Lectures on Cosmological Correlations

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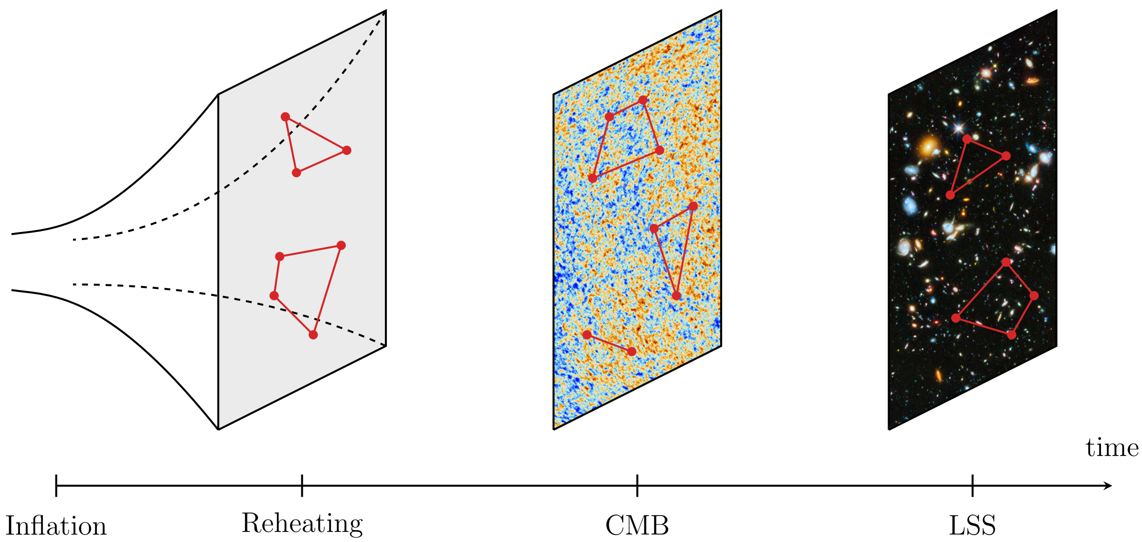

OUTLINE

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III. Inflationary Fluctuations

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V. Wavefunction Approach

VI. Cosmological Bootstrap

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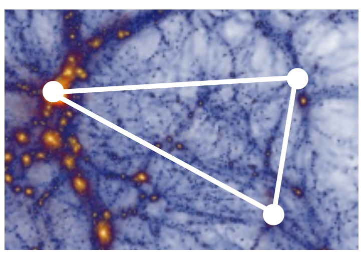
Lecture notes and lecture scripts can be found at:
https://github.com/ddbaumann/cosmo-correlators/Cargese-Lectures.pdf

I. MOTIVATION

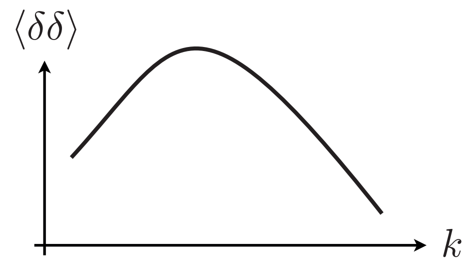
One of the biggest questions in all of science is the origin of structure in the Universe. What created everything that we see around us? An important clue lies in the fact that the structures in the Universe aren't distributed randomly, but displays large-scale correlations. These correlations are a fossil record of the early universe, and by measuring them we hope to uncover how the cosmological perturbations formed and evolved. In this course, we will give an introduction to these "cosmological correlations".

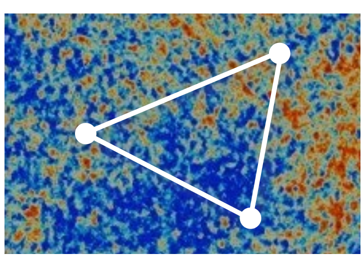
1.1 Practical Motivation

In cosmology, we measure **spatial correlations**:

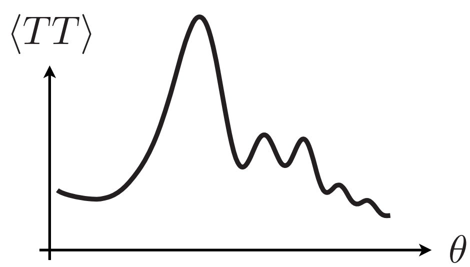


$$\langle \delta\rho(\mathbf{x}_1)\cdots\delta\rho(\mathbf{x}_N)\rangle$$

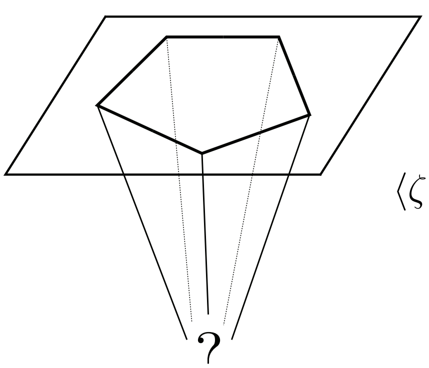




$$\langle \delta T(\theta_1)\cdots\delta T(\theta_N)\rangle$$



These correlations can be traced back to the origin of the hot Big Bang:



$$\langle \zeta(\mathbf{x}_1)\cdots\zeta(\mathbf{x}_N)\rangle$$

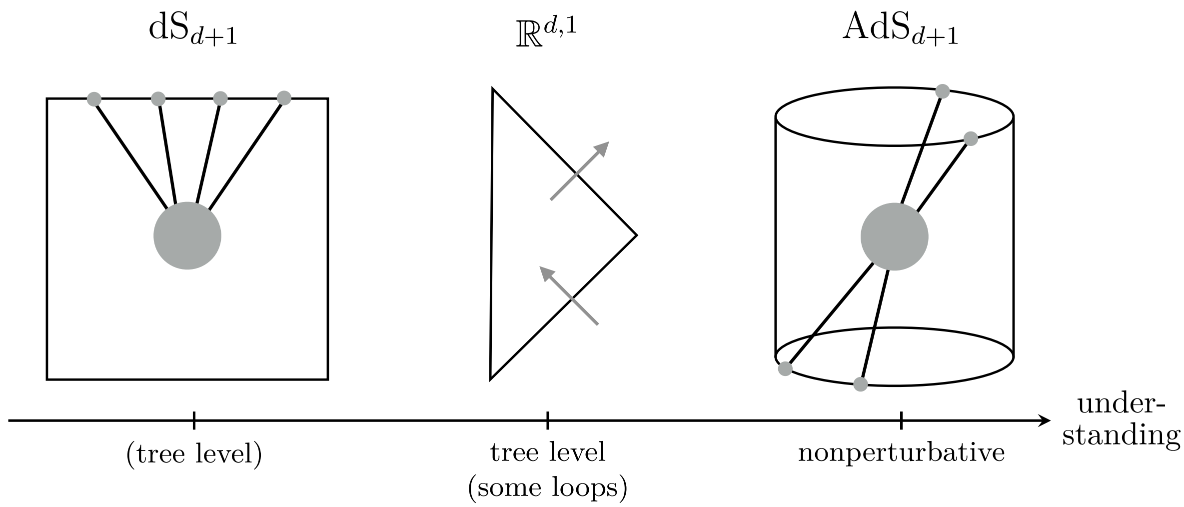
Where did the primordial correlations come from?

- Clue 1: The correlations span superhorizon scales.
- Clue 2: They are scale-invariant.

This suggests that the fluctuations were created **before the hot Big Bang**, during a phase of approximate time-translation invariance (= **inflation**).

1.2 Conceptual Motivation

The study of cosmological correlators is also of conceptual interest:



Our understanding of quantum field theory in de Sitter space (cosmology) is still rather underdeveloped $\Rightarrow$ Opportunity for you to make progress!

II. PRIMORDIAL STATISTICS

In cosmology, we only predict the statistical properties of the Universe. This means that we can't predict whether a galaxy will be found at a specific position $\mathbf{x}$. Instead, our theories can only tell us about the probability $P(\mathbf{x})$ of finding a galaxy at $\mathbf{x}$, or the conditional probability $P(\mathbf{x}_2, \mathbf{x}_1)$ that a second galaxy is at $\mathbf{x}_2$ given a galaxy at $\mathbf{x}_1$. Using these probabilities, we can derive spatial correlation functions, which are the main observables in cosmology.

2.1 Random Fields

Cosmological fluctuations $(\delta\rho, \delta g_{\mu\nu}, \cdots)$ are "random fields" $\equiv \Phi(t, \mathbf{x})$.

- *Correlation functions*:

$$\langle \Phi_1 \Phi_2 \cdots \Phi_N \rangle \equiv \int \Phi_1 \Phi_2 \cdots \Phi_N \, P(\Phi_1, \cdots, \Phi_N) \, \mathcal{D}\Phi_1 \cdots \mathcal{D}\Phi_N \,,$$

  where $\Phi_i \equiv \Phi(t, \mathbf{x}_i)$.

- *Ergodic hypothesis*:
  Average over statistical ensemble = spatial average over one realization.

$$\langle \Phi(t, \mathbf{x}) \rangle = \int \Phi P(\Phi) \mathrm{d}\Phi \underset{\substack{\uparrow \\ \text{ergodicity}}}{=} \frac{1}{V} \int \mathrm{d}^3 x \, \Phi(t, \mathbf{x}) \; \equiv \; \bar{\Phi}(t) \,.$$

- *Symmetries*:

  The correlations are statistically homogeneous and isotropic:

$$\langle \Phi(\mathbf{x}_1) \Phi(\mathbf{x}_2) \rangle \underset{\substack{\uparrow \\ \text{homogeneity}}}{=} \xi_\Phi(\mathbf{x}_1 - \mathbf{x}_2) \underset{\substack{\uparrow \\ \text{isotropy}}}{=} \xi_\Phi(|\mathbf{x}_1 - \mathbf{x}_2|) \equiv \xi_\Phi(x_{12}) \,,$$

- *Fourier space*:

  Reasons cosmologists like Fourier space:

  1) Fourier modes "decouple" (in linear theory)
  2) Correlators "diagonalize"

  Fourier expansion of the field is

$$\Phi(t,\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, \underbrace{\Phi_{\mathbf{k}}(t)}_{\text{complex}\,:\, \Phi^*_{\mathbf{k}} = \Phi_{-\mathbf{k}}}$$

  ⇒ Equal-time correlators: $\langle \Phi_{\mathbf{k}_1}\Phi_{\mathbf{k}_2}\cdots\Phi_{\mathbf{k}_N}\rangle$.
  ⇒ Two-point function:

$$\begin{aligned}\langle \Phi_{\mathbf{k}_1}\Phi_{\mathbf{k}_2}\rangle &= \int \mathrm{d}^3x_1\mathrm{d}^3x_2\, e^{-i\mathbf{k}_1\cdot\mathbf{x}_1}e^{-i\mathbf{k}_2\cdot\mathbf{x}_2}\,\langle\Phi(\mathbf{x}_1)\Phi(\mathbf{x}_2)\rangle \\ &= \underbrace{\int \mathrm{d}^3x_2\, e^{-i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}_2}}_{(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2)\ \uparrow\ \text{homogeneity}}\ \underbrace{\int \mathrm{d}^3x_{12}\, e^{-i\mathbf{k}_1\cdot\mathbf{x}_{12}}\,\xi_\Phi(x_{12})}_{P_\Phi(k_1)\ \uparrow\ \text{power spectrum}}\end{aligned}$$

- *Power spectrum*:

$$\langle \Phi_{\mathbf{x}_1}\Phi_{\mathbf{x}_2}\rangle \equiv \xi_\Phi(|\mathbf{x}_1-\mathbf{x}_2|) \quad\Rightarrow\quad \langle \Phi_{\mathbf{k}_1}\Phi_{\mathbf{k}_2}\rangle = (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2)P_\Phi(k_1)\,,$$

  where

$$P_\Phi(k) = \int \mathrm{d}^3x\, e^{-i\mathbf{k}\cdot\mathbf{x}}\,\xi_\Phi(x)\,,$$

$$\underbrace{\xi_\Phi(0)}_{\uparrow\ \text{variance}} = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,P_\Phi(k) = \int \mathrm{d}\ln k\, \frac{k^3}{2\pi^2}P_\Phi(k) \equiv \int \mathrm{d}\ln k\, \underbrace{\Delta^2_\Phi(k)}_{\uparrow\ \text{dimensionless power spectrum}}$$

  The spectrum is *scale-invariant* if $\Delta^2_\Phi(k)$ is a constant.

- For *Gaussian random fields*, all correlators are functions of $P_\Phi(k)$.
  ⇒ "connected correlators" vanish for $N > 2$.

2.2 Cosmological Perturbations

Write the metric and stress-energy tensor as

$$g_{\mu\nu}(t, \mathbf{x}) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t, \mathbf{x}) \, ,$$
$$T_{\mu\nu}(t, \mathbf{x}) = \bar{T}_{\mu\nu}(t) + \delta T_{\mu\nu}(t, \mathbf{x}) \, ,$$

and then expand $\nabla^\mu T_{\mu\nu} = 0$ and $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ to linear order.

- The form of the perturbations depends on the choice of coordinates.
- In the *comoving gauge* ($T_{0i} \equiv 0$), we can write the spatial metric as

$$g_{ij} = a^2 e^{2\zeta(t,\mathbf{x})} \delta_{ij} \, ,$$

  where ζ is the *comoving curvature perturbation*: $a^2 R_{(3)} = 4\nabla^2\zeta$.
- During inflation:

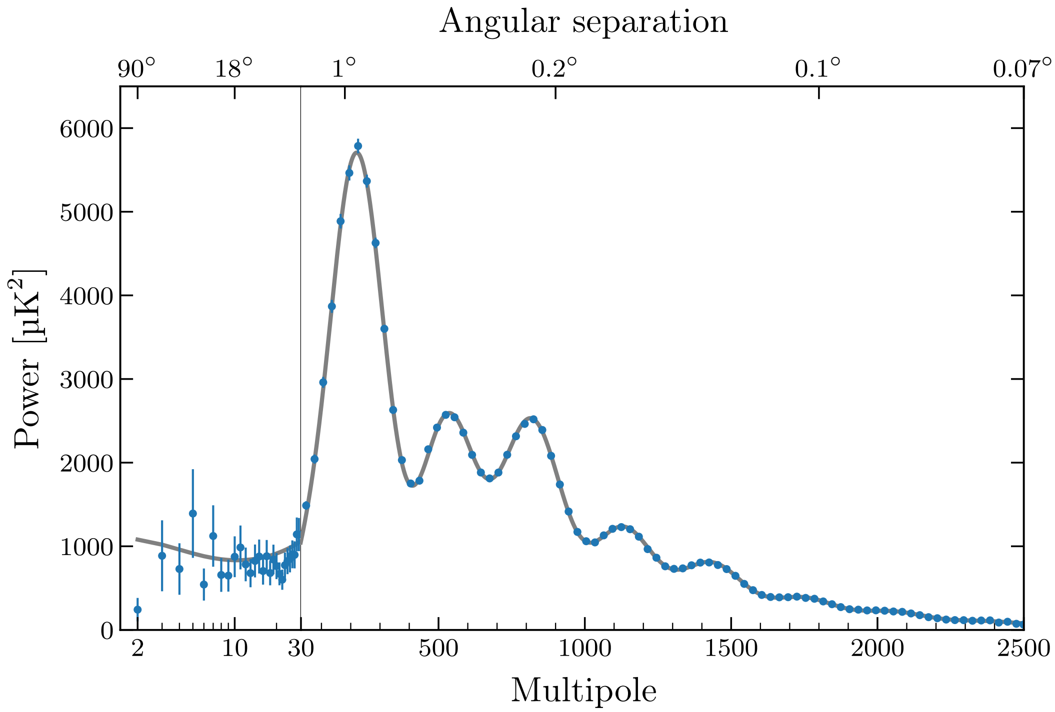
$$\zeta = -\frac{H}{\dot{\phi}} \delta\phi \, ,$$

  where $\delta\phi$ is the inflaton perturbation in spatially flat gauge.
- After inflation: $\zeta = \text{const}$ for $k < (aH)$ (superhorizon scales).
- Each observable $\mathcal{O}$ is related to $\zeta_{\mathbf{k}}(t_i)$ by a *transfer function*:

$$\mathcal{O}(\mathbf{x}, t) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \, T_{\mathcal{O}}(k; t, t_i) \, \zeta_{\mathbf{k}}(t_i) \, e^{i\mathbf{k}\cdot\mathbf{x}} \, .$$

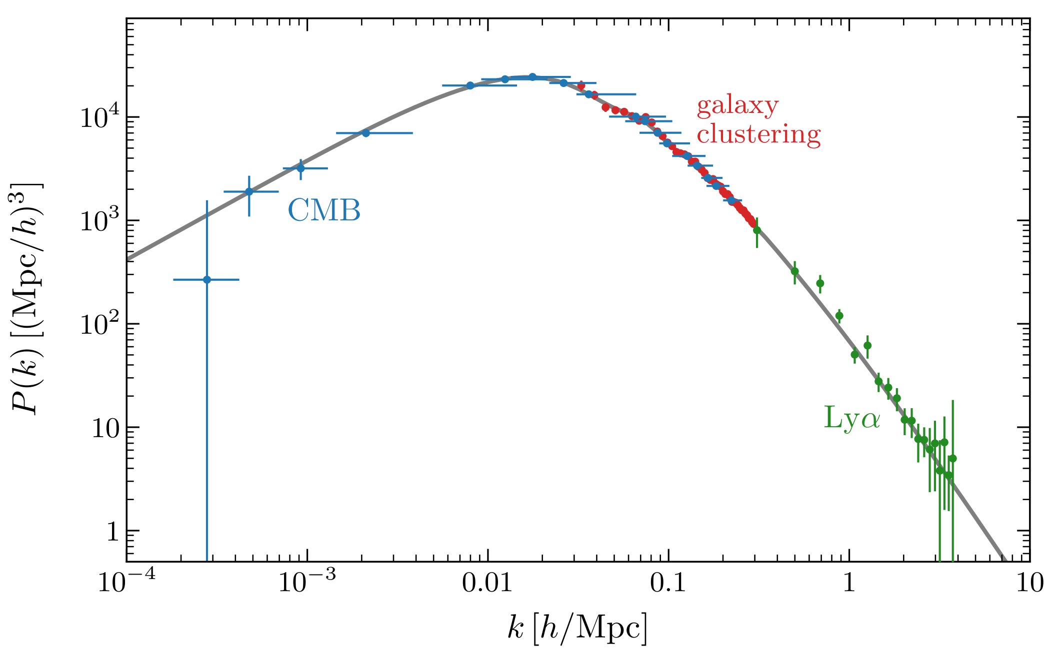
$\Rightarrow$ Fluctuations can be traced back to $\zeta_{\mathbf{k}}(t_i)$.
$\Rightarrow$ Physics before the hot Big Bang ($t < t_i$) in encoded in $\langle \zeta_{\mathbf{k}_1} \cdots \zeta_{\mathbf{k}_N} \rangle$.

2.3 Observed Correlations



The shape of the CMB power spectrum depends on

$$\{\, H_0\,,\, \Omega_K\,,\, \Omega_\Lambda\,,\, \Omega_m\,,\, \Omega_b\,,\, \Omega_\nu\,,\, A_s e^{-2\tau}\,,\, n_s \,\}\,.$$



The shape of the matter power spectrum depends on $\{\, \Omega_m\,,\, \sigma_8 \,\}$.

2.4 Initial Conditions

We have learned **four important facts** about the initial conditions:

1. *Superhorizon*

   The fluctuations spanned superhorizon scales at photon decoupling.
   $\Rightarrow$ They must have been created before the hot Big Bang.

2. *Scale-invariant*

   The primordial fluctuations were scale invariant.
   $\Rightarrow$ This is a natural prediction of inflation.

3. *Adiabatic*

   The fluctuations were adiabatic (i.e. no fluctuations in composition).
   $\Rightarrow$ All fluctuations are sourced by a single scalar mode:

$$\zeta(\mathbf{x}) = \frac{1}{3}\left(\frac{\delta\rho}{\bar{\rho}+\bar{P}}\right)_i , \quad i = \text{dark matter, baryons, photons}$$

4. *Gaussian*

   The primordial fluctuations were highly Gaussian:

$$\frac{\langle\zeta\zeta\zeta\rangle}{\langle\zeta\zeta\rangle^{3/2}} < 0.1\%$$

   $\Rightarrow$ This a natural prediction for quantum fluctuations of free fields.
   $\Rightarrow$ A lot of the physics of inflation is encoded in non-Gaussian correlations.

III. INFLATIONARY FLUCTUATIONS

One of the most remarkable features of inflation is that it provides a natural mechanism for creating the primordial density fluctuations that seeded the structure in the Universe. In this chapter, we will derive the spectrum of quantum fluctuations produced during inflation.

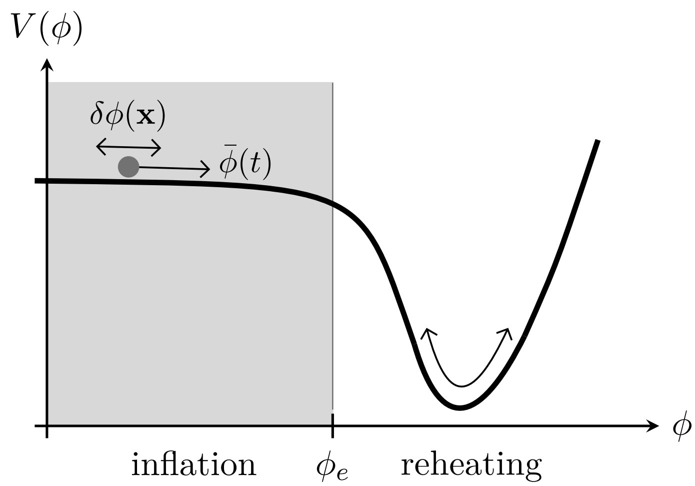
3.1 Basics of Inflation

Consider the action

$$S = \int \mathrm{d}^4x \sqrt{-g} \left(\frac{M_{\mathrm{Pl}}^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right),$$

where $M_{\mathrm{Pl}} \equiv (8\pi G)^{-1/2}$.

- Leads to accelerated expansion ($\ddot{a} > 0$) when the potential $V(\phi)$ is flat.
- Inflation ends when the potential steepens.
- Quantum fluctuations in the field create density perturbations.



Varying the inflaton action with respect to the metric gives

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - V(\phi) \right).$$

- Assuming $\phi = \phi(t)$ for the homogeneous background, we get

$$\rho_\phi = -T^0{}_0 = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,.$$

- The Friedmann equation becomes

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho_\phi}{3M_{\rm Pl}^2} = \frac{1}{3M_{\rm Pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right).$$

- Slow-roll inflation occurs when $\frac{1}{2}\dot{\phi}^2 \ll V$.
- During inflation, $H \approx$ const $\Rightarrow a(t) \approx e^{Ht}$.

Varying the inflaton action with respect to the field gives

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{\mathrm{d}V}{\mathrm{d}\phi}\,.$$

- Sustained slow-roll inflation occurs when $\ddot{\phi} \ll 3H\dot{\phi}$.

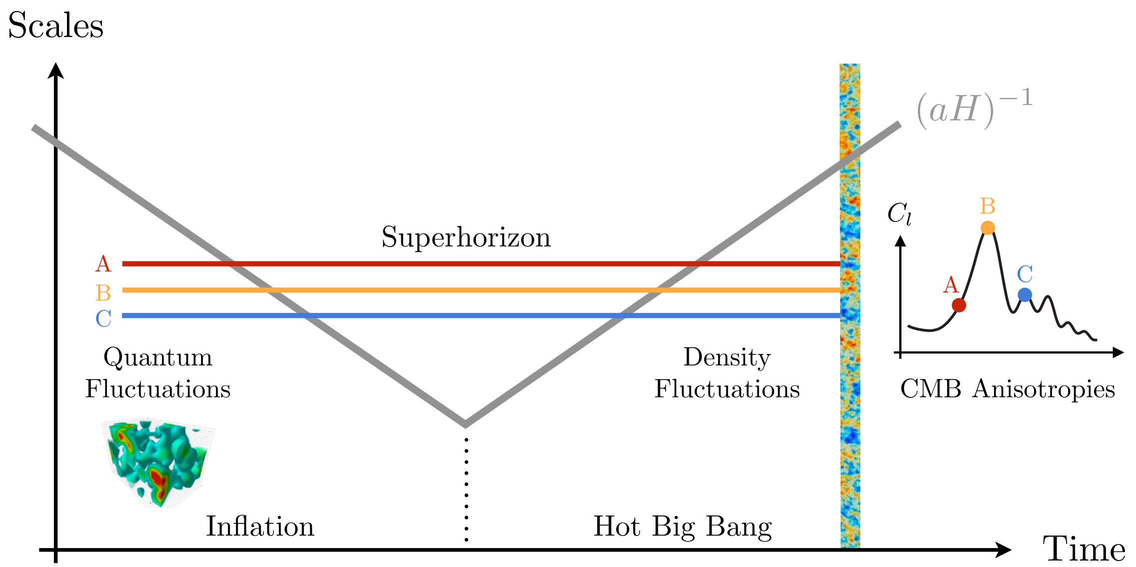
Accelerated expansion implies a shrinking comoving Hubble radius:

$$\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{a})^{-1} = -\frac{\ddot{a}}{(\dot{a})^2} < 0\,.$$

This has important consequences for the evolution of perturbations:

- Small-scale perturbations are stretched to large scales.
- Quantum fluctuations become classical perturbations.
- Perturbations are frozen on super-Hubble scales.

See diagram on the next page.



The shrinking Hubble radius can also be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}(1-\varepsilon)\,,$$

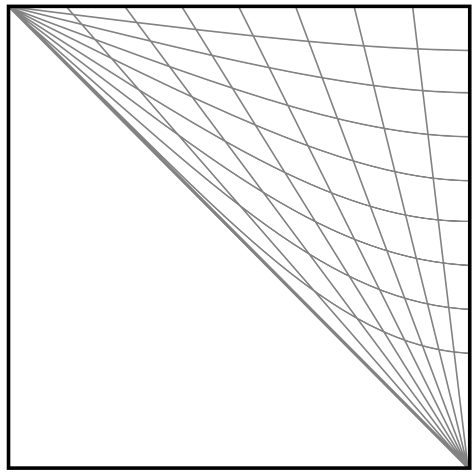
where the "slow-roll parameter" is

$$\varepsilon \equiv -\frac{\dot{H}}{H^2}\,.$$

- Inflation requires $\varepsilon < 1$.
- Scale-invariant perturbations require $\varepsilon \ll 1$.

3.2 Free Scalars in De Sitter

The spacetime during inflation is approximately **de Sitter space**.



In conformal time, $\mathrm{d}\eta = \mathrm{d}t/a(t)$, the de Sitter metric is

$$\mathrm{d}s^2 = \frac{-\mathrm{d}\eta^2 + \mathrm{d}\mathbf{x}^2}{(H\eta)^2}\,,$$

where $-\infty < \eta < 0$. The primordial correlations live on the **future boundary** at $\eta_* \approx 0$. We consider a free, massless scalar in this background.

Classical dynamics

The action is

$$\begin{aligned} S &= \frac{1}{2}\int \mathrm{d}\eta \mathrm{d}^3x\, a^2\Big[(\phi')^2 - (\nabla\phi)^2\Big] \\ &= \frac{1}{2}\int \mathrm{d}\eta \mathrm{d}^3x \left[(u')^2 - (\nabla u)^2 + \frac{a''}{a}u^2\right], \end{aligned}$$

where $u \equiv a(\eta)\phi$.

- The classical equation of motion then is

$$u'' - \nabla^2 u - \frac{a''}{a}u = 0 \qquad \Longrightarrow \qquad \boxed{u_{\mathbf{k}}'' + \left(k^2 - \frac{2}{\eta^2}\right)u_{\mathbf{k}} = 0}\,.$$

- At early times ($|k\eta| \gg 1$):

$$u_{\mathbf{k}}'' + k^2 u_{\mathbf{k}} = 0 \qquad \Longrightarrow \qquad u_{\mathbf{k}} = \frac{1}{\sqrt{2k}}\,e^{\pm ik\eta}\,.$$

- At late times ($|k\eta| \ll 1$):

$$u''_{\mathbf{k}} - \frac{2}{\eta^2} u_{\mathbf{k}} = 0 \qquad \Longrightarrow \qquad u_{\mathbf{k}} = A_{\mathbf{k}} \eta^{-1} + B_{\mathbf{k}} \eta^2 \,.$$

- Most general solution:

$$\boxed{u_{\mathbf{k}}(\eta) = C_{\mathbf{k}} \left(1 - \frac{i}{k\eta}\right) \frac{e^{-ik\eta}}{\sqrt{2k}} + D_{\mathbf{k}} \left(1 + \frac{i}{k\eta}\right) \frac{e^{ik\eta}}{\sqrt{2k}}} \,.$$

Canonical quantization

- Introduce **operators** $\hat{u}, \hat{\pi} = \partial_\eta \hat{u}$.
- Impose **commutation relations**:

$$[\hat{u}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{x}')] = i\delta(\mathbf{x} - \mathbf{x}') \quad \Longrightarrow \quad [\hat{u}_{\mathbf{k}}(\eta), \hat{\pi}_{\mathbf{k}'}(\eta)] = i(2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') \,.$$

- Define **mode expansion**:

$$\hat{u}(\eta, \mathbf{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left(u_k(\eta)\, \hat{a}_{\mathbf{k}} + u_k^*(\eta)\, \hat{a}^\dagger_{-\mathbf{k}} \right) e^{i\mathbf{k}\cdot\mathbf{x}} \,,$$

where

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') \qquad \Longrightarrow \qquad u_k(\eta)\, \partial_\eta u_k^*(\eta) - u_k^*(\eta) \partial_\eta u_k(\eta) = i \,.$$

- Define **vacuum**:

$$\hat{a}_{\mathbf{k}} |0\rangle = 0 \,.$$

Bunch–Davies vacuum

Choose the minimum energy state at early times:

$$\lim_{k\eta \to -\infty} u_k(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta} \qquad \Longrightarrow \qquad \boxed{u_k(\eta) = \left(1 - \frac{i}{k\eta}\right) \frac{e^{-ik\eta}}{\sqrt{2k}}} \,.$$

Zero-point fluctuations

The variance of the field operator is

$$\begin{aligned}
\langle|\hat{u}|^2\rangle &\equiv \langle 0|\hat{u}(\eta,\mathbf{0})\hat{u}(\eta,\mathbf{0})|0\rangle \\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\int \frac{\mathrm{d}^3k'}{(2\pi)^3}\,\langle 0|\big(u_k^*(\eta)\hat{a}^\dagger_{-\mathbf{k}} + u_k(\eta)\hat{a}_{\mathbf{k}}\big)\big(u_{k'}(\eta)\hat{a}_{\mathbf{k}'} + u_{k'}^*(\eta)\hat{a}^\dagger_{-\mathbf{k}'}\big)|0\rangle \\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\int \frac{\mathrm{d}^3k'}{(2\pi)^3}\,u_k(\eta)u_{k'}^*(\eta)\,\langle 0|[\hat{a}_{\mathbf{k}},\hat{a}^\dagger_{-\mathbf{k}'}]|0\rangle \\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,|u_k(\eta)|^2 \\
&= \int \mathrm{d}\log k\,\,\frac{k^3}{2\pi^2}|u_k(\eta)|^2\ .
\end{aligned}$$

We define the (dimensionless) **power spectrum** as

$$\Delta_u^2(k,\eta) \equiv \frac{k^3}{2\pi^2}|u_k(\eta)|^2\,.$$

Substituting the Bunch–Davies mode function, we find

$$\Delta_\phi^2(k,\eta) = \frac{\Delta_u^2(k,\eta)}{a^2(\eta)} = \left(\frac{H}{2\pi}\right)^2\Big[1+(k\eta)^2\Big] \xrightarrow{k\eta\to 0} \boxed{\left(\frac{H}{2\pi}\right)^2}\ .$$

Since $H \approx$ const, the spectrum is **scale-invariant**.

3.3 Curvature Perturbations

Now consider the coupled inflaton-metric fluctuations.

- ADM metric:

$$ds^2 = -N^2 dt^2 + h_{ij}(N^i dt + dx^i)(N^j dt + dx^j)\,,$$

  so that

$$g_{\mu\nu} = \left(\begin{array}{c|c} -N^2 + N_i N^i & N_j \\ \hline N_i & h_{ij} \end{array}\right), \qquad g^{\mu\nu} = \frac{1}{N^2}\left(\begin{array}{c|c} -1 & N^j \\ \hline N^i & N^2 h^{ij} - N^i N^j \end{array}\right),$$

  and $\sqrt{-g} = N\sqrt{h}$.

- Intrinsic curvature: $R^{(3)}_{ij}$
- Extrinsic curvature:

$$K_{ij} \equiv \frac{1}{2N}\left(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i\right) \equiv \frac{1}{N} E_{ij}\,.$$

- Four-dimensional Ricci scalar:

$$R = R^{(3)} + N^{-2}\left(E^{ij}E_{ij} - E^2\right) + \text{total derivative},$$

  where $E \equiv h^{ij}E_{ij}$.

- Inflaton action:

$$S = \frac{1}{2}\int d^4x \sqrt{h} N \Big[R^{(3)} + N^{-2}\big(E^{ij}E_{ij} - E^2\big) + N^{-2}\big(\dot{\phi} - N^i\partial_i\phi\big)^2 - h^{ij}\partial_i\phi\partial_j\phi - 2V \Big]\,,$$

  where $M_{\rm Pl} \equiv 1$.

- Varying the action with respect to N and N^i, we find

$$R^{(3)} - 2V - h^{ij}\partial_i\phi\partial_j\phi - N^{-2}[E_{ij}E^{ij} - E^2 - (\dot{\phi} - N^i\partial_i\phi)^2] = 0\,,$$
$$\nabla_i[N^{-1}(E^i_j - E\delta^i_j)] = 0\,.$$

  These are constraint equations for N and N^i.

We will work in **comoving gauge**:

$$h_{ij} = a^2 e^{2\zeta(t,\mathbf{x})}\delta_{ij}\,, \quad N \equiv 1 + \alpha(t,\mathbf{x})\,, \quad N_i \equiv \partial_i \beta(t,\mathbf{x})\,.$$

Ex: Show that

$$\begin{aligned} R^{(3)} &= -2a^{-2}e^{-2\zeta}\left[2\partial^2\zeta + (\partial\zeta)^2\right], \\ E_{ij} &= a^2 e^{2\zeta}(H + \dot{\zeta})\delta_{ij} - \partial_{(i}N_{j)} + 2N_{(i}\partial_{j)}\zeta - N_k\partial_k\zeta\delta_{ij}\,. \end{aligned}$$

Ex: Show that

$$\alpha = \frac{\dot{\zeta}}{H}\,, \quad \partial^2\beta = -\frac{\partial^2\zeta}{H} + a^2\frac{\dot{\phi}^2}{2H^2}\dot{\zeta}\,.$$

Plugging these results into the action, expanding to second order, performing integrations by parts and using the background equations of motion, we get

$$\boxed{S_2 = \int \mathrm{d}t\,\mathrm{d}^3x\, a^3\varepsilon\left(\dot{\zeta}^2 - \frac{1}{a^2}(\partial_i\zeta)^2\right)}\,.$$

- Note that $\mathcal{L}_2 \propto \varepsilon \implies \zeta$ is pure gauge in the dS limit ($\varepsilon \to 0$)
- Note that ζ has no mass term $\implies$ frozen superhorizon modes

In conformal time, we have

$$S_2 = \frac{1}{2}\int \mathrm{d}\eta\mathrm{d}^3x\, z^2\left[(\zeta')^2 - (\partial_i\zeta)^2\right], \quad z \equiv a\sqrt{2\varepsilon}\,.$$

Defining $u \equiv z\zeta$, the equation of motion is

$$u'' - \nabla^2 u - \frac{z''}{z}u = 0 \quad \implies \quad \boxed{u_k'' + \left(k^2 - \frac{z''}{z}\right)u_k = 0}$$

Ex: Show that

$$\frac{z''}{z} = \frac{1}{\eta^2}\left[2 + 3\left(\varepsilon + \frac{1}{2}\kappa\right)\right],$$

where $\kappa \equiv \varepsilon'/(\mathcal{H}\varepsilon)$.

The equation of motion becomes

$$u_k'' + \left(k^2 - \frac{\nu^2 - 1/4}{\eta^2}\right) u_k = 0\,, \quad \text{where} \quad \nu \equiv \frac{3}{2} + \varepsilon + \frac{1}{2}\kappa\,,$$

whose Bunch–Davies solution is

$$\boxed{u_k(\eta) = \frac{\sqrt{\pi}}{2}\sqrt{-\eta}\, H_\nu^{(1)}(-k\eta)}\,.$$

Ex: Show that

$$z(\eta) = z_*(\eta/\eta_*)^{\frac{1}{2}-\nu}\,,$$

where $\eta_* = -k_*^{-1}$.

The dimensionless power spectrum of ζ is

$$\begin{aligned}\Delta_\zeta^2(k) &= \frac{k^3}{2\pi^2}\frac{|u_k(\eta)|^2}{z^2(\eta)} \\ &= \frac{k^3}{2\pi^2}\frac{1}{2\varepsilon_* a_*^2}(-k_*\eta)^{2\nu-1}\,\frac{\pi}{4}(-\eta)|H_\nu^{(1)}(-k\eta)|^2\,.\end{aligned}$$

- In the late-time limit, we get

$$\lim_{k\eta\to 0}|H_\nu^{(1)}(-k\eta)|^2 \approx \frac{2}{\pi}(-k\eta)^{-2\nu} \qquad \Longrightarrow \qquad \boxed{\Delta_\zeta^2(k) = \frac{1}{8\pi^2\varepsilon_*}\frac{H_*^2}{M_{\rm Pl}^2}\,(k/k_*)^{3-2\nu}}\,.$$

- The scalar tilt is

$$\boxed{n_{\rm s} - 1 \equiv \frac{{\rm d}\ln\Delta_\zeta^2}{{\rm d}\ln k} = -2\varepsilon_* - \kappa_*}\,,$$

which was first measured by WMAP: $n_{\rm s} = 0.965 \pm 0.004$.

3.4 Gravitational Waves

Consider tensor metric perturbations:

$$\mathrm{d}s^2 = a^2(\eta)\big[-\mathrm{d}\eta^2 + (\delta_{ij} + h_{ij})\mathrm{d}x^i\mathrm{d}x^j\big]\,,$$

where $\partial^i h_{ij} = h^i{}_i = 0$.

- Expanding the Einstein–Hilbert action to second order gives

$$S = \frac{M_{\rm Pl}^2}{2}\int \mathrm{d}^4x\sqrt{-g}R \;=\; \frac{M_{\rm Pl}^2}{8}\int \mathrm{d}\eta\mathrm{d}^3x\, a^2\left[(h_{ij}')^2 - (\nabla h_{ij})^2\right] + \cdots\,.$$

- Use rotational symmetry to write $\mathbf{k} = (0,0,k)$ and

$$\frac{M_{\rm Pl}}{\sqrt{2}}a\,h_{ij} \equiv \begin{pmatrix} u_+ & u_\times & 0 \\ u_\times & -u_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- The action becomes

$$S = \frac{1}{2}\sum_{\lambda=+,\times}\int \mathrm{d}\eta\mathrm{d}^3x\left[(u_\lambda')^2 - (\nabla u_\lambda)^2 + \frac{a''}{a}u_\lambda^2\right].$$

- The equation of motion for each polarization mode is

$$u_k'' + \left(k^2 - \frac{a''}{a}\right)u_k = 0\,,$$

  where the effective mass can be written as

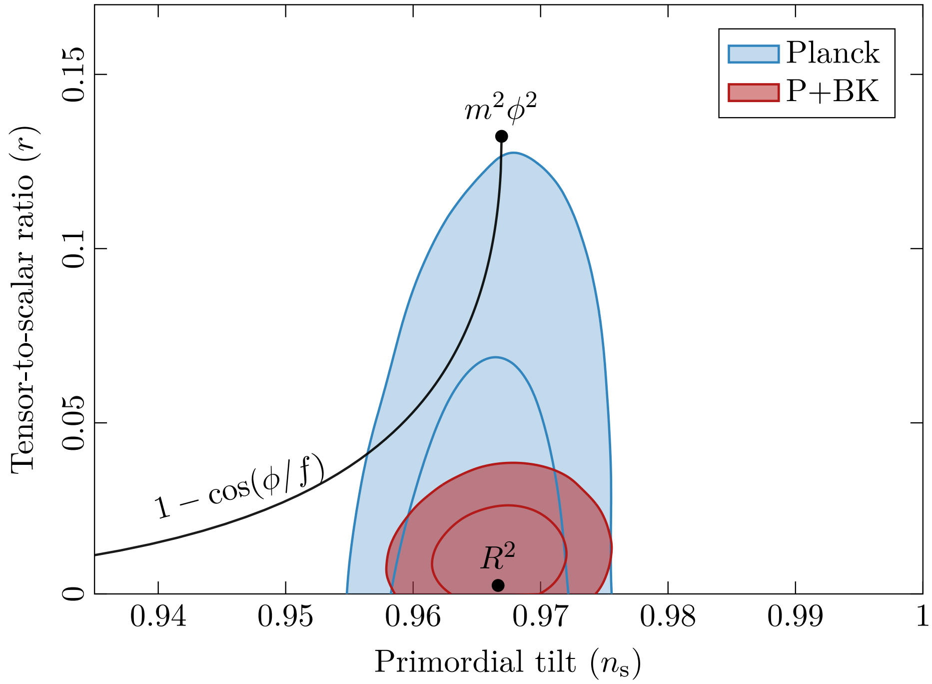
$$\frac{a''}{a} = \frac{\nu^2 - 1/4}{\eta^2}\,, \quad \text{with} \quad \nu \approx \frac{3}{2} + \varepsilon\,.$$

- Bunch–Davies mode function is the same as before.
- Superhorizon limit of the power spectrum:

$$\Delta_h^2(k) = 2\times\left(\frac{2}{aM_{\rm Pl}}\right)^2 \lim_{k\eta\to 0}\frac{k^3}{2\pi^2}|u_k(\eta)|^2 = \boxed{\frac{2}{\pi^2}\frac{H_*^2}{M_{\rm Pl}^2}(k/k_*)^{3-2\nu}}\,.$$

- Observations are expressed in terms of the tensor-to-scalar ratio:

$$\boxed{r \equiv \frac{\Delta_h^2(k_*)}{\Delta_\zeta^2(k_*)} = 16\varepsilon_*}\,.$$

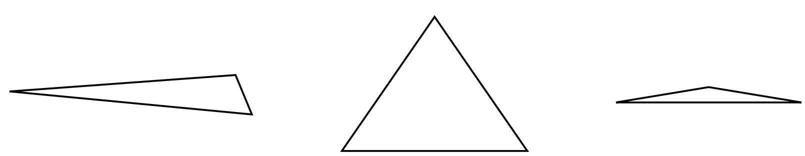


3.5 Primordial Non-Gaussianity

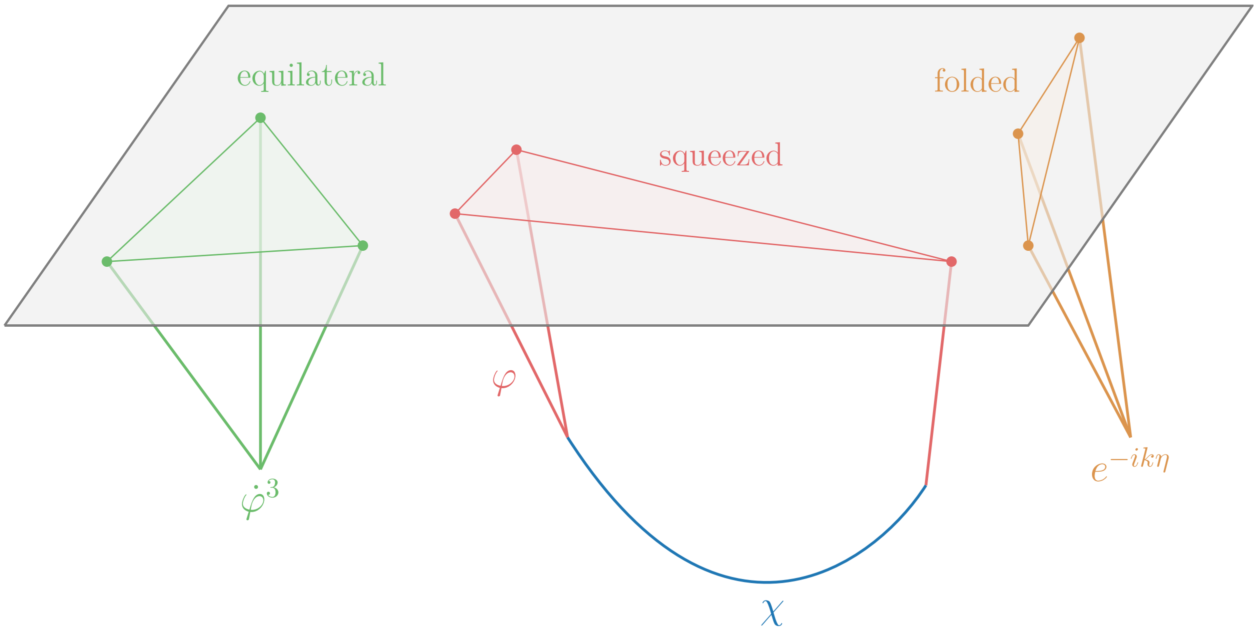
The main diagnostic of primordial non-Gaussianity is the **bispectrum**:

$$\langle \zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle = B_\zeta(k_1,k_2,k_3) \times (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\,.$$

- amplitude: $f_{\rm NL} \equiv \dfrac{5}{18}\dfrac{B_\zeta(k,k,k)}{P_\zeta^2(k)}$
- shape:



- effect: new particles — new interactions — excited states
- Planck constraints: $|f_{\rm NL}^{\rm loc}| < 5$ — $|f_{\rm NL}^{\rm equil}| < 40$ — $|f_{\rm NL}^{\rm flat}| < 20$



In the following, we describe three methods for computing these higher-point correlations:

- In-In Formalism
- Wavefunction Approach
- Cosmological Bootstrap
