III. INFLATIONARY FLUCTUATIONS

One of the most remarkable features of inflation is that it provides a natural mechanism for creating the primordial density fluctuations that seeded the structure in the Universe. In this chapter, we will derive the spectrum of quantum fluctuations produced during inflation.

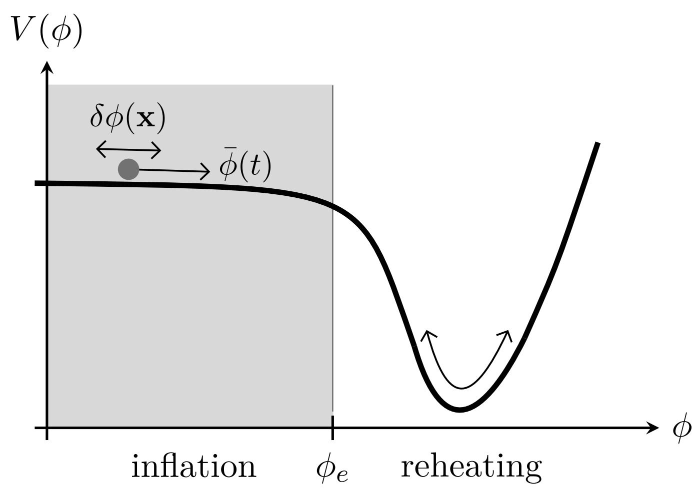
3.1 Basics of Inflation

Consider the action

$$S = \int \mathrm{d}^4x \sqrt{-g} \left(\frac{M_{\mathrm{Pl}}^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) ,$$

where $M_{\mathrm{Pl}} \equiv (8\pi G)^{-1/2}$.

- Leads to accelerated expansion ($\ddot{a} > 0$) when the potential $V(\phi)$ is flat.
- Inflation ends when the potential steepens.
- Quantum fluctuations in the field create density perturbations.



- The homogeneous background satisfies

$$\delta_g S = 0 \quad \Longrightarrow \quad H^2 = \frac{1}{3M_{\mathrm{Pl}}^2} \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right) , \quad H \equiv \dot{a}/a \, ,$$

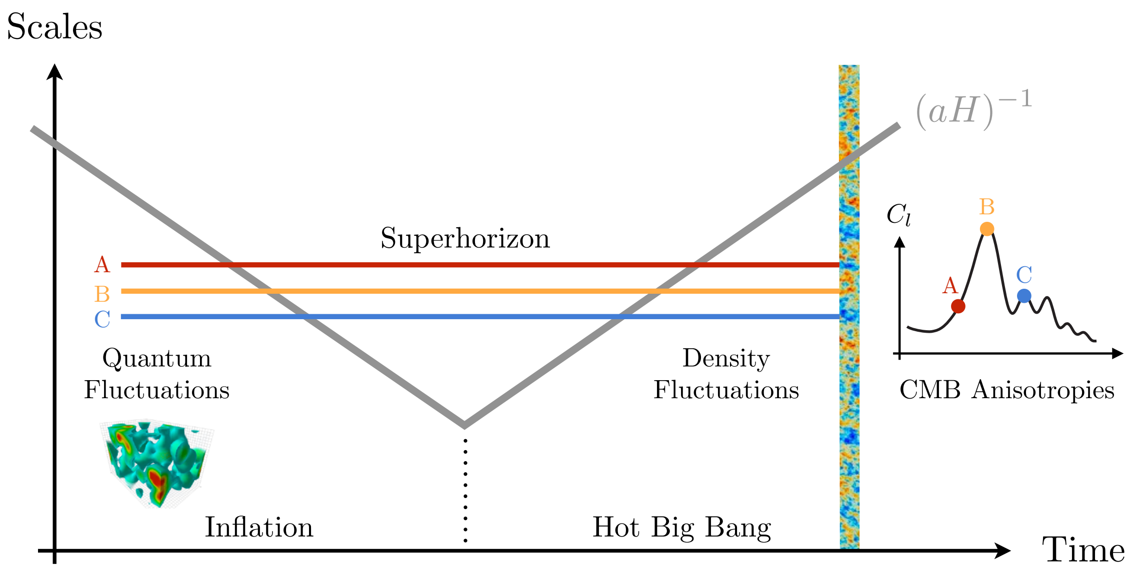
$$\delta_\phi S = 0 \quad \Longrightarrow \quad \ddot{\phi} + 3H\dot{\phi} = -\frac{\mathrm{d}V}{\mathrm{d}\phi} \, .$$

- Slow-roll inflation occurs when $\frac{1}{2}\dot{\phi}^2 \ll V$.
- Slow-roll inflation lasts when $\ddot{\phi} \ll 3H\dot{\phi}$.
- During inflation, $H \approx \text{const} \Rightarrow a(t) \approx e^{Ht}$.

Accelerated expansion implies a shrinking comoving Hubble radius:

$$\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{a})^{-1} = -\frac{\ddot{a}}{(\dot{a})^2} < 0\,.$$

This has important consequences for the evolution of perturbations:



The shrinking Hubble radius can also be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}(1-\varepsilon)\,,$$

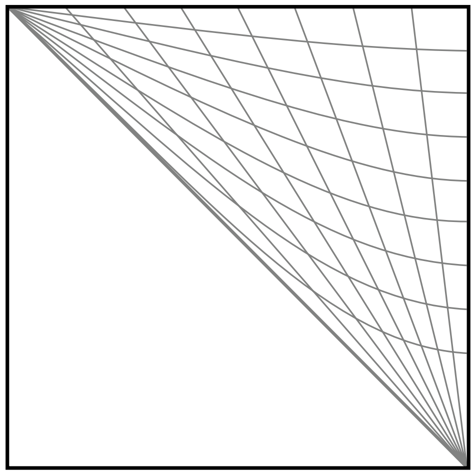
where the "slow-roll parameter" is

$$\varepsilon \equiv -\frac{\dot{H}}{H^2}\,.$$

- Inflation requires $\varepsilon < 1$.
- Scale-invariant perturbations require $\varepsilon \ll 1$.

3.2 Free Scalars in De Sitter

The spacetime during inflation is approximately **de Sitter space**.



In conformal time, $\mathrm{d}\eta = \mathrm{d}t/a(t)$, the de Sitter metric is

$$\mathrm{d}s^2 = \frac{-\mathrm{d}\eta^2 + \mathrm{d}\mathbf{x}^2}{(H\eta)^2}\,,$$

where $-\infty < \eta < 0$. The primordial correlations live on the **future boundary** at $\eta_* \approx 0$. We consider a free, massless scalar in this background.

Classical dynamics

The action is

$$\begin{aligned} S &= \frac{1}{2}\int \mathrm{d}\eta\,\mathrm{d}^3x\, a^2\left[(\phi')^2 - (\nabla\phi)^2\right] \\ &= \frac{1}{2}\int \mathrm{d}\eta\,\mathrm{d}^3x\left[(u')^2 - (\nabla u)^2 + \frac{a''}{a}u^2\right], \end{aligned}$$

where $u \equiv a(\eta)\phi$.

- The classical equation of motion then is

$$u'' - \nabla^2 u - \frac{a''}{a}u = 0 \qquad \Longrightarrow \qquad \boxed{u_{\mathbf{k}}'' + \left(k^2 - \frac{2}{\eta^2}\right)u_{\mathbf{k}} = 0}\,.$$

- At early times ($|k\eta| \gg 1$):

$$u''_{\mathbf{k}} + k^2 u_{\mathbf{k}} = 0 \qquad \Longrightarrow \qquad u_{\mathbf{k}} = A_{\mathbf{k}} \frac{1}{\sqrt{2k}}\, e^{-ik\eta} + B_{\mathbf{k}} \frac{1}{\sqrt{2k}}\, e^{ik\eta}\,.$$

  After quantization, the minimum energy solution has $B_{\mathbf{k}} = 0$.
- At late times ($|k\eta| \ll 1$):

$$u''_{\mathbf{k}} - \frac{2}{\eta^2} u_{\mathbf{k}} = 0 \qquad \Longrightarrow \qquad u_{\mathbf{k}} = C_{\mathbf{k}} \eta^{-1} + D_{\mathbf{k}} \eta^2\,.$$

- Bunch–Davies solution:

$$\boxed{u_{\mathbf{k}}(\eta) = A_{\mathbf{k}} \left(1 - \frac{i}{k\eta}\right) \frac{e^{-ik\eta}}{\sqrt{2k}}}$$

What is $A_{\mathbf{k}}$?

Canonical quantization

- Introduce **operators** $\hat{u}, \hat{\pi} = \partial_\eta \hat{u}$.
- Impose **commutation relations**:

$$[\hat{u}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{x}')] = i\hbar\, \delta(\mathbf{x} - \mathbf{x}') \quad \Longrightarrow \quad [\hat{u}_{\mathbf{k}}(\eta), \hat{\pi}_{\mathbf{k}'}(\eta)] = i\hbar\, (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}')\,.$$

- Define **mode expansion**:

$$\hat{u}(\eta, \mathbf{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left(u_k(\eta)\, \hat{a}_{\mathbf{k}} + u^*_k(\eta)\, \hat{a}^\dagger_{-\mathbf{k}} \right) e^{i\mathbf{k}\cdot\mathbf{x}}\,,$$

  where $[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}')$.
- Define **vacuum**:

$$\hat{a}_{\mathbf{k}} |0\rangle = 0\,.$$

Zero-point fluctuations

The variance of the field operator is

$$
\begin{aligned}
\langle|\hat{u}|^2\rangle &\equiv \langle 0|\hat{u}(\eta,\mathbf{0})\hat{u}(\eta,\mathbf{0})|0\rangle \\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\int \frac{\mathrm{d}^3k'}{(2\pi)^3}\,\langle 0|\big(u_k^*(\eta)\hat{a}^\dagger_{-\mathbf{k}} + u_k(\eta)\hat{a}_{\mathbf{k}}\big)\big(u_{k'}(\eta)\hat{a}_{\mathbf{k}'} + u_{k'}^*(\eta)\hat{a}^\dagger_{-\mathbf{k}'}\big)|0\rangle \\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\int \frac{\mathrm{d}^3k'}{(2\pi)^3}\,u_k(\eta)u_{k'}^*(\eta)\,\langle 0|[\hat{a}_{\mathbf{k}},\hat{a}^\dagger_{-\mathbf{k}'}]|0\rangle \\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,|u_k(\eta)|^2 \\
&= \int \mathrm{d}\log k\;\frac{k^3}{2\pi^2}|u_k(\eta)|^2\;.
\end{aligned}
$$

We define the (dimensionless) **power spectrum** as

$$\Delta_u^2(k,\eta) \equiv \frac{k^3}{2\pi^2}|u_k(\eta)|^2\,.$$

Substituting the Bunch–Davies mode function, we find

$$\Delta_\phi^2(k,\eta) = \frac{\Delta_u^2(k,\eta)}{a^2(\eta)} = \left(\frac{H}{2\pi}\right)^2\left[1+(k\eta)^2\right] \xrightarrow{k\eta\to 0} \boxed{\left(\frac{H}{2\pi}\right)^2}\;.$$

Since $H \approx \mathrm{const}$, the spectrum is **scale-invariant**.

3.3 Curvature Perturbations

Now consider the coupled inflaton-metric fluctuations.

• ADM metric:

$$ds^2 = -N^2 dt^2 + h_{ij}(N^i dt + dx^i)(N^j dt + dx^j)\,.$$

• Comoving gauge ($\delta\phi \equiv 0$):

$$h_{ij} = a^2 e^{2\zeta(t,\mathbf{x})}\delta_{ij}\,, \quad N \equiv 1 + \alpha(t,\mathbf{x})\,, \quad N_i \equiv \partial_i \beta(t,\mathbf{x})\,.$$

• Einstein equations lead to

$$\alpha = \frac{\dot\zeta}{H}\,, \quad \partial^2\beta = -\frac{\partial^2\zeta}{H} + a^2\frac{\dot\phi^2}{2H^2}\dot\zeta\,.$$

• Plugging these results into the action, expanding to second order, performing integrations by parts and using the background equations of motion, we get

$$\boxed{S_2 = \int dt\, d^3x\, a^3\varepsilon\left(\dot\zeta^2 - \frac{1}{a^2}(\partial_i\zeta)^2\right)}\,, \quad \varepsilon = \frac{\frac{1}{2}\dot\phi^2}{M_{\rm Pl}^2 H^2}\,.$$

• Note that $\mathcal{L}_2 \propto \varepsilon \implies \zeta$ is pure gauge in the dS limit ($\varepsilon \to 0$)
• Note that ζ has no mass term $\implies$ frozen superhorizon modes

In conformal time, we have

$$S_2 = \frac{1}{2}\int d\eta d^3x\, z^2\left[(\zeta')^2 - (\partial_i\zeta)^2\right]\,, \quad z \equiv a\sqrt{2\varepsilon}\,.$$

Defining $u \equiv z\zeta$, the equation of motion is

$$u'' - \nabla^2 u - \frac{z''}{z}u = 0 \quad \implies \quad \boxed{u_k'' + \left(k^2 - \frac{z''}{z}\right)u_k = 0}$$

Ex: Show that

$$\frac{z''}{z} = \frac{1}{\eta^2}\left[2 + 3\left(\varepsilon + \frac{1}{2}\kappa\right)\right]\,,$$

where $\kappa \equiv \varepsilon'/(\mathcal{H}\varepsilon)$.

The equation of motion becomes

$$u_k'' + \left(k^2 - \frac{\nu^2 - 1/4}{\eta^2}\right) u_k = 0\,, \quad \text{where} \quad \nu \equiv \frac{3}{2} + \varepsilon + \frac{1}{2}\kappa\,,$$

whose Bunch–Davies solution is

$$\boxed{u_k(\eta) = \frac{\sqrt{\pi}}{2}\sqrt{-\eta}\, H_\nu^{(1)}(-k\eta)}\,.$$

Ex: Show that

$$z(\eta) = z_*(\eta/\eta_*)^{\frac{1}{2}-\nu}\,,$$

where $\eta_* = -k_*^{-1}$.

The dimensionless power spectrum of ζ is

$$\begin{aligned}\Delta_\zeta^2(k) &= \frac{k^3}{2\pi^2}\frac{|u_k(\eta)|^2}{z^2(\eta)} \\ &= \frac{k^3}{2\pi^2}\frac{1}{2\varepsilon_* a_*^2}(-k_*\eta)^{2\nu-1}\,\frac{\pi}{4}(-\eta)|H_\nu^{(1)}(-k\eta)|^2\,.\end{aligned}$$

- In the late-time limit, we get

$$\lim_{k\eta\to 0}|H_\nu^{(1)}(-k\eta)|^2 \approx \frac{2}{\pi}(-k\eta)^{-2\nu} \qquad \Longrightarrow \qquad \boxed{\Delta_\zeta^2(k) = \frac{1}{8\pi^2\varepsilon_*}\frac{H_*^2}{M_{\rm Pl}^2}\,(k/k_*)^{3-2\nu}}\,.$$

- The scalar tilt is

$$\boxed{n_{\rm s} - 1 \equiv \frac{{\rm d}\ln\Delta_\zeta^2}{{\rm d}\ln k} = -2\varepsilon_* - \kappa_*}\,,$$

  which was first measured by WMAP: $n_{\rm s} = 0.965 \pm 0.004$.

3.4 Gravitational Waves

Consider tensor metric perturbations:

$$ds^2 = a^2(\eta)\big[-d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j\big]\,,$$

where $\partial^i h_{ij} = h^i{}_i = 0$.

- Expanding the Einstein–Hilbert action to second order gives

$$S = \frac{M_{\rm Pl}^2}{2}\int d^4x\sqrt{-g}R \;=\; \frac{M_{\rm Pl}^2}{8}\int d\eta d^3x\, a^2\left[(h_{ij}')^2 - (\nabla h_{ij})^2\right] + \cdots\,.$$

- Use rotational symmetry to write $\mathbf{k} = (0,0,k)$ and

$$\frac{M_{\rm Pl}}{\sqrt{2}}a\, h_{ij} \equiv \begin{pmatrix} u_+ & u_\times & 0 \\ u_\times & -u_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- The action becomes

$$S = \frac{1}{2}\sum_{\lambda=+,\times}\int d\eta d^3x\left[(u_\lambda')^2 - (\nabla u_\lambda)^2 + \frac{a''}{a}u_\lambda^2\right].$$

- The equation of motion for each polarization mode is

$$u_k'' + \left(k^2 - \frac{a''}{a}\right)u_k = 0\,,$$

  where the effective mass can be written as

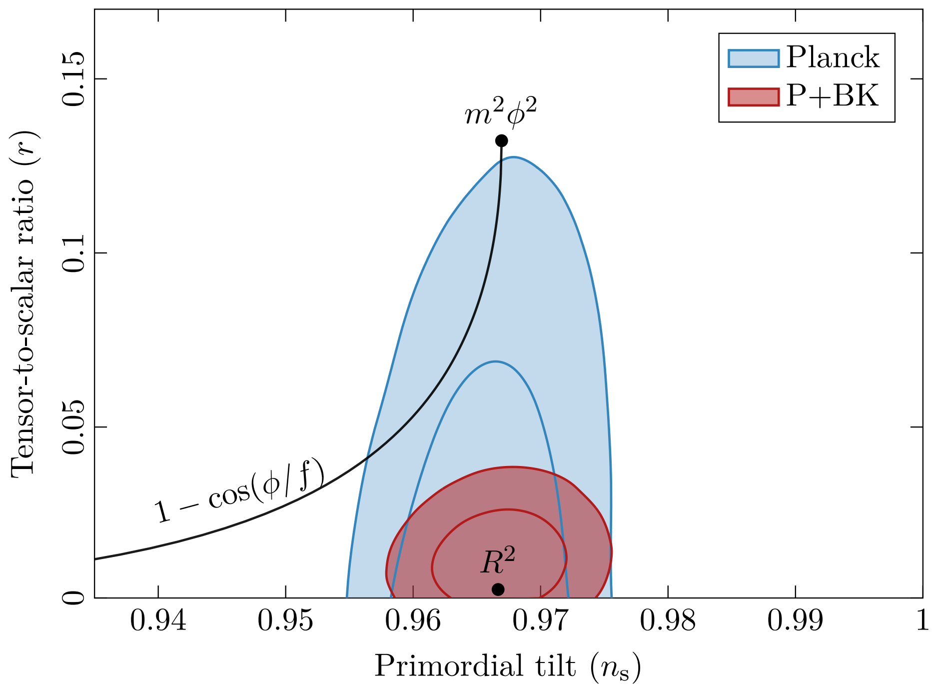
$$\frac{a''}{a} = \frac{\nu^2 - 1/4}{\eta^2}\,, \quad \text{with} \quad \nu \approx \frac{3}{2} + \varepsilon\,.$$

- Bunch–Davies mode function is the same as before.
- Superhorizon limit of the power spectrum:

$$\Delta_h^2(k) = 2\times\left(\frac{2}{aM_{\rm Pl}}\right)^2 \lim_{k\eta\to 0}\frac{k^3}{2\pi^2}|u_k(\eta)|^2 = \boxed{\frac{2}{\pi^2}\frac{H_*^2}{M_{\rm Pl}^2}(k/k_*)^{3-2\nu}}\,.$$

- Observations are expressed in terms of the tensor-to-scalar ratio:

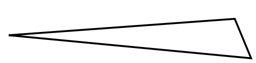
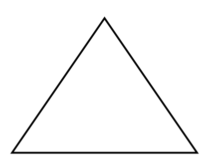
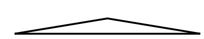
$$\boxed{r \equiv \frac{\Delta_h^2(k_*)}{\Delta_\zeta^2(k_*)} = 16\varepsilon_*}\,.$$

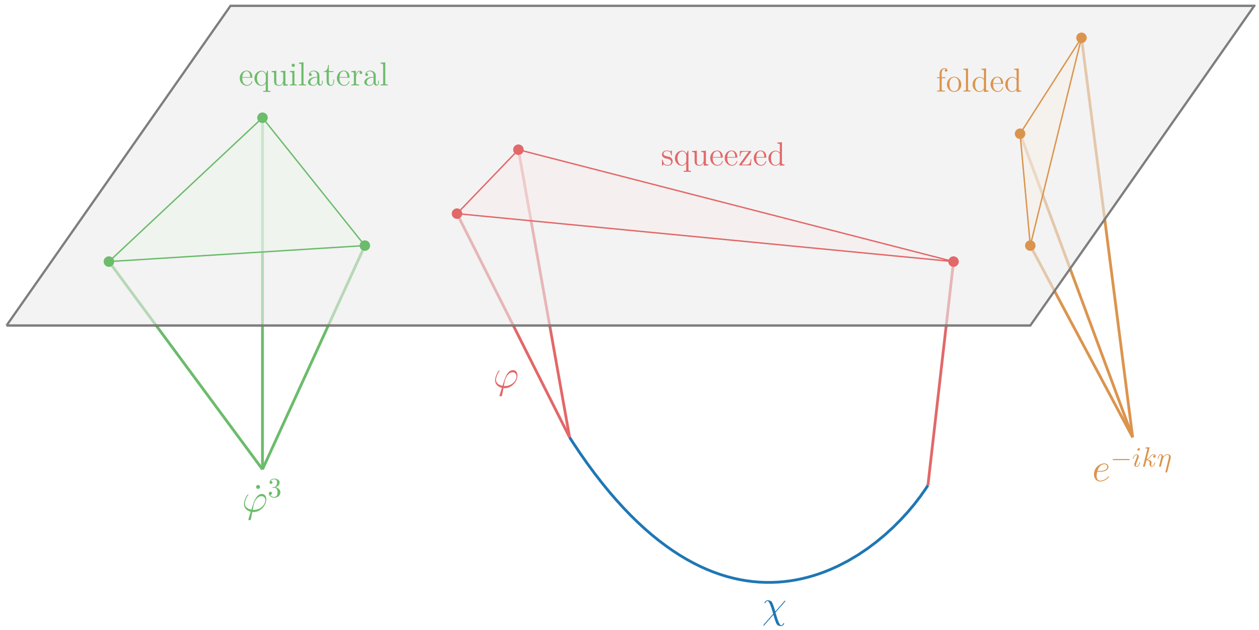


3.5 Primordial Non-Gaussianity

The main diagnostic of primordial non-Gaussianity is the **bispectrum**:

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle = (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)\, B_\zeta(k_1, k_2, k_3)\,.$$

- amplitude: $f_{\rm NL} \equiv \dfrac{5}{18} \dfrac{B_\zeta(k,k,k)}{P_\zeta^2(k)}$
- shape:   
- effect: new particles — new interactions — excited states
- Planck constraints: $|f_{\rm NL}^{\rm loc}| < 5$ — $|f_{\rm NL}^{\rm equil}| < 40$ — $|f_{\rm NL}^{\rm flat}| < 20$



In the following, we describe three methods for computing these higher-point correlations:

- In-In Formalism
- Wavefunction Approach
- Cosmological Bootstrap

IV. IN-IN FORMALISM

In particle physics, the main observables are scattering amplitudes which predict the transition probabilities between some initial ("in") and final ("out") states (defined in terms free particles far in the past and future). In cosmology, however, there are no well-defined asymptotic regions where particles behave freely, which prevents a clean definition of scattering amplitudes. Instead, the primary observables are correlation functions, defined as expectation values at a fixed time in a quantum state evolved from an initial "in" state.

4.1 Master Formula

We are interested in computing

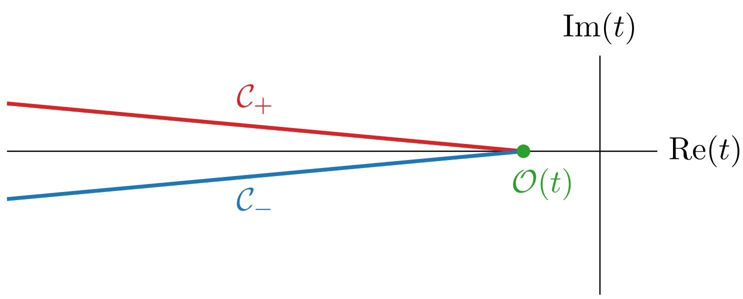
$$\langle \hat{\mathcal{O}}(t) \rangle \equiv \langle \Omega | \hat{\mathcal{O}}(t) | \Omega \rangle ,$$

where $|\Omega\rangle$ the vacuum of the interacting theory.

In the interaction picture, we get

$$\boxed{\langle \hat{\mathcal{O}}(t) \rangle = \langle 0 | \, \bar{T} e^{i \int_{-\infty}^{t} \mathrm{d}t' \hat{H}_{\rm int}} \, \hat{\mathcal{O}}_I(t) \, T e^{-i \int_{-\infty}^{t} \mathrm{d}t' \hat{H}_{\rm int}} \, |0\rangle} \, ,$$

where $H_{\rm int}$ is the interaction Hamiltonian and the free vacuum $|0\rangle$ is selected by the "$i\epsilon$ prescription": $t \to t(1 - i\epsilon)$



In perturbation theory, we find

$$\langle \hat{\mathcal{O}}(t) \rangle = 2 \, \mathrm{Im} \left(\int_{-\infty}^{t} \mathrm{d}t' \, \langle 0 | \, \hat{\mathcal{O}}_I(t) \hat{H}_{\rm int}(t') | 0 \rangle \right) + \cdots$$

In the notes, we apply this to many examples.

4.2 Slow-Roll Inflation

The cubic action for ζ is

$$S_2 = \int \mathrm{d}\eta\, \mathrm{d}^3x\, a^2\varepsilon \Big[(\zeta')^2 - (\partial_i\zeta)^2\Big]\,,$$

$$S_3 = \int \mathrm{d}\eta\, \mathrm{d}^3x\, \left\{a^2\varepsilon^2\Big[\zeta(\zeta')^2 + \zeta(\partial_i\zeta)^2 - 2\zeta'\partial_i\zeta\partial_i^{-1}\zeta'\Big] - \frac{1}{2}\partial_\eta\left(a^2\varepsilon\kappa\,\zeta^2\zeta'\right)\right\}.$$

Exercise: Show that

$$\boxed{\langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle' = \frac{1}{32\varepsilon^2}\frac{H^4}{(k_1k_2k_3)^3}\left[\varepsilon\bigg(\sum_{i\neq j} k_i k_j^2 + \frac{8}{K}\sum_{i>j} k_i^2 k_j^2\bigg) + (\kappa-\varepsilon)\sum_i k_i^3\right]}\,.$$

The amplitude of the bispectrum is

$$f_{\rm NL} \equiv \frac{5}{18}\frac{B_\zeta(k,k,k)}{\Delta_\zeta^4(k)} = \boxed{\frac{5}{12}(9\varepsilon+\kappa)} \ll 1\,.$$

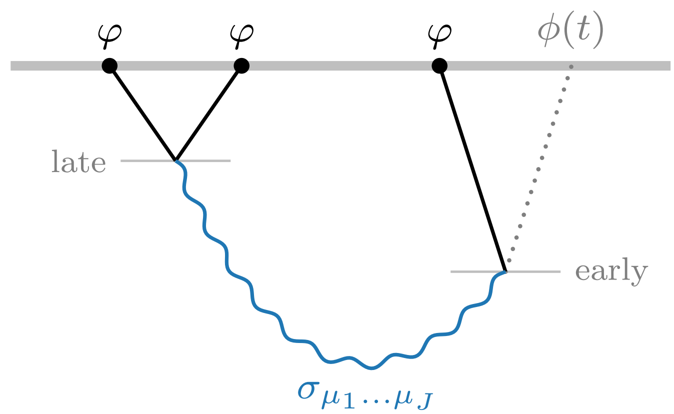
It is also interesting to take the *squeezed limit*, $k_3 \ll k_1 \approx k_2$, of the bispectrum:

$$\begin{aligned}
\lim_{k_3\to 0}\langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle' &= \frac{1}{32\varepsilon^2}\frac{H^4}{k_1^6k_3^3}\left[\varepsilon\left(2k_1^3 + \frac{8}{2k_1}k_1^4\right) + (\kappa-\varepsilon)2k_1^3\right] \\
&= (2\varepsilon+\kappa)\left(\frac{H^2}{4\varepsilon}\frac{1}{k_1^3}\right)\left(\frac{H^2}{4\varepsilon}\frac{1}{k_3^3}\right) \\
&= (2\varepsilon+\kappa)\, P_\zeta(k_1)P_\zeta(k_3) \\
&= \boxed{(1-n_{\rm s})P_\zeta(k_1)P_\zeta(k_3)}\,,
\end{aligned}$$

where $P_\zeta(k)$ is the power spectrum. This result (called "single-field consistency relation") applies to all models of single-field inflation, not just slow-roll models.

4.3 Cosmological Collider

New particles during inflation can lead to a violation of the single-field consistency relation:



Consider

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 - V(\phi) - \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}M^2\sigma^2 + \frac{\sigma(\partial_\mu \phi)^2}{\Lambda} \, .$$

The associated bispectrum is hard to compute (see “cosmological bootstrap”). In the squeezed limit, we find

$$\boxed{\lim_{k_3 \to 0} \frac{\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle'}{P_\zeta(k_1) P_\zeta(k_3)} \sim \frac{\varepsilon}{8\pi} \frac{M_{\rm Pl}^2}{\Lambda^2} \, e^{-\pi M/H} \left(\frac{k_3}{k_1}\right)^{3/2} \sin\left[\frac{M}{H} \log\left(\frac{k_3}{k_1}\right)\right]}$$