From motives to differential equations for loop integrals

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I.: **Feynman integrals beyond multiple polylogarithms**

II: **The two-loop sunrise integral**

III: **The second-order differential equation**

S. Müller-Stach, S.W., R. Zayadeh, arXiv:1112:4360

One-loop amplitudes

All one-loop amplitudes can be expressed as a sum of algebraic functions of the momenta and masses times two transcendental functions, whose arguments are again algebraic functions of the momenta and the masses.

The two transcendental functions are the logarithm and the dilogarithm:

$$\begin{aligned}
\mathrm{Li}_1(x) &= -\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \\
\mathrm{Li}_2(x) &= \sum_{n=1}^{\infty} \frac{x^n}{n^2}
\end{aligned}$$

Generalisations of the logarithm

Beyond one-loop, at least the following generalisations occur:

Polylogarithms:

$$\text{Li}_m(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^m}$$

Multiple polylogarithms:

$$\text{Li}_{m_1,m_2,\ldots,m_k}(x_1,x_2,\ldots,x_k) = \sum_{n_1>n_2>\ldots>n_k>0}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \cdot \frac{x_2^{n_2}}{n_2^{m_2}} \cdot \ldots \cdot \frac{x_k^{n_k}}{n_k^{m_k}}$$

Multiple polylogarithms have a rich algebraic structure:
two Hopf algebras (shuffle and quasi-shuffle, both with co-product and antipode), convolution and conjugation.

Feynman integrals beyond multiple polylogarithms

Empirical fact:

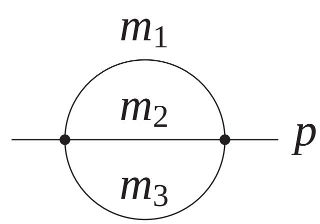
- All two-loop integrals with massless particles calculated up to now can be expressed in terms of multiple polylogarithms.
- In two-loop integrals with massive particles new functions beyond multiple polylogarithms appear.

To learn more, study two-loop integrals with massive particles.

Simplest non-trivial case: Two-loop self-energy.

The two-loop sunrise integral

$$S\left(p^2, m_1^2, m_2^2, m_3^2\right) \quad = $$



- Two-loop contribution to the self-energy of massive particles.
- In the unequal mass case not known analytically up to now.

The two-loop sunrise integral: Prior art

Integration-by-parts identities allow to derive a coupled system of 4 first-order differential equations for S and S_1, S_2, S_3, where

$$S_i = \frac{\partial}{\partial m_i^2} S$$

(Caffo, Czyz, Laporta, Remiddi, 1998).

This system reduces to a single second-order differential equation in the case of equal masses $m_1 = m_2 = m_3$.

Dimensional recurrence relations relate integrals in $D = 4$ dimensions and $D = 2$ dimensions

(Tarasov, 1996, Baikov, 1997, Lee, 2010).

Analytic result known in the equal mass case, result involves elliptic functions

(Laporta, Remiddi, 2004).

The two-loop sunrise integral

Is the system of 4 coupled first-order differential equations generic for the unequal mass case or can we do better ?

Yes, we can !
Also in the unequal mass case there is a single second-order differential equation.

The second-order differential equation follows from algebraic geometry.

(S. Müller-Stach, S.W., R. Zayadeh, arXiv:1112:4360)

Algebraic geometry

Algebraic geometry studies the zero sets of polynomials.

Example:

$$x_1x_2 + x_2x_3 + x_3x_1 \quad = \quad 0.$$

This is actually an equation in projective space $\mathbb{P}^2$.

Study integrals where polynomials appear in the denominator:

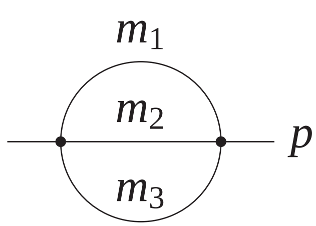
$$\int d^3x\, \delta\left(1 - \sum_{i=1}^{3} x_3\right) \frac{1}{x_1x_2 + x_2x_3 + x_3x_1}$$

What happens in the points $(1,0,0)$, $(0,1,0,)$ or $(0,0,1)$?

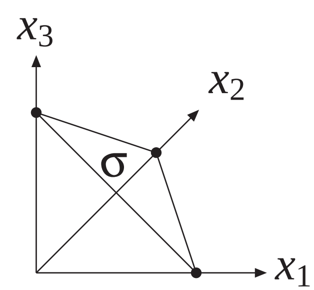
The two-loop sunrise integral

The two-loop sunrise integral with unequal masses in two-dimensions ($t = p^2$):

$$S(t) \quad = $$



$$= \int_\sigma \frac{\omega}{\mathcal{F}},$$



$$\begin{aligned} \omega &= x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2, \\ \mathcal{F} &= -x_1 x_2 x_3 t + \left(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2 \right) \left(x_1 x_2 + x_2 x_3 + x_3 x_1 \right) \end{aligned}$$

For physicists it's just another Feynman integral.

For mathematicians it's a period of a variation of a mixed Hodge structure !

Hodge structures

Hodge structures have their origin in the study of compact Kähler manifolds.
For every k one has

$$H^k(X,\mathbb{C}) \quad = \quad \bigoplus_{p+q=k} H^{p,q}(X), \qquad \overline{H^{p,q}(X)} = H^{q,p}(X).$$

This is an example of a pure Hodge structure of weight k.

Algebraic varieties have a generalisation: Mixed Hodge structures

(Deligne, 1971).

Variation of a Hodge structure: Family of Hodge structures, parametrised by a manifold

(Griffiths, 1968).

Hodge structures

Key fact: If a Hodge structure varies smoothly with some parameters, then

$$\dim H^{p,q}$$

remains constant.

Translated to plain english: If the two-loop sunrise integral

- corresponds to a Hodge structure,
- varies smoothly with the masses and the momenta,
- has a second-order differential equation in the equal mass case,

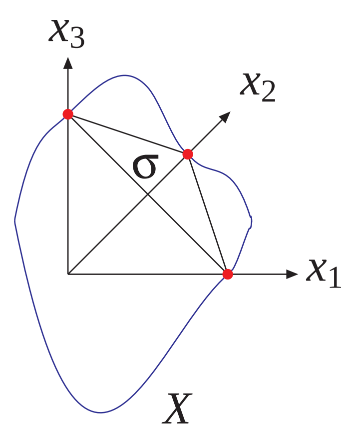
then it also must have a second-order differential equation in the unequal mass case.

The two-loop sunrise integral

From the point of view of algebraic geometry there are two objects of interest:

- the domain of integration σ,
- the zero set X of $\mathcal{F} = 0$.

X and σ intersect at three points:



The motive

P: Blow-up of $\mathbb{P}^2$ in the three points, where X intersects σ.

Y: Strict transform of the zero set X of $\mathcal{F} = 0$.

B: Total transform of $\{x_1 x_2 x_3 = 0\}$.

Mixed Hodge structure:

$$H^2\left(P \backslash Y, B \backslash B \cap Y\right)$$

(S. Bloch, H. Esnault, D. Kreimer, 2006)

We need to analyse $H^2(P \backslash Y, B \backslash B \cap Y)$.
We can show that essential information is given by $H^1(X)$.

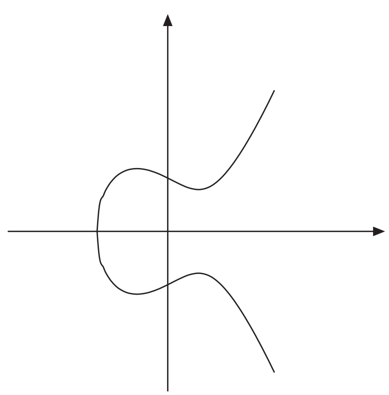
The elliptic curve

Algebraic variety X defined by the polynomial in the denominator:

$$-x_1x_2x_3t + \left(x_1m_1^2 + x_2m_2^2 + x_3m_3^2\right)\left(x_1x_2 + x_2x_3 + x_3x_1\right) \quad = \quad 0.$$

This defines an elliptic curve.

Change of coordinates $\longrightarrow$ Weierstrass normal form



$y^2 = x^3 - x + 1$

$$y^2z - x^3 - a_2(t)xz^2 - a_3(t)z^3 \quad = \quad 0.$$

In the chart $z = 1$ this reduces to

$$y^2 - x^3 - a_2(t)x - a_3(t) \quad = \quad 0.$$

The curve varies with t.

The elliptic curve

In the Weierstrass normal form $H^1(X)$ is generated by

$$\eta = \frac{dx}{y} \quad \text{and} \quad \dot{\eta} = \frac{d}{dt}\eta.$$

$\ddot{\eta} = \frac{d^2}{dt^2}\eta$ must be a linear combination of η and $\dot{\eta}$:

$$p_0(t)\ddot{\eta} + p_1(t)\dot{\eta} + p_2(t)\eta \quad = \quad 0.$$

Picard-Fuchs operator:

$$L^{(2)} \quad = \quad p_0(t)\frac{d^2}{dt^2} + p_1(t)\frac{d}{dt} + p_2(t)$$

The Picard-Fuchs operator

We can show that applying the Picard-Fuchs operator to the integrand gives an exact form:

$$L^{(2)}\left(\frac{\omega}{\mathcal{F}}\right) = d\beta$$

Integrating over σ yields

$$L^{(2)}S(t) = \int_{\sigma} d\beta$$

and using Stokes:

$$\int_{\sigma} d\beta = \int_{\partial\sigma} \beta$$

Integration of β over $\partial\sigma$ is elementary.

Result: The second-order differential equation

$$\left[p_0(t)\frac{d^2}{dt^2}+p_1(t)\frac{d}{dt}+p_2(t)\right]S(t) \quad = \quad p_3(t)$$

p_0, p_1, p_2 and p_3 are polynomials in t, for example

$$\begin{aligned}
p_0(t) &= t\left[t-(m_1+m_2+m_3)^2\right]\left[t-(-m_1+m_2+m_3)^2\right]\left[t-(m_1-m_2+m_3)^2\right] \\
&\quad \left[t-(m_1+m_2-m_3)^2\right]\left[3t^2-2M_{100}t-M_{200}+2M_{110}\right] \\
p_2(t) &= 3t^5-7M_{100}t^4+(2M_{200}+16M_{110})t^3+(6M_{300}-14M_{210})t^2 \\
&\quad -(5M_{400}-8M_{310}+6M_{220}-8M_{211})t+(M_{500}-3M_{410}+2M_{320}+8M_{311}-10M_{221})
\end{aligned}$$

$M_{\lambda_1\lambda_2\lambda_3}$: monomial symmetric polynomials in the variables m_1^2, m_2^2 and m_3^2.

$$M_{\lambda_1\lambda_2\lambda_3} \quad = \quad \sum_\sigma \left(m_1^2\right)^{\sigma(\lambda_1)}\left(m_2^2\right)^{\sigma(\lambda_2)}\left(m_3^2\right)^{\sigma(\lambda_3)}$$

Result: The second-order differential equation

In the equal mass case the second-order differential equation reduces to the well-known result from Laporta and Remiddi:

$$\left[p_0(t)\frac{d^2}{dt^2}+p_1(t)\frac{d}{dt}+p_2(t)\right]S(t) \quad = \quad p_3(t)$$

$$\begin{aligned}
p_0(t) &= t\left(t-m^2\right)\left(t-9m^2\right), \\
p_1(t) &= 3t^2-20tm^2+9m^4, \\
p_2(t) &= t-3m^2, \\
p_3(t) &= -6\mu^2.
\end{aligned}$$

Summary

The method is not specific to the two-loop sunrise integral:

- To find a differential equation for a Feynman integral, view the Feynman integral as a period of a variation of a mixed Hodge structure.
- The fibre will be the complement of an algebraic variety.
- The algebraic variety is given by the zero set of the graph polynomial.