}^{(n_f)} = \bar{q}_0^{(n_f)} \Gamma_n \tau q_0^{(n_f)} \quad \Gamma_n = \gamma^{[\mu_1} \dots \gamma^{\mu_n]}$$

3-loop anomalous dimensions: J. A. Gracey (2000)

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Bare and renormalized decoupling

$$\zeta_n(\mu, \mu') = \frac{Z_n^{(n_l)}(\alpha_s^{(n_l)}(\mu'))}{Z_n^{(n_f)}(\alpha_s^{(n_f)}(\mu))} \zeta_n^0 \quad j_{n0}^{(n_f)} = \zeta_n^0 j_{n0}^{(n_l)}$$

# Matching

On-shell matrix elements

$$Z_q^{\text{os}(n_f)} \left( 1 + \Gamma_n^{(n_f)} \right) = \zeta_n^0 Z_q^{\text{os}(n_l)} \left( 1 + \Gamma_n^{(n_l)} \right)$$

# Matching

On-shell matrix elements

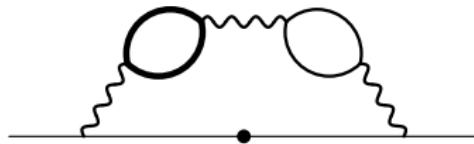
$$Z_q^{\text{os}(n_f)} \left( 1 + \Gamma_n^{(n_f)} \right) = \zeta_n^0 Z_q^{\text{os}(n_l)} \left( 1 + \Gamma_n^{(n_l)} \right)$$

$$\zeta_n^0 = Z_q^{\text{os}}(\alpha_{s0}^{(n_f)}, a_0^{(n_f)}) \left( 1 + \Gamma_n(\alpha_{s0}^{(n_f)}, a_0^{(n_f)}) \right)$$

AGG, A.V. Smirnov, V.A. Smirnov (2006)

# Vertex function

$$\begin{aligned}\Gamma_n &= G_h \left( n_b m_{b0}^{-4\varepsilon} + n_c m_{c0}^{-4\varepsilon} \right) C_F T_F \left( \frac{\alpha_{s0}^{(n_f)}}{\pi} \Gamma(\varepsilon) \right)^2 \\ &+ \left[ (G_{hg} + G_{hl} T_F n_l) \left( n_b m_{b0}^{-6\varepsilon} + n_c m_{c0}^{-6\varepsilon} \right) \right. \\ &\quad \left. + G_{hh} T_F \left( n_b^2 m_{b0}^{-6\varepsilon} + n_c^2 m_{c0}^{-6\varepsilon} \right) \right. \\ &\quad \left. + G_{bc} \left( \frac{m_{c0}}{m_{b0}} \right) T_F n_b n_c (m_{b0} m_{c0})^{-3\varepsilon} \right] C_F T_F \left( \frac{\alpha_{s0}^{(n_f)}}{\pi} \Gamma(\varepsilon) \right)^3 + \dots\end{aligned}$$



# Particular cases

$n = 1$

$$\zeta_1^0 = 1 \quad Z_1^{(n_f)} = Z_1^{(n_l)} = 1 \quad \zeta_1(\mu, \mu') = 1$$

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$n = 0$

$$\zeta_0^0 = \zeta_m^0 \quad \zeta_0(\mu, \mu') = \zeta_m(\mu', \mu)$$

$n = 4, 3$

$$(\bar{q} \gamma_5^{\text{HV}} \tau q)_\mu = \frac{i}{4!} \varepsilon_{\alpha\beta\gamma\delta} j_4^{\alpha\beta\gamma\delta}(\mu)$$

$$(\bar{q} \gamma_5^{\text{HV}} \gamma_\delta \tau q)_\mu = \frac{i}{3!} \varepsilon_{\alpha\beta\gamma\delta} j_3^{\alpha\beta\gamma}(\mu)$$

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$$(\bar{q}\gamma_5^{\text{AC}}\tau q)_\mu = Z_P(\alpha_s(\mu)) (\bar{q}\gamma_5^{\text{HV}}\tau q)_\mu$$

$$(\bar{q}\gamma_5^{\text{AC}}\gamma^\alpha\tau q)_\mu = Z_P(\alpha_s(\mu)) (\bar{q}\gamma_5^{\text{HV}}\gamma^\alpha\tau q)_\mu$$

S. A. Larin, J. A. M. Vermaseren (1991); S. A. Larin (1992)

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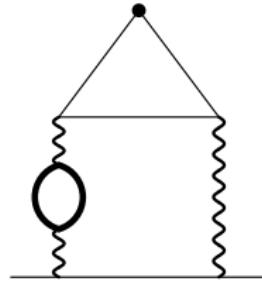
$$(\bar{q}\gamma_5^{\text{AC}}\gamma^\alpha\tau q)_\mu = Z_P(\alpha_s(\mu)) (\bar{q}\gamma_5^{\text{HV}}\gamma^\alpha\tau q)_\mu$$

S. A. Larin, J. A. M. Vermaseren (1991); S. A. Larin (1992)

$$\frac{\zeta_4(\mu, \mu')}{\zeta_0(\mu, \mu')} = \frac{Z_P^{(n_l)}(\alpha_s^{(n_l)}(\mu'))}{Z_P^{(n_f)}(\alpha_s^{(n_f)}(\mu))}$$

$$\frac{\zeta_3(\mu, \mu')}{\zeta_1(\mu, \mu')} = \frac{Z_A^{(n_l)}(\alpha_s^{(n_l)}(\mu'))}{Z_A^{(n_f)}(\alpha_s^{(n_f)}(\mu))}$$

# Light singlet currents



$n$  odd

$n = 1$  — trivial

$n = 3$

$$A_s C_F T_F^2 n_l \left( n_b m_{b0}^{-6\varepsilon} + n_c m_{c0}^{-6\varepsilon} \right) \left( \frac{\alpha_{s0}^{(n_f)}}{\pi} \Gamma(\varepsilon) \right)^3$$

- ▶ Anomalous dimension: S. A. Larin (1992);  
K. G. Chetyrkin, J. H. Kühn (1993)
- ▶ Decoupling: K. G. Chetyrkin, O. V. Tarasov (1993);  
S. A. Larin, T. van Ritbergen, J. A. M. Vermaseren (1995)

# Heavy currents

Vector  $\mathcal{O}(1/m^2)$

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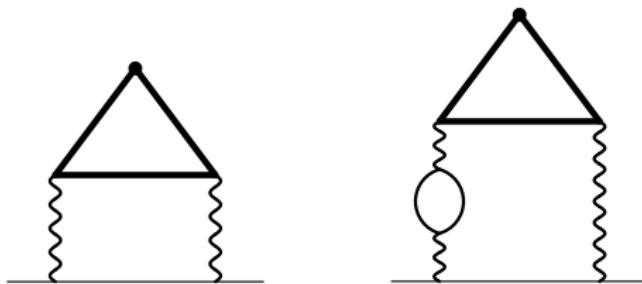
Axial

$$j_{b0} = \bar{b}_0 \gamma^{[\alpha} \gamma^\beta \gamma^{\gamma]} b_0 \quad j_{q0} = \bar{q}_0 \gamma^{[\alpha} \gamma^\beta \gamma^{\gamma]} q_0 \quad j_{b0} = \zeta_A^0 j_{q0}$$

K. G. Chetyrkin, J. H. Kühn (1993)

K. G. Chetyrkin, O. V. Tarasov (1993)

# Heavy axial currents



$$\zeta_A^0 = \left[ A + (A_g + A_l T_F n_l + A_h T_F n_b + A_{bc}(x) T_F n_c) \frac{\alpha_{s0}}{\pi} \Gamma(\varepsilon) m_{b0}^{-2\varepsilon} + \dots \right] C_F T_F n_b \left( \frac{\alpha_{s0}}{\pi} \Gamma(\varepsilon) m_{b0}^{-2\varepsilon} \right)^2$$

$$A_{bc}(1) = A_h \quad A_{bc}^{\text{hard}}(x \rightarrow 0) = A_l \quad A_{bc}^{\text{hard}}(x \rightarrow \infty) \rightarrow A_s x^{-6\varepsilon}$$

# Conclusion

Fields and parameters of the Lagrangian

- ▶  $\alpha_s, m_q$
- ▶  $a, A, c, q$

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Fields and parameters of the Lagrangian

- ▶  $\alpha_s, m_q$
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Bilinear quark currents

- ▶ Light non-singlet
- ▶ Light singlet
- ▶ Heavy