Bound-Unbound Universality

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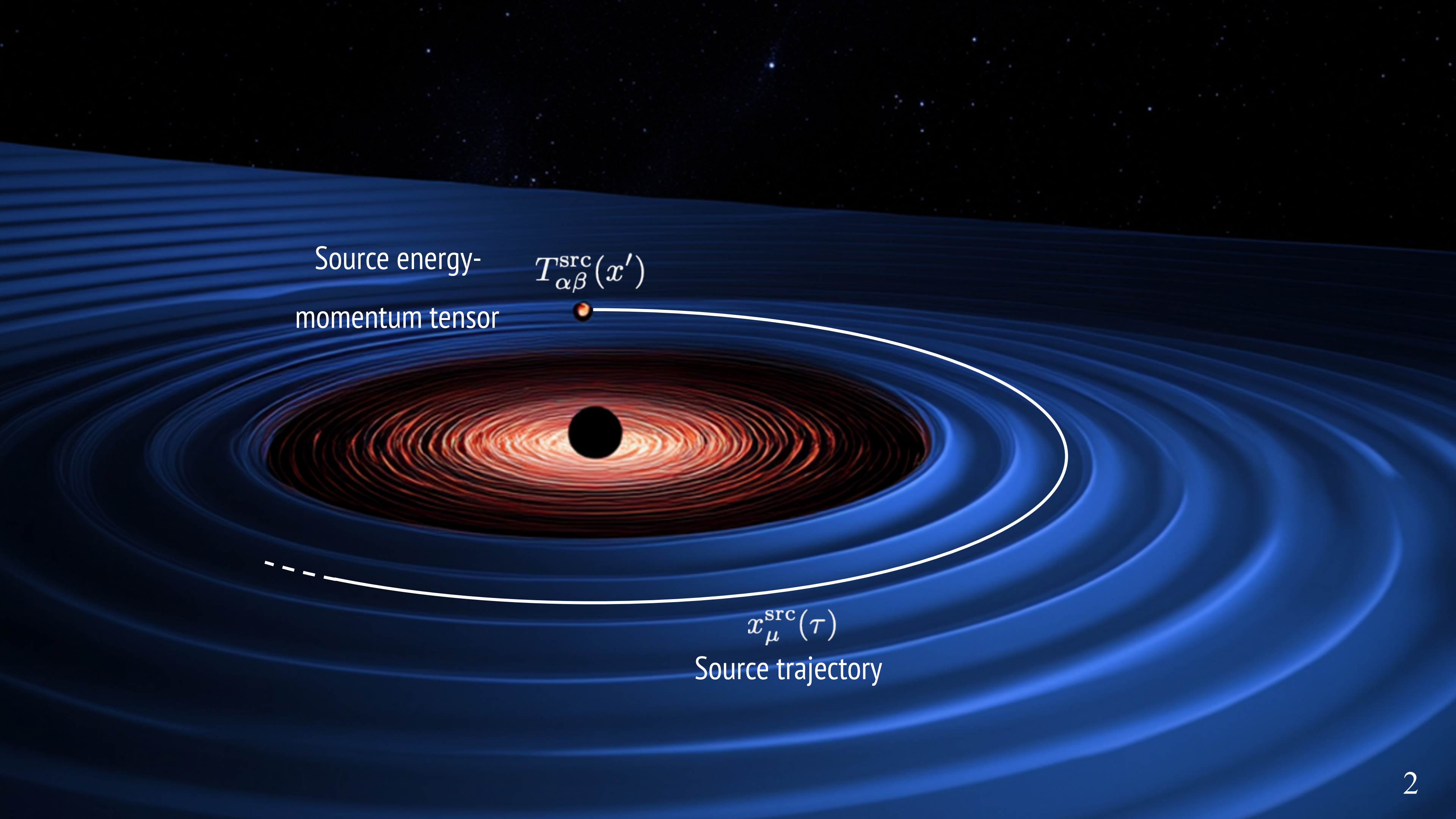
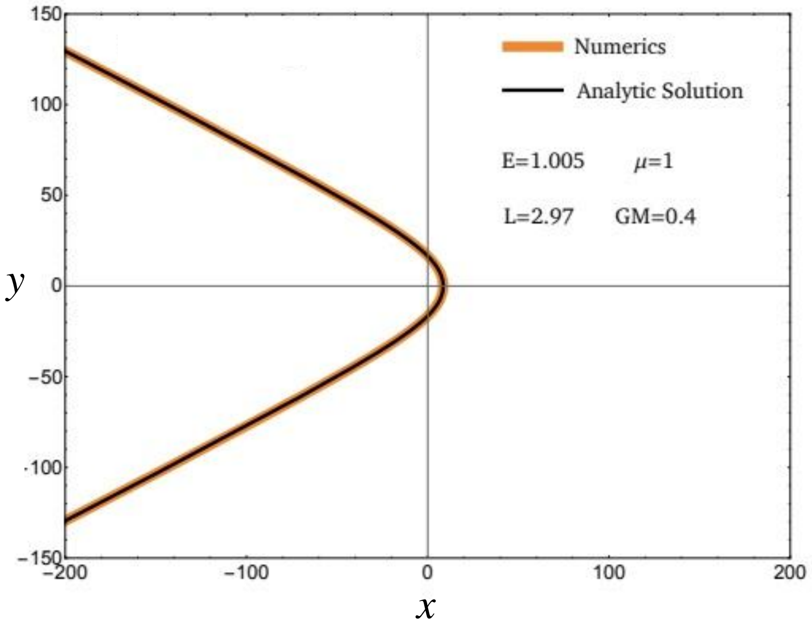

Source energy-
momentum tensor
$T^{\mathrm{src}}_{\alpha\beta}(x')$
$x^{\mathrm{src}}_{\mu}(\tau)$
Source trajectory

Time-Dependent Observables

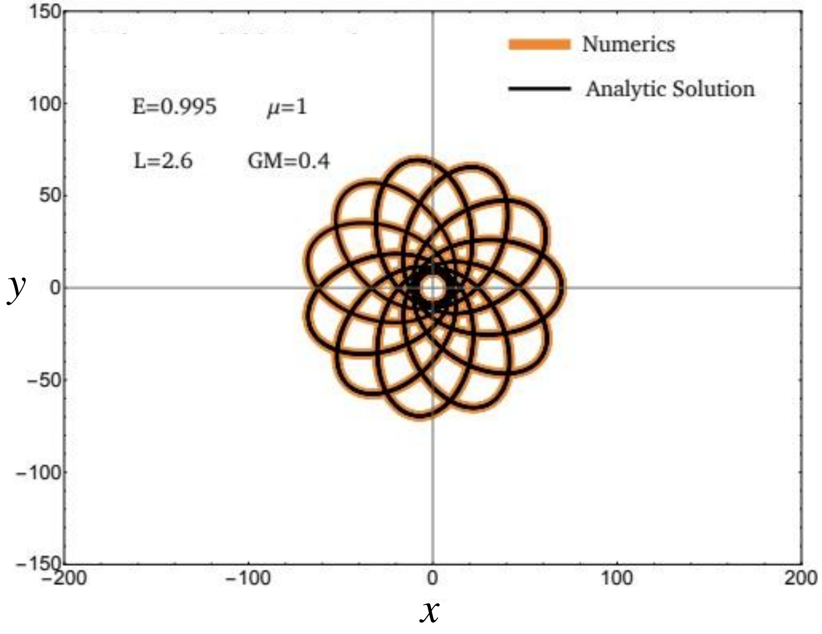
Consider a time-dependent observable $\mathcal{O}(t)$: radius, azimuthal angle, etc...

Clearly, $\mathcal{O}(t)$ is very different between the unbound ($E > \mu$) and bound ($E < \mu$) cases.

$E > \mu$, unbound motion



$E < \mu$, bound motion



Here we...

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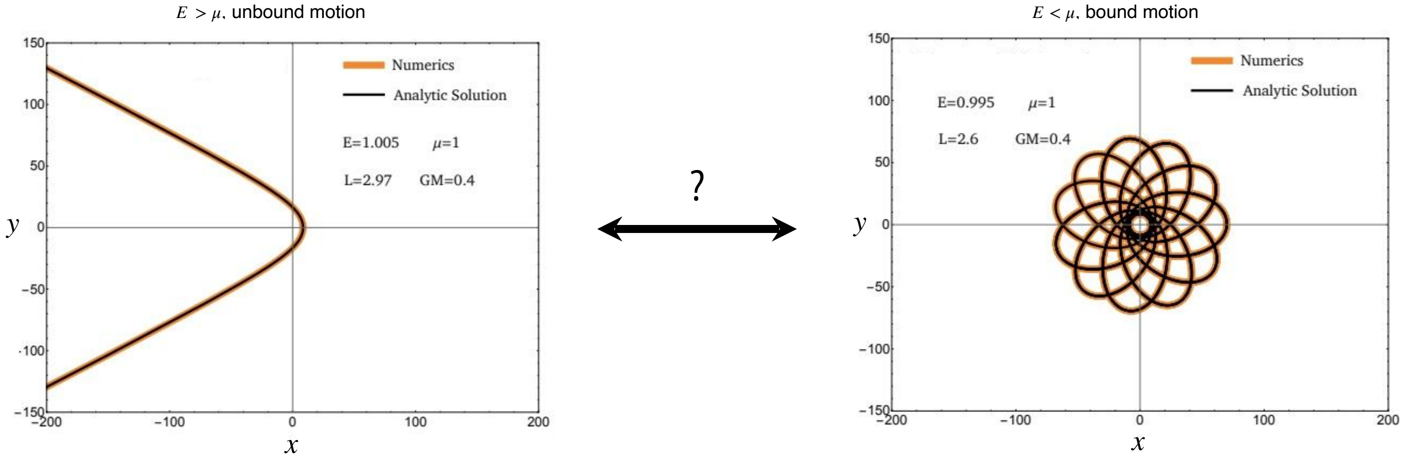
Is there a way to compute and **analytically continue** $\mathcal{O}(t)$ between the unbound and bound regimes?



This is a crucial question for the computation of (bound) black hole inspirals with (unbound) scattering amplitude methods. We answer it in the **probe limit**.

Laplace Observables

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We found a representation for any $\mathcal{O}(t)$ which is automatically analytically continued between unbound and bound motion: its **Laplace transform**

Laplace: $$\hat{\mathcal{O}}\left(s_L\right)=\frac{\Omega_r}{2\pi}\int_{-\infty}^{\infty} dt\ \mathcal{O}(t)\, e^{-s_L \Omega_r t}$$

Inverse Laplace: $$\mathcal{O}(t)=-i\mathcal{P}\int_{\Gamma_r-i\infty}^{\Gamma_r+i\infty} \hat{\mathcal{O}}\left(s_L\right) e^{s_L \Omega_r t} ds_L$$

Γ_r - real value in convergence strip of $\hat{\mathcal{O}}(s_L)$

$\mathcal{P}$ – Principal value

$\Omega_r^b = i\Omega_r = \dfrac{2\pi}{T_r}$ is the radial fundamental frequency with $T_r = 2\int_{r_{min}}^{r_*} \dfrac{\gamma(r')\mu}{\sqrt{U_r(r')}}\, dr'$

Ω_r^b is positive real for bound. Ω_r is positive real for unbound

Laplace Observables

In the bound regime, the Laplace transform becomes a **Fourier transform**

Fourier: $$\widetilde{\mathcal{O}}(s_L) = i\hat{\mathcal{O}}(s_L) = \frac{\Omega_r^b}{2\pi}\int_{-\infty}^{\infty} dt\; \mathcal{O}(t)\, e^{is_L\Omega_r^b t}$$

Inverse Fourier: $$\mathcal{O}(t) = \mathcal{P}\int_{-\infty}^{\infty} ds_L\; \widetilde{\mathcal{O}}(s_L)\; e^{-is_L\Omega_r^b t}$$

The inverse Fourier can be computed by contour integration, picking up the poles of $\hat{\mathcal{O}}(s_L)$ on the real line and becoming a **Fourier series**

$$\mathcal{O}(t) = 2\pi\,\mathrm{sign}(t)\sum \mathrm{Res}_{s_L}\left[\hat{\mathcal{O}}(s_L)\, e^{-is_L\Omega_r^b t}\right]$$

Computing $\hat{\mathcal{O}}(s_L)$ as a complex function $\leftrightarrow$ Solving the full dynamics both in bound and unbound

The Quantum Spectral Method

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Master equation:

$$\hat{\mathcal{O}}(s_L) = \sum_{\Delta\ell,\Delta m} (-1)^{\Delta\ell} \lim_{\hbar\to 0} \langle j_r', \ell', m' | \mathcal{O} | j_r, \ell, m\rangle$$

Classical Laplace observable

"Quantum" Matrix element between Klein-Gordon "wavefunctions"

Klein-Gordon (KG) "wavefunctions" – solutions to the KG equation with the potential

In the point-particle (classical) limit $\hbar \to 0$, the "quantum numbers" j_r, ℓ, m go to infinity while their products with ħ are the finite, dimensionful **action variables** for point-particle motion:

$$(j_r, \ell, m) = \hbar^{-1}(J_r, L, L_z) \quad (j_r', \ell', m') = (j_r, \ell, m) - (\Delta j_r, \Delta\ell, \Delta m)$$

$$\Delta j_r = -s_L - f_\varphi \Delta\ell$$

$$f_\varphi = L/\sqrt{L^2 - K^2}$$

Sample QSM Calculation: Azimuthal Angle

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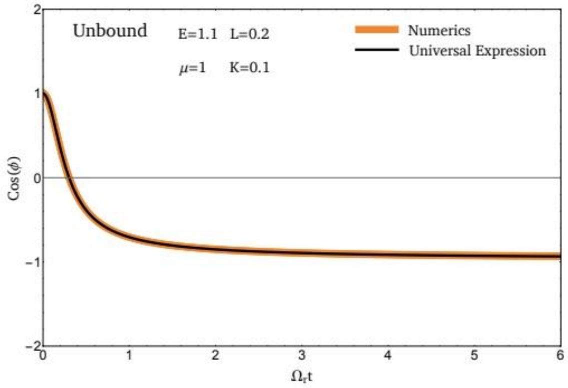
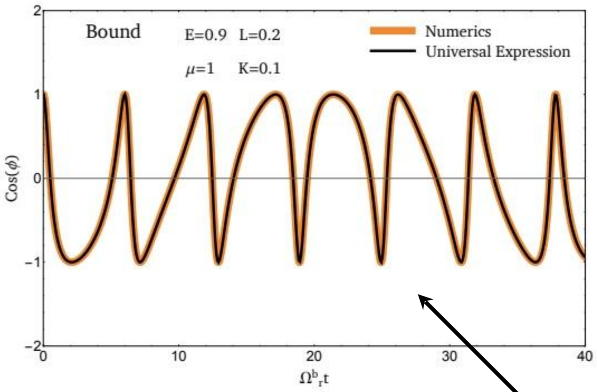


(a) Unbound



(b) Bound

Can see periastron advance

Cos ϕ(t)

Same analytic Laplace observable: bound - unbound universality