# V. WAVEFUNCTION APPROACH

Cosmological correlations can also be computed by means of the so-called "wavefunction of the universe." The wavefunction is a slightly more primitive object than correlators themselves—the relation between them is roughly the same as that between the $S$-matrix and scattering cross-sections. As a result, the wavefunction is somewhat simpler than correlators in certain ways that we will see.

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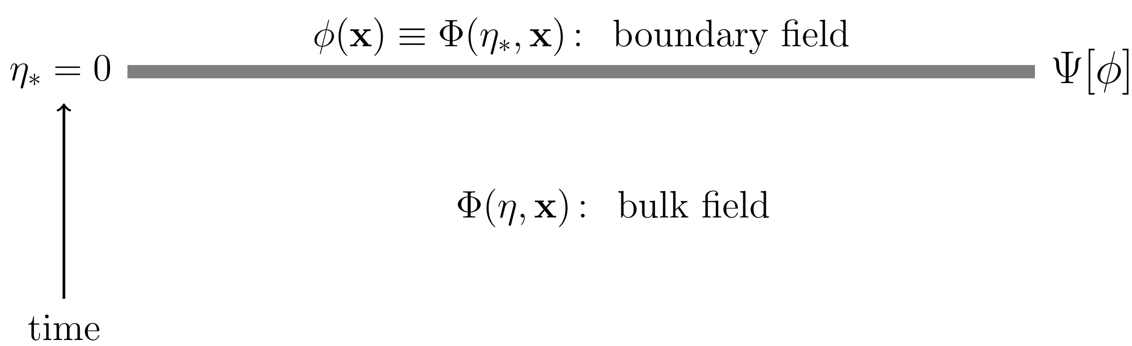
## 5.1 Wavefunction of the Universe




The "wavefunction of the universe" is $\Psi[\phi] \equiv \langle \phi(\mathbf{x}) | \Omega \rangle$.
It defines boundary correlators

$$\langle \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_N) \rangle = \int \mathcal{D}\phi \, \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_N) \, |\Psi[\phi]|^2 \, .$$

The perturbative expansion of the wavefunction (in momentum space) is

$$\Psi[\phi] \; = \; \exp\left( - \sum_{N=2}^{\infty} \frac{1}{N!} \int \mathrm{d}^3 k_1 \cdots \mathrm{d}^3 k_N \, \Psi_N(\underline{\mathbf{k}}) \, \phi_{\mathbf{k}_1} \cdots \phi_{\mathbf{k}_N} \right) ,$$

where the "wavefunction coefficients" are

$$\begin{aligned} \Psi_N(\underline{\mathbf{k}}) &= (2\pi)^3 \delta(\mathbf{k}_1 + \cdots + \mathbf{k}_N) \, \psi_N(\underline{\mathbf{k}}) \\ &= (2\pi)^3 \delta(\mathbf{k}_1 + \cdots + \mathbf{k}_N) \, \langle O_{\mathbf{k}_1} \cdots O_{\mathbf{k}_N} \rangle' \, . \end{aligned}$$

$\uparrow$

dual operators: $\phi \to O$, $\gamma_{ij} \to T_{ij}$

The relation between correlators and wavefunction coefficients is

$$\langle\phi\phi\rangle' = \frac{1}{2\text{Re}\langle OO\rangle'}\,,$$
$$\langle\phi\phi\phi\rangle' = \frac{2\text{Re}\langle OOO\rangle'}{\prod_{a=1}^{3} 2\text{Re}\langle O_a O_a\rangle'}\,,$$
$$\langle\phi\phi\phi\phi\rangle' = \frac{\langle OOOO\rangle'}{(\langle OO\rangle^4} + \frac{\langle OOX\rangle^3}{\langle XX\rangle\langle OO\rangle^4}\,.$$

The wavefunction has the following path integral representation:

$$\Psi[\phi] = \int_{\substack{\Phi(\eta_*)=\phi \\ \Phi(-\infty^-)=0}} \mathcal{D}\Phi\, e^{iS[\Phi]} \underset{\substack{\uparrow \\ \text{tree level}}}{\approx} e^{iS[\Phi_{\rm cl}]}\,,$$

where $-\infty^- \equiv -\infty(1-i\epsilon)$. [Note: Opposite $i\epsilon$ to canonical quantization!]

To find the wavefunction, we therefore need to find the classical solution for the bulk field with the correct boundary conditions.

## 5.2 Warmup in Quantum Mechanics

Consider our old friend the **simple harmonic oscillator**:

$$S[\Phi] = \int \mathrm{d}t \left(\frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}\omega^2\Phi^2\right).$$

- The classical solution (with the correct boundary conditions) is

$$\Phi_{\rm cl}(t) = \phi e^{i\omega t}\,.$$

- The on-shell action becomes

$$\begin{aligned} S[\Phi_{\rm cl}] &= \int_{t_i}^{t_*} \mathrm{d}t \left[\frac{1}{2}\partial_t(\dot{\Phi}_{\rm cl}\Phi_{\rm cl}) - \frac{1}{2}\Phi_{\rm cl}\underbrace{(\ddot{\Phi}_{\rm cl} + \omega^2\Phi_{\rm cl})}_{=0}\right] \\ &= \frac{1}{2}\dot{\Phi}_{\rm cl}\Phi_{\rm cl}\Big|_{t=t_*} \\ &= \frac{i\omega}{2}\phi^2\,. \end{aligned}$$

- The wavefunction then is

$$\Psi[\phi] \approx \exp(iS[\Phi_{\rm cl}]) = \exp\left(-\frac{\omega}{2}\phi^2\right).$$

- The quantum variance of the oscillator therefore is

$$\boxed{\langle\phi^2\rangle = \frac{1}{2\omega}}\ .$$

- In QFT, the same result applies for each Fourier mode:

$$\boxed{\langle\phi_{\mathbf{k}}\phi_{-\mathbf{k}}\rangle' = \frac{1}{2\omega_k}}\ ,$$

where $\omega_k = \sqrt{k^2+m^2}$.

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To make this more interesting, consider a **time-dependent oscillator**:

$$S[\Phi] = \int {\rm d}t \left(\frac{1}{2}A(t)\dot\Phi^2 - \frac{1}{2}B(t)\Phi^2\right).$$

- The classical solution is

$$\Phi_{\rm cl} = \phi K(t)\,, \quad \text{with} \quad \begin{array}{l} K(0) = 1 \\ K(-\infty) \sim e^{i\omega t} \end{array}$$

- The on-shell action becomes

$$\begin{aligned} S[\Phi_{\rm cl}] &= \int_{t_i}^{t_*} {\rm d}t \left[\frac{1}{2}\partial_t(A\dot\Phi_{\rm cl}\Phi_{\rm cl}) - \frac{1}{2}\Phi_{\rm cl}\underbrace{(\partial_t(A\dot\Phi_{\rm cl}) + B\Phi_{\rm cl})}_{=0}\right] \\ &= \frac{1}{2}A\dot\Phi_{\rm cl}\Phi_{\rm cl}\Big|_{t=t_*} \\ &= \frac{1}{2}A\phi^2\partial_t \log K\Big|_{t=t_*}. \end{aligned}$$

- The wavefunction then is

$$\Psi[\phi] \approx \exp(iS[\Phi_{\rm cl}]) = \exp\left(\frac{i}{2}(A\partial_t\log K)\Big|_*\,\phi^2\right),$$

which implies

$$|\Psi[\phi]|^2 = \exp\left(-{\rm Im}(A\partial_t\log K)\big|_*\,\phi^2\right) \quad\Longrightarrow\quad \boxed{\langle\phi^2\rangle = \frac{1}{2\,{\rm Im}(A\partial_t\log K)\big|_*}}\ .$$

## 5.3 Free Fields in de Sitter

Consider a **massless field in de Sitter**:

$$S = \int \mathrm{d}\eta \mathrm{d}^3x\, a^2(\eta) \left[(\Phi')^2 - (\nabla\Phi)^2\right]$$
$$= \frac{1}{2}\int \mathrm{d}\eta \frac{\mathrm{d}^3k}{(2\pi)^3}\left[\frac{1}{(H\eta)^2}\Phi'_{\mathbf{k}}\Phi'_{-\mathbf{k}} - \frac{k^2}{(H\eta)^2}\Phi_{\mathbf{k}}\Phi_{-\mathbf{k}}\right],$$

which is the same as the time-dependent oscillator.

- The classical solution is

$$\Phi_{\rm cl} = \phi K(\eta)\,, \quad \text{with} \quad \begin{array}{l} K(0) = 1 \\ K(-\infty) \sim e^{ik\eta} \end{array}$$

  The function $K(\eta)$ is the *bulk-to-boundary propagator.*

- For a massless field, we have

$$K(\eta) = (1 - ik\eta)e^{ik\eta}\,,$$
$$\log K(\eta) = \log(1 - ik\eta) + ik\eta\,,$$

  and hence

$$\mathrm{Im}(A\partial_\eta \log K)\big|_{\eta=\eta_*} = \frac{1}{(H\eta_*)^2}\,\mathrm{Im}\left(\frac{-ik}{1-ik\eta_*} + ik\right)$$
$$= \frac{1}{(H\eta_*)^2}\,\mathrm{Im}\left(\frac{k^2\eta_* + ik^3\eta_*^2}{1+k^2\eta_*^2}\right) \xrightarrow{\eta_*\to 0} \boxed{\frac{k^3}{H^2}}\,,$$

- The two-point function then is

$$\boxed{\langle\phi_{\mathbf{k}}\phi_{-\mathbf{k}}\rangle' = \frac{H^2}{2k^3}}\,.$$

The result for a massive field is derived in the lecture notes.

### 5.4 Anharmonic Oscillator

Consider the following **anharmonic oscillator**:

$$S[\Phi] = \int \mathrm{d}t \left( \frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}\omega^2\Phi^2 - \frac{1}{3}\lambda\Phi^3 \right).$$

- The classical equation of motion is

$$\ddot{\Phi}_{\rm cl} + \omega^2\Phi_{\rm cl} = -\lambda\Phi_{\rm cl}^2\,.$$

- A formal solution is

$$\Phi_{\rm cl}(t) = \phi K(t) + i\int \mathrm{d}t'\, G(t,t')\big(-\lambda\Phi_{\rm cl}^2(t')\big),$$

  where

$$\begin{aligned} K(t) &= e^{i\omega t}\,, \\ G(t,t') &= \frac{1}{2\omega}\left(e^{-i\omega(t-t')}\theta(t-t') + e^{i\omega(t-t')}\theta(t'-t) - e^{i\omega(t+t')}\right). \end{aligned}$$

- Computing the on-shell action is now a bit more subtle.

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As before, we first write

$$\begin{aligned} S[\Phi] &= \int_{t_i}^{t_*} \mathrm{d}t \left[\frac{1}{2}\partial_t(\Phi\dot{\Phi}) - \frac{1}{2}\Phi(\ddot{\Phi}+\omega^2\Phi) - \frac{\lambda}{3}\,\Phi^3\right] \\ &= \frac{1}{2}\Phi\dot{\Phi}\Big|_{t=t_*} + \int \mathrm{d}t \left[ -\frac{1}{2}\Phi(\ddot{\Phi}+\omega^2\Phi) - \frac{\lambda}{3}\,\Phi^3\right]. \end{aligned}$$

Since

$$\begin{aligned} &\lim_{t\to 0} G(t,t') = 0\,, \\ &\lim_{t\to 0} \partial_t G(t,t') = -ie^{i\omega t'} \neq 0\,, \end{aligned}$$

the boundary term is

$$\begin{aligned} \frac{1}{2}\Phi_{\rm cl}\dot{\Phi}_{\rm cl}\Big|_{t=t_*} &= \frac{1}{2}\phi\left(i\omega\phi - i\lambda\int \mathrm{d}t'\,\big(-ie^{i\omega t'}\big)\,\Phi_{\rm cl}^2(t')\right) \\ &= \frac{i\omega}{2}\phi^2 - \frac{\lambda}{2}\phi\int \mathrm{d}t'\, e^{i\omega t'}\,\Phi_{\rm cl}^2(t')\,. \end{aligned}$$

The action then becomes

$$\begin{aligned} S[\Phi_{\rm cl}] \;&=\; \frac{i\omega}{2}\phi^2 - \frac{\lambda}{2}\phi\int {\rm d}t\, e^{i\omega t}\,\Phi_{\rm cl}^2 \\ &\quad + \int {\rm d}t\left[-\frac{1}{2}\left(\phi e^{i\omega t} - i\lambda\int {\rm d}t'\, G(t,t')\,\Phi_{\rm cl}^2(t')\right)\Big(-\lambda\Phi_{\rm cl}^2(t)\Big) - \frac{\lambda}{3}\,\Phi_{\rm cl}^3\right]. \end{aligned}$$

The terms linear in $\phi$ cancel.

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- The final on-shell action is

$$\boxed{S[\Phi_{\rm cl}] = \frac{i\omega}{2}\phi^2 - \frac{\lambda}{3}\int {\rm d}t\,\Phi_{\rm cl}^3(t) - \frac{i\lambda^2}{2}\int {\rm d}t\,{\rm d}t'\,G(t,t')\,\Phi_{\rm cl}^2(t')\,\Phi_{\rm cl}^2(t)}\;.$$

- We then write the classical solution as $\Phi_{\rm cl}(t) = \sum \lambda^n \Phi^{(n)}(t)$, where

$$\Phi^{(0)}(t) = \phi e^{i\omega t}\,,$$

$$\begin{aligned} \Phi^{(1)}(t) &= i\int {\rm d}t'\,G(t,t')\left(-(\Phi^{(0)}(t'))^2\right) \\ &= i\int {\rm d}t'\,G(t,t')\left(-\phi^2 e^{2i\omega t'}\right) = \frac{\phi^2}{3\omega^2}\,\left(e^{2i\omega t} - e^{i\omega t}\right). \end{aligned}$$

- With this, the wavefunction becomes

$$\boxed{\Psi[\phi] \approx e^{iS[\Phi_{\rm cl}]} = \exp\left(-\frac{\omega}{2}\phi^2 - \frac{\lambda}{9\omega}\phi^3 + \frac{\lambda^2}{72\omega^3}\phi^4 + \cdots\right)}\;.$$

From this, we can compute $\langle\phi^3\rangle$, $\langle\phi^4\rangle$, etc.

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### 5.5 Wavefunction in Field Theory

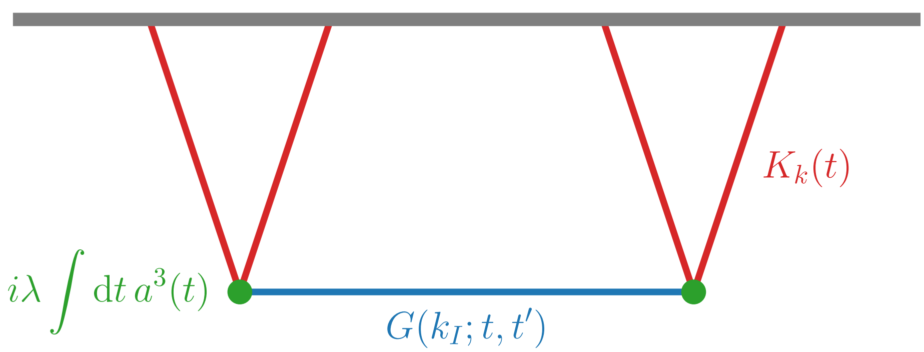
Back to field theory:

$$S[\Phi] = \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}m^2\Phi^2 - \frac{1}{3}\lambda\Phi^3\right).$$

The analysis is similar to that of the anharmonic oscillator (⇒ lecture notes).

In the interest of time, we jump directly to **Feynman rules** for WF coefficients:

- bulk-to-boundary propagator $K$ for every external line
- bulk-to-bulk propagator $G$ for every internal line
- integrate each vertex over time.




Given a mode function $f_k(t)$, the bulk-to-boundary and bulk-to-bulk propagators are

$$K_k(t) = \frac{f_k(t)}{f_k(t_*)}\,,$$

$$G(k;t,t') = \underbrace{f_k^*(t)f_k(t')\,\theta(t-t') + f_k^*(t')f_k(t)\,\theta(t'-t)}_{=\,G_F(k;t,t')} - \frac{f_k^*(t_*)}{f_k(t_*)}f_k(t)f_k(t')\,.$$

## 5.6 Examples in Flat Space

The wavefunction is an interesting object even in flat space:

$$S = \int \mathrm{d}^4x \left( -\frac{1}{2}(\partial\Phi)^2 - \frac{\lambda}{3!}\Phi^3 \right) .$$

- Evaluate correlators at $t_* \equiv 0$.
- The flat-space mode function is

$$f_k(t) = \frac{1}{\sqrt{2k}} e^{ikt} .$$

- Using this, the relevant propagators are

$$K_k(t) = e^{ikt} ,$$

$$G(k; t, t') = \frac{1}{2k} \left( e^{-ik(t-t')}\, \theta(t-t') + e^{ik(t-t')}\, \theta(t'-t) - e^{ik(t+t')} \right) .$$

We will compute the simplest tree-level correlators in this theory.

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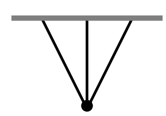
- The three-point wavefunction coefficient in $\Phi^3$ theory is

$\langle O_1 O_2 O_3 \rangle' \equiv$ 

$$= i\lambda \int_{-\infty}^{0} \mathrm{d}t\, e^{i(k_1+k_2+k_3)t}$$

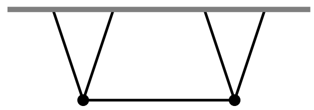
$$= \frac{\lambda}{k_1+k_2+k_3} .$$

- This is easily generalized to $N$-point wavefunction coefficients:

$$\langle O_1 O_2 \ldots Q_N \rangle' = i\lambda \int_{-\infty}^{0} \mathrm{d}t\, e^{i(k_1+k_2+\cdots+k_N)t}$$

$$= \frac{\lambda}{k_1+k_2+\cdots+k_N}$$

• The four-point wavefunction coefficient in $\Phi^3$ theory is

$\langle O_1 O_2 O_3 O_4 \rangle' \;\equiv$ 

$$= -\lambda^2 \int_{-\infty}^{0} \mathrm{d}t\,\mathrm{d}t'\, e^{ik_{12}t}\, G(k_I; t, t')\, e^{ik_{34}t'}$$

$$= -\frac{\lambda^2}{2k_I} \int_{-\infty}^{0} \mathrm{d}t\,\mathrm{d}t'\, e^{ik_{12}t} \left( e^{-ik_I(t-t')}\, \theta(t-t') + e^{ik_I(t-t')}\, \theta(t'-t) - e^{ik_I(t+t')} \right) e^{ik_{34}t'}$$

$$\begin{aligned} = &-\frac{\lambda^2}{2k_I} \int_{-\infty}^{0} \mathrm{d}t \int_{-\infty}^{t} \mathrm{d}t'\, e^{i(k_{12}-k_I)t} e^{i(k_{34}+k_I)t'} \\ &-\frac{\lambda^2}{2k_I} \int_{-\infty}^{0} \mathrm{d}t' \int_{-\infty}^{t'} \mathrm{d}t\, e^{i(k_{12}+k_I)t} e^{i(k_{34}-k_I)t'} \\ &+\frac{\lambda^2}{2k_I} \int_{-\infty}^{0} \mathrm{d}t \int_{-\infty}^{0} \mathrm{d}t'\, e^{i(k_{12}+k_I)t} e^{i(k_{34}+k_I)t'} \end{aligned}$$

$$= \frac{\lambda^2}{2k_I} \left[ \frac{1}{(k_{12}+k_{34})(k_{34}+k_I)} + \frac{1}{(k_{12}+k_{34})(k_{12}+k_I)} - \frac{1}{(k_{12}+k_I)(k_{34}+k_I)} \right]$$

$$= \boxed{\frac{\lambda^2}{(k_{12}+k_{34})(k_{12}+k_I)(k_{34}+k_I)}}\,.$$

The answer has an interesting singularity structure. More about this later.

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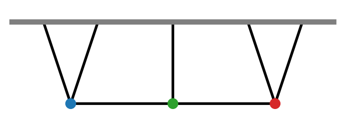
**Exercise**: Show that

$\langle O_1 O_2 O_3 O_4 O_5 \rangle' \;\equiv$ 

$$= \boxed{\frac{\lambda^3}{E E_L E_M E_R} \left[ \frac{1}{k_{123}+k_I'} + \frac{1}{k_{345}+k_I} \right]}\,.$$

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### 5.7 Examples in De Sitter

In general, correlators in de Sitter cannot be computed analytically (some exceptions are presented in the lecture notes).

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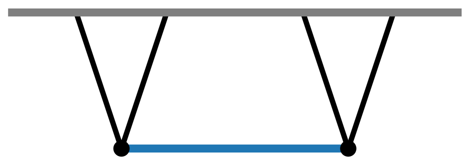
### 5.8 Cosmological Collider Physics

Consider the four-point function of a **conformally coupled scalar** (with $m^2 = 2H^2$) mediated by the exchange of a **massive scalar**:

$$F \equiv \langle O_1 O_2 O_3 O_4 \rangle \equiv$$



$$= -\lambda^2 \int \frac{\mathrm{d}\eta}{\eta^2} \int \frac{\mathrm{d}\eta'}{\eta'^2}\, e^{ik_{12}\eta} e^{ik_{34}\eta'}\, G(k_I; \eta, \eta')\,.$$

↑ products of Hankel functions

The time integrals cannot be performed analytically.

Is there a way to obtain an analytic understanding of this (and other) correlators in de Sitter space?

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