L1

Cargese 2025 EFT Lectures

Tim Cohen (arXiv: 1903.03622)

Riccardo (RR) explained "Physics is EFT"

One of my main goals is to present a more mundane version of This statement

"Renormalization is EFT"

* RR already emphasized symmetry

Quick review of things RR said:

- EFTs emerge when there is "large seperation of scales": $\Lambda_{IR} \ll \Lambda_{UV}$. EFTs are "universal".
- EFT is dimensional analysis + Taylor expansions
  $\Rightarrow$ Catagorization "relevant", "marginal", "irrelevant"
- RR presented "Wilsonian EFT"; I want to introduce you to "Continuum EFT"
  relies on dim reg in essential way
- Visualization of EFT

Λ_{UV} ——— Full theory ——— μ

EFT(Λ_{UV})

$\downarrow$ RG

EFT(Λ_{IR})

Λ_{IR} ——————— m

"power counting"

$\lambda \sim m/M \ll 1$

EFT involves a dual perturbation theory:

1) loop ($\hbar$) expansion

2) E/M expansion

Clearly distinguished at tree, but this becomes much more subtle at loop level.

EFTs and the SM:

- $E \ll m_e$: Euler-Heisenberg for photons
- $E \ll \Lambda_{QCD}$: Chiral $\mathcal{L}$ for light mesons
- $E \ll m_W$: Fermi theory for quarks and leptons (lately called "LEFT")
- $E \ll \Lambda_{\text{new physics}}$: SMEFT (w/ $\Lambda_{np} \gg v$)

  HEFT (w/ $\Lambda_{NP} \sim v$)

EFTs w/ kinematic restrictions ("mode based")

- HQET: $m_Q \gg \Lambda_{QCD}$ $\quad p \ll m_Q$
- NRQCD: $m_Q \gg \Lambda_{QCD}$ $\quad p \ll m_Q$ $\quad p \sim mv$, $E \sim \frac{1}{2} m v^2$, $p \sim E$
- SCET: $p \ll \sqrt{\hat{s}}$ w/ P "collinear" or "soft"

Plan: I. Heavy Particle Decoupling

II. EFT Geometry

I. Heavy Particle Decoupling

"EFT sums IR logs" (But RG is about UV logs?)

Ex: $\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^4} \to \frac{i}{8\pi^2}\int_{\Lambda_{IR}}^{\Lambda_{UV}} \frac{d\ell}{\ell} = \frac{i}{16\pi^2}\log\left(\frac{\Lambda_{UV}^2}{\Lambda_{IR}^2}\right)$

- Scaleless integrals vanish in dim reg $(d = 4-2\varepsilon)$

Notation $(d\ell) = \frac{d^d\ell}{(2\pi)^d}$ $\qquad \bar{\mu}^2 = 4\pi e^{-\gamma_E}\mu^2$

$$I = \mu^{2\varepsilon}\int (d\ell)\frac{1}{\ell^4} = \underbrace{\mu^{2\varepsilon}\int(d\ell)\frac{1}{\ell^2(\ell^2-m^2)}}_{I_{UV}} - \underbrace{\mu^{2\varepsilon}\int(d\ell)\frac{m^2}{\ell^4(\ell^2-m^2)}}_{I_{IR}}$$

$$I_{UV} = \frac{i}{16\pi^2}\left(\frac{1}{\varepsilon_{UV}} + \log\frac{\bar{\mu}_{UV}^2}{m^2} + 1\right) + \dots$$

$$I_{IR} = \frac{i}{16\pi^2}\left(\frac{1}{\varepsilon_{IR}} + \log\frac{\bar{\mu}_{IR}^2}{m^2} + 1\right) + \dots$$

$\Rightarrow I = \frac{i}{16\pi^2}\left(\frac{1}{\varepsilon_{UV}} - \frac{1}{\varepsilon_{IR}}\right) = 0$ when $\varepsilon_{UV} = \varepsilon_{IR}$, $\mu_{UV} = \mu_{IR}$ $\leftarrow$ This is the true magic!!

- In general, Scaleless integrals vanish in dim reg. This will be critical for the sucess of the continuum EFT formalism.

(Note we did something truely perverse here: we had to analytically continue $\varepsilon_{IR} \to -\varepsilon_{IR}$ since I_{IR} converges for $d = 4 + 2\varepsilon_{IR}$. Weird...)

Summary: Dim reg maps IR logs to UV logs $\Rightarrow$ Can use RG to resum them!

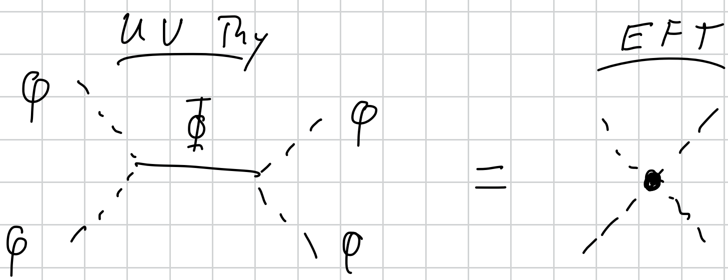
But RR insisted that there is a hierarchy paradox ?!?

Let us discover the hierarchy problem (no apologies!) using continuum EFT

This requires understanding the notion of "matching" which requires us to keep track of finite parts of loop integrals.

- Let's start with a tree-level example. The UV theory has scalar field φ w/ mass m, Φ w/ mass M, and $\mathcal{L}_{int} = \frac{g}{2}\varphi^2\Phi$

"Matching" is simply equating

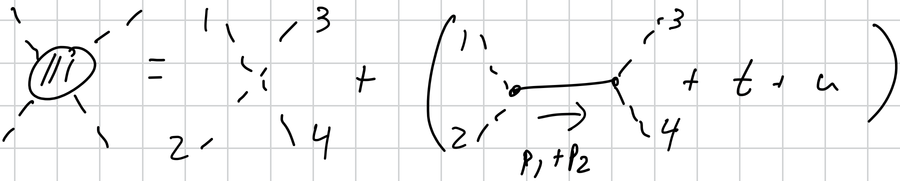


(* Modern approach relies on "functional methods": integrate out heavy physics in path integral directly)

Let's be a bit more systematic

Ex: $\mathcal{L}_{int} = \frac{\lambda}{4!}\phi^4 + \frac{g}{2}\phi^2\Phi$

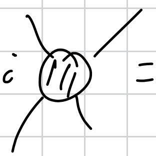
Focus on 4-point:



$$= -i\lambda - ig^2\left[\frac{1}{s-M^2} + \frac{1}{t-M^2} + \frac{1}{u+M^2}\right]$$

Assume $E \ll M$: $\frac{1}{p^2-M^2+i\varepsilon} = -\frac{1}{M^2} - \frac{p^2}{M^2} + \ldots$

note $s+t+u = 4m^2$
$\Rightarrow$ connection to field redefs

$\Rightarrow -i$  $= \lambda + \frac{g^2}{M^2} + \frac{g^2}{M^4}(s+t+u) + \ldots$

$$\Rightarrow \mathcal{L}_{eff}^{(4)} = -\frac{1}{4!}\left(\lambda - \frac{3g^2}{M^2}\right)\phi^4 - \frac{g^2}{8M^4}\phi^2\Box\phi^2 + \ldots$$

What are we doing? Shrinking heavy line to point:

$\Rightarrow$ local!

etc.

Power Counting

Integrating out Φ generates an ∞ # of terms

Can we organize them?

Assume: Fundamental params * UV hypothesis!!

$g \sim M_\Phi \equiv M$ & $\lambda \sim \mathcal{O}(1)$ — * role of symmetry

$$\Rightarrow \mathcal{L}_{eff} \sim \sum_{n,m} \frac{1}{M^{n+m-4}} \partial^n \phi^m$$ w/ only even powers due to Symmetries

eg at $\mathcal{O}(1/M^2)$: $\phi^6,\ \partial^2\phi^4,\ \partial^4\phi^2$

Truncating to $\mathcal{O}(1/M^2) \Rightarrow$ computing amplitudes to accuracy E^2/M^2.

$\Rightarrow$ Power counting determines accuracy of calculation

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Note: Assuming we are working with mass eigenstates, integrating out particle at tree-level only impacts couplings in EFT. To see hierarchy problem, need to match at loop level.

Technical Aside on RGs $\qquad \mathcal{L}_{eff} = \frac{C_n}{n!}\varphi^n$

Renormalization Group Equations $\frac{d}{d\log\tilde{\mu}^2} C_n^r = \gamma_{nm} C_m^r$

anomalous dimension

If $\gamma = \text{const} \Rightarrow \int \frac{1}{C^r} dC^r = \int \gamma \, d\log\tilde{\mu}^2$

$$\Rightarrow C^r(\tilde{\mu}_H) = C^r(\tilde{\mu}_L)\exp\left(\gamma \log\frac{\tilde{\mu}_H^2}{\tilde{\mu}_L^2}\right) = \left(\frac{\tilde{\mu}_H^2}{\tilde{\mu}_L^2}\right)^{2\gamma} C^r(\tilde{\mu}_L)$$

When $\gamma_{nm} \neq 0$ w/ $n \neq m \Rightarrow$ operator mixing

Derive equation for γ_{nm}:

In pert theory $Z = 1 + \mathcal{O}(C^r, \alpha^r)$

Bare Lagrangian must be $\tilde{\mu}$ independent (Callan-Symanzik Eq)

$$0 = \tilde{\mu}\frac{d}{d\tilde{\mu}} C^0 = \tilde{\mu}\frac{d}{d\tilde{\mu}}\left(Z\tilde{\mu}^{n\varepsilon} C^r\right)$$

Ex: $\mathcal{L} \supset \frac{1}{4!} C_4 \varphi^4 + \frac{1}{6!}\frac{1}{M^2} C_6 \varphi^6$

Tree level $0 = \tilde{\mu}\left(\frac{1}{Z_4}\cancelto{0}{\frac{dZ_4}{d\tilde{\mu}}} + \frac{1}{C_4^r}\frac{dC_4^r}{d\tilde{\mu}} + \frac{1}{\tilde{\mu}^{2\varepsilon}} 2\varepsilon\tilde{\mu}^{2\varepsilon-1}\right)\cancelto{1}{Z_4} C_4^r \tilde{\mu}^{2\varepsilon}$

$$\Rightarrow \frac{dC_4^r}{d\log\tilde{\mu}^2} = -\varepsilon C_4^r \Rightarrow \gamma_{44}^{\text{classical}} = -\varepsilon$$

Tree level change in dimension of operator w/ $d \neq 4$

Similarly, $\gamma_6^{\text{Classical}} = -2\varepsilon$

At one-loop: $Z_4 = Z_4(C_4^r, C_6^r)$

$$\Rightarrow 0 = \frac{d}{d\log\bar{\mu}^2}\left(Z_4(C_4^r, C_6^r)\,\bar{\mu}^{2\varepsilon}C_4^r\right)$$

$$= \frac{1}{2}\left(\frac{\partial Z_4}{\partial C_4^r}\frac{\bar{\mu}}{Z_4}\frac{dC_4^r}{d\bar{\mu}} + \frac{\partial Z_6}{\partial C_6^r}\frac{\bar{\mu}}{Z_4}\frac{dC_6^r}{d\bar{\mu}} + \frac{\bar{\mu}}{C_4^r}\frac{dC_4^r}{d\bar{\mu}} + 2\varepsilon\right) Z_4\,\bar{\mu}^{2\varepsilon}C_4^r$$

Then truncate $\frac{1}{Z_4} = 1$ and use leading sols $\Rightarrow$

$$\frac{dC_4^r}{d\log\bar{\mu}^2} = \left(\varepsilon (C_4^r)^2\frac{\partial Z_4}{\partial C_4^r} - \varepsilon C_4^r + 2\varepsilon C_4^r C_6^r\frac{\partial Z_4}{\partial C_6^r}\right)$$

$$\Rightarrow \gamma_{44} = \lim_{\varepsilon\to 0}\left(\varepsilon C_4^r\frac{\partial Z_4}{\partial C_4^r} - \varepsilon\right) \quad \text{and} \quad \gamma_{46} = \lim_{\varepsilon\to 0}\left(2\varepsilon C_4^r\frac{\partial Z_4}{\partial C_6^r}\right)$$

- Practically, one can differentiate fixed order result to get γ, and then plug back in to resum

- Solving this equation "resums logs"

- Now we have two simeltaneous expansions

  - Count $(\alpha\log\lambda)^n$ differently then α

    $(\alpha\log\lambda)^n \leftarrow N^nLL$ $\alpha \leftarrow N^nLO$ Be careful about double counting

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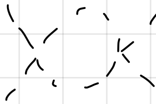
Summing Logs (no longer careful w/ "r" and $\bar{\mu}$)

Ex: $\mathcal{L} \supset -\frac{1}{4!}C_4\phi^4$ is defined at some high scale μ_H

Want to predict $\phi\phi\to\phi\phi$ at $\mu_L^2 \sim m^2$ (threshold)

Tree:  $= -i C_4(\mu_H)$

1-loop: (s+t+u channels)  $= \frac{3i}{32\pi^2}\mu_H^{2\varepsilon} C_4^2\left(\frac{1}{\varepsilon} + \log\frac{\mu_H^2}{m^2} + \frac{2}{3}\right)$

$$\Rightarrow Z_4 = 1 + \frac{3}{32\pi^2} C_4 \frac{1}{\varepsilon} \Rightarrow \frac{d C_4}{d\log\mu^2} = \frac{3}{32\pi^2} C_4^2$$

$$\Rightarrow C_4(\mu_L) = \frac{C_4(\mu_H)}{1 + C_4(\mu_H)\frac{3}{32\pi^2}\log\frac{\mu_H^2}{\mu_L^2}}$$

Landau pole

$\mu = \Lambda$ when $C_4(\Lambda) \to \infty$

$\Lambda = m \,\mathrm{Exp}\left(1 \Big/ \tfrac{3}{32\pi^2} C_4(m)\right)$

$\Rightarrow$ Theory breaks down

Dim transmutation!

Then computing using running coupling @ low scale to 1-loop

$$A = -C_4(\mu_L)\left(1 - \frac{3}{32\pi^2} C_4(\mu_L)\left(\log\frac{\mu_L^2}{m^2} + \frac{2}{3}\right)\right)$$

$$= -C_4(\mu_H)\left\{\left(1 - C_4(\mu_H)\frac{3}{32\pi^2}\log\frac{\mu_H^2}{\mu_L^2}\right)\right.$$

$$\left.\times\left[1 - \frac{3}{32\pi^2} C_4(\mu_H)\left(\log\frac{\mu_L^2}{m^2}\right)\right] + \dots\right.$$

$$= -C_4(\mu_H)\left\{1 - C_4(\mu_H)\left(\log\frac{\mu_H^2}{m^2} + \frac{2}{3}\right) + \dots\right\}$$

$\Rightarrow$ Reproduces non-improved result.

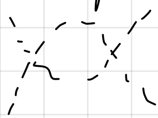
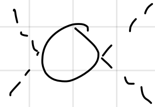
Heavy Particle Decoupling @ 1-loop

$$\mathcal{L}^{Full} \supset -\frac{1}{4}\varkappa\,\varphi^2\Phi^2 - \frac{1}{4!}\eta\,\varphi^4$$

Process $\varphi\varphi \to \varphi\varphi$ @ threshold

Assume $m^2 \ll M^2$

First compute NLO process to uncover apparent non-decoupling

3x  + 3x 

$$\Rightarrow \mathcal{A}^{Full} = \eta + \frac{3}{32\pi^2}\eta^2\left(\log\frac{\mu^2}{m^2} + \frac{2}{3}\right) + \frac{3}{32\pi^2}\varkappa^2\log\frac{\mu^2}{M^2} + \dots$$

Any choice of μ results in a large log.

RGEs do not solve the problem:

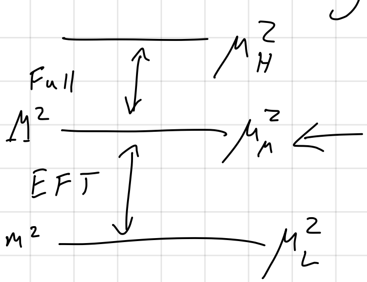
$$\frac{d\eta}{d\log\mu^2} = \frac{3}{32\pi^2}\left(\eta^2 - \varkappa^2\right) \qquad \frac{d\varkappa}{d\log\mu^2} = \frac{1}{8\pi^2}\varkappa^2 + \frac{1}{32\pi^2}\varkappa\eta$$

(no mass scales)

Must <u>match</u> onto an EFT at scale μ_M

Need IR of EFT to be same as Full Thy

$\Rightarrow$ Build EFT using φ w/ mass m^2



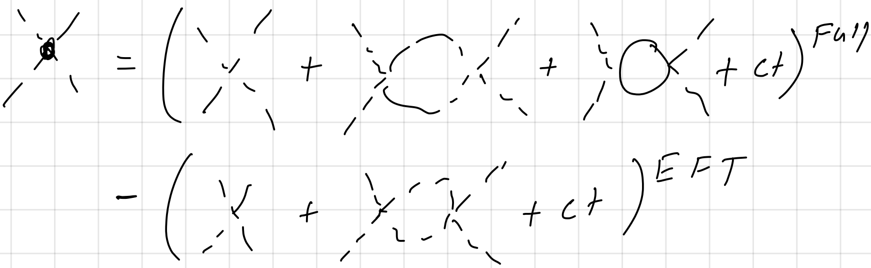
"Intermediate Matching scale"

$$\mathcal{L}^{EFT} \supset -\frac{C_4}{4!}\varphi^4$$

$$A_{match} = \left[A_{Full} - A^{CT}_{Full}\right] - \left[A_{EFT} - A^{CT}_{EFT}\right]$$

mention on-shell w.f. factors

$iA^{Match} =$



At tree level $C_4(\mu_M) = \eta(\mu_M)$

One-loop: $C_4(\mu_M) = \eta(\mu_M) - \frac{3}{32\pi^2}(\alpha(\mu_M))^2 \log\frac{\mu_M^2}{M^2} + \dots$

This is BC for RGE within EFT

$$\frac{dC_4}{d\log\mu^2} = \frac{3}{32\pi^2}(C_4)^2 \Leftarrow \text{Heavy particle decoupled}$$

No large logs!

Expanding RGE improved couplings reproduces original A

$$iA^{EFT}_{Expanded} = -i\eta(\mu_H) + \frac{3i}{32\pi^2}\left[\eta^2\log\frac{\mu_H^2}{\mu_M^2} + (C_4)^2\left(\log\frac{\mu_M^2}{m^2}+\frac{2}{3}\right)\right] + \frac{3i}{32\pi^2}\alpha^2\log\frac{\mu_H^2}{M^2}$$

The Hierarchy Problem

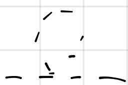
No quad div in dim reg

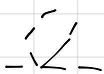
$$\mathcal{L}_{Full} \ni -\frac{1}{2}m^2\varphi^2 - M^2\Phi^2 - \frac{1}{4!}\eta\varphi^4 - \frac{1}{4}\lambda\varphi^2\Phi^2$$

$$\mathcal{L}_{EFT} \ni -m^2\varphi^2 - \frac{1}{4!}C_4\varphi^4$$

Matching $A_{match} = \left[A_{Full} - A^{CT}_{Full}\right] - \left[A_{EFT} - A^{CT}_{EFT}\right]$

at high scale $\bar{\mu}_H \sim M$

Full  $= \frac{\eta}{2}\mu_H^{2\varepsilon}\int (d\ell)\,\frac{1}{\ell^2 - m^2} = i\frac{\eta}{32\pi^2}m^2\left[\frac{1}{\varepsilon} + \log\frac{\bar{\mu}_H^2}{m^2} + 1 + \mathcal{O}(\varepsilon)\right]$

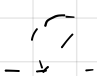
EFT  $= \eta \to C_4$

Use $\varphi\varphi \to \varphi\varphi$ to match at tree-level $-iC_4 = -i\eta$

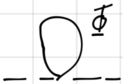
$$\Rightarrow -im^2_{match} = -im_r^2 + \frac{i\eta}{32\pi^2}m_r^2\left(\log\frac{\bar{\mu}_H^2}{m_r^2} + 1\right) - \left[\eta \to C_4\right] = 0$$

So 1-loop matching gives no contribution. Note same prescription on both sides of matching

Then in EFT at μ_L

 w/ $\mu_H \to \mu_L \Rightarrow$ Proportional to $m^2 \Rightarrow$ no tuning problem!

Now what about the Heavy loop?



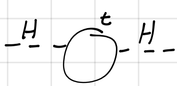
$$= \frac{1}{2} \lambda \mu_H^{2\varepsilon} \int (d\ell) \frac{1}{\ell^2 - M^2} = \frac{i\lambda}{32\pi^2} M^2 \left[\frac{1}{\varepsilon} + \log \frac{\bar{\mu}_H^2}{M^2} + 1 + \mathcal{O}(\varepsilon) \right]$$

No loop in EFT

$$\Rightarrow -i M^2_{match} = \left(-i m_r^2 + \frac{i\lambda}{32\pi^2} M^2 \left[\log \frac{\bar{\mu}_H^2}{M^2} + 1 \right] \right) - \left(-i m_r^2 \right)$$

Take natural $\bar{\mu}_H = M \Rightarrow \quad m^2_{match} = \frac{-\lambda}{32\pi^2} M^2$

Physical "quadratic divergence"

- Fine tuning at matching scale
- Complain: 

$\sim y_t^2 \int_0^\Lambda \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 - m_t^2} \sim \frac{y_t^2}{16\pi^2} \Lambda^2$

Not physical! No tuning problem in SM alone

<u>But</u> if another physical scale, will make quad div physical

Always happens in calculable models of Higgs mass Need new Symmetry!

Read TASI lectures for context

(also see RR lectures)

A heavy-light integral & the Method of Regions

One extreamly useful diagnostic for correctness of matching calculations is to see that matching corrections (capturing physics @ Λ_{UV}) are analytic in the limit $\Lambda_{IR} \to 0$

This example will show how EFT can seperate scales for physical logs $\log m/M$ and resum them.

Ex: $\mathcal{L}_{Full} \ni \frac{\rho}{3!} \varphi^3 \Phi$

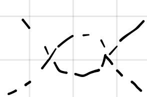
Process $\varphi\varphi \to \varphi\varphi$ @ threshold

Power counting $\lambda = m/M$

Symmetries Lorentz, $\mathbb{Z}_2$ w/ $\varphi \to -\varphi$ & $\Phi \to -\Phi$

$$\mathcal{L}_{EFT} \ni -\frac{C_4}{4!}\varphi^4 - \frac{C_6}{M^2 6!}\varphi^6$$

$$\begin{array}{ccc} M^2 & \overset{\text{Full}}{\underset{\text{EFT}}{\overline{\hphantom{xxxxxxxx}}}} & \mu_M^2 \\ & \updownarrow \text{ RGE} & \\ m^2 & \overline{\hphantom{xxxxxxxx}} & \mu_L^2 \end{array}$$

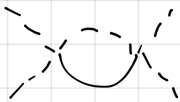
Will use C_4 running from 

Compute $\varphi\varphi \to \varphi\varphi$ in Full Theory $\sim \alpha^2$ IR finite as $m \to 0$

$\Rightarrow$ can't have $\log m$ alone (need $m \log m$)

$P_c = (m, \vec{0})$ & $p_1 + p_2 = p_3 + p_4$

t-channel + u-channel



$$= 2\rho^2 \mu^{2\varepsilon} \int (dl) \frac{1}{(l^2 - m^2)(l^2 - M^2)}$$

$$= 2\alpha^2 \mu^{2\varepsilon} \int (dl) \int_0^1 dx \frac{1}{\left[l^2 - (x m^2 + (1-x) M^2\right]^2} = \frac{i 2\rho^2}{16\pi^2} \int_0^1 dx \left[\frac{1}{\varepsilon} + \log\left(\frac{\bar{\mu}_H^2}{x m^2 + (1-x) M^2}\right)\right]$$

$$= \frac{i 2\rho^2}{16\pi^2}\left(\frac{1}{\varepsilon} - \frac{m^2}{M^2 - m^2}\log\left(\frac{\bar{\mu}_H^2}{m^2}\right) + \frac{M^2}{M^2 - m^2}\log\left(\frac{\bar{\mu}_H^2}{M^2}\right)\right) \leftarrow \text{note } m \leftrightarrow M \text{ invariance}$$

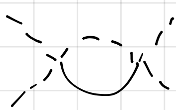
$$= \frac{i 2\rho^2}{16\pi^2}\left(\frac{1}{\varepsilon} + \log\frac{\bar{\mu}_H^2}{m^2} + \frac{m^2}{M^2}\log\frac{m^2}{M^2} + \mathcal{O}(\lambda^4)\right)$$

where I used

$$\left(\frac{-m^2}{M^2 - m^2}\log\frac{\bar{\mu}_H^2}{m^2} + \frac{M^2}{M^2 - m^2}\log\frac{\bar{\mu}_H^2}{M^2}\right) + \left(\frac{m^2}{M^2 - m^2}\log\frac{\bar{\mu}_H^2}{M^2} - \frac{m^2}{M^2 - m^2}\log\frac{\bar{\mu}_H^2}{M^2}\right)$$

$$= \log\frac{\bar{\mu}_H^2}{M^2} + \frac{m^2}{M^2 - m^2}\log\frac{m^2}{M^2} = \log\frac{\bar{\mu}_H^2}{M^2} + \frac{m^2}{M^2}\log\frac{m^2}{M^2} + \mathcal{O}(\lambda^4)$$

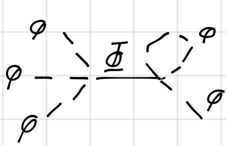
s-channel



$$= \rho^2 \mu^{2\varepsilon} \int (dl) \frac{1}{((l + p_1 + p_2)^2 - m^2)(l^2 - M^2)}$$

$$= \frac{i\rho^2}{16\pi^2}\left[\frac{1}{\varepsilon} + \log\frac{\bar{\mu}_H^2}{M^2} + 1 + \frac{m^2}{M^2}\left(\log\frac{m^2}{M^2} + 2\right) + \mathcal{O}(\lambda^4)\right]$$

Another set of diagrams $\qquad$ ~~1~~ LPI



$$= \rho^2 \frac{4}{2} \frac{1}{m^2 - M^2} \mu_H^{2\varepsilon} \int (dl) \frac{1}{l^2 - m^2} = -i\frac{\rho^2}{8\pi^2}\frac{m^2}{M^2}\left(\frac{1}{\varepsilon} - \log\frac{m^2}{\bar{\mu}_H^2} + 1\right)$$

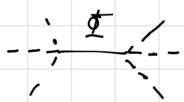
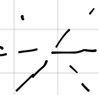
$$\Rightarrow \mathcal{A}(\varphi\varphi\to\varphi\varphi)^{\text{renorm}} = \frac{i\rho^2}{16\pi^2}\left[3\log\frac{\bar{\mu}_H^2}{M^2} + 3 + \frac{m^2}{M^2}\left(3\log\frac{m^2}{M^2} + 2\log\frac{m^2}{M^2}\right)\right] \leftarrow * \text{ Keep this}$$

resum by running η

"large log" (technically never gets large due to λ^2 suppression)

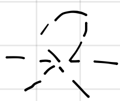
EFT Separate scales

Match at tree level: $C_4 = 0$

C_6: $\frac{1}{2}\binom{6}{3}$  $= -i\,10\rho^2\left(\frac{-1}{M^2}\right) + \mathcal{O}(\lambda) =$  $= -i\frac{C_6}{M^2}$

$$\Rightarrow C_6 = -10\rho^2$$

Need power supressed contribution to quartic

 $$= \frac{1}{2}\frac{C_6}{M^2}\mu_H^{2\varepsilon}\int (d\ell)\frac{1}{\ell^2 - m^2} = \frac{i}{32\pi^2}\frac{C_6}{M^2}m^2\left[\frac{1}{\varepsilon} + \log\frac{\bar{\mu}_H^2}{m^2} + 1 + \mathcal{O}(\varepsilon)\right]$$

$$\Rightarrow -i(Z_4 - 1) \ni -i\frac{1}{C_4}\frac{C_6}{32\pi^2}\frac{m^2}{M^2}\frac{1}{\varepsilon} \Rightarrow \gamma_{46} = \frac{1}{16\pi^2}\frac{m^2}{M^2}$$

$$-iC_4^{\text{match}}(\bar{\mu}_H) = \text{Full} - \text{EFT}$$

$$= \frac{i\rho^2}{16\pi^2}\left[3\log\frac{\bar{\mu}_H^2}{M^2} + 1 + \frac{m^2}{M^2}\left(3\log\frac{m^2}{M^2} + 2\log\frac{m^2}{\bar{\mu}_H^2}\right)\right]$$

$$- \left(-i\frac{5\rho^2}{16\pi^2}\frac{m^2}{M^2}\left[\log\frac{\bar{\mu}_H^2}{m^2} + 1\right]\right)$$

$$= -i\frac{1}{10}\frac{C_6}{16\pi^2}\left(3\log\frac{\bar{\mu}_H^2}{M^2} + 1 + \frac{m^2}{M^2}\left(3\log\frac{\bar{\mu}_H^2}{M^2} + 5\right)\right) \leftarrow \text{no } \log m\,!$$

- Note now $\log \frac{\bar{\mu}_H^2}{M^2}$ absorbed by matching (separate scales)
- Note non-trivial reshuffeling of tree and loop effects
- Next we run to the low scale

$$\frac{d}{d\log\bar{\mu}} C_4 = \frac{3}{32\pi^2} C_4^2 + \frac{1}{32\pi^2}\frac{m^2}{M^2} C_6$$

Need to solve this consistent w/ power counting Can't have ∞ insertions of C_6

Write $C_4 = C_4^{(0)} + C_4^{(1)}$ w/ superscripts tracking $\mathcal{O}$ in λ.

$$\frac{d}{d\log\bar{\mu}}\left(C_4^{(0)} + C_4^{(1)}\right) = \frac{3}{32\pi^2}\left(C_4^{(0)}\right)^2 + \frac{1}{32\pi^2}\frac{m^2}{M^2} C_6$$

Zeroth order eq $\Rightarrow$ running of C_4 from yesterday

The $C_4^{(0)} C_4^{(2)}$ term is higher order

$$\Rightarrow \frac{d}{d\log\bar{\mu}} C_4^{(2)} = \frac{1}{32\pi^2}\frac{m^2}{M^2} C_6$$

$$\Rightarrow C_4^{\text{resum}}(\bar{\mu}_L) = \frac{C_4^{(0)}(\bar{\mu}_H)}{1 - C_4^{(0)}\frac{3}{32\pi^2}\log\frac{\bar{\mu}_H}{\bar{\mu}_L}} + C_4^{(2)}(\bar{\mu}_H) + \frac{1}{32\pi^2}\frac{m^2}{M^2} C_6 \log\frac{\bar{\mu}_L}{\bar{\mu}_H}$$

1-loop at low scale

$$\Rightarrow i\,A_{EFT} = -i\,C_4(\bar{\mu}_L)\left[1 - \frac{3}{32\pi^2} C_4(\mu_L)\log\frac{\bar{\mu}_L^2}{m^2}\right] + \frac{i\,C_6}{32\pi^2}\frac{m^2}{M^2}\left(\log\frac{\bar{\mu}_L^2}{m^2} + 1\right)$$

Put it all together

$$iA^{Full} = \frac{iC_6}{32\pi^2}\frac{1}{10}\frac{m^2}{M^2}\left(3\log\frac{m^2}{M^2} + 2\log\frac{m^2}{\bar{\mu}_H^2}\right)$$

$$iA^{EFT}_{Exp} = i\frac{1}{10}\frac{1}{16\pi^2}C_6\frac{m^2}{M^2}\left(3\log\frac{\bar{\mu}_H^2}{M^2} + 2\log\frac{\bar{\mu}_H^2}{\bar{\mu}_H^2}\right) \leftarrow \text{matching @ } \bar{\mu}_H$$

$$-i\frac{1}{32\pi^2}\frac{m^2}{M^2}C_6\log\frac{\bar{\mu}_H^2}{\bar{\mu}_L^2} \leftarrow \text{RGE}$$

$$-i\frac{1}{32\pi^2}\frac{m^2}{M^2}C_6\log\frac{\bar{\mu}_L^2}{m^2} \leftarrow \text{Finite terms @ } \bar{\mu}_L$$

They agree!

$$= i\frac{C_6}{32\pi^2}\frac{1}{10}\frac{m^2}{M^2}\left[3\log\frac{m^2}{M^2} + 2\log\frac{m^2}{\bar{\mu}_H^2}\right]$$

- Reduced multi-scale problem to series of single scale problems

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Recap:

- Integrating out a heavy particle leads to naive expectations from dimensional analysis: all scales are set by Λ_{UV}
- To see physics of EFT requires matching $\Rightarrow$ modifies coeffs of local ops $\Rightarrow$ EFT is renormilization
- EFT seperates scales and resums potentially large logs by converting IR divergences into UV divergences

- Method of Regions: heavy-light integral

$$I = \int \frac{d^4\ell}{(\ell^2-m^2)(\ell^2-M^2)}$$

Assume various scallings for ℓ using λ
Keep ones that contribute to order of interest

$$\ell_h \sim M(1,1,1,1) \Rightarrow I \to \int \frac{d^4\ell}{(\ell^2-M^2)\lambda^2} + \frac{m^2 d^4\ell}{(\ell^2-M^2)\lambda^4}$$

$$\ell_s \sim M(\lambda,\lambda,\lambda,\lambda) \Rightarrow I \to \frac{d^4\ell}{-M^2(\ell^2-m^2)}$$

Evaluate them

$$I_h^{(0)} = \mu^{2\varepsilon}\int (d\ell)\frac{1}{(\ell^2-M^2)\ell^2} = \frac{i}{16\pi^2}\left(\frac{1}{\varepsilon} + \log\frac{\bar{\mu}^2}{M^2} + 1\right)$$

$$I_h^{(1)} = \mu^{2\varepsilon}\int (d\ell)\frac{m^2}{(\ell^2-M^2)\ell^4} = \frac{i}{16\pi^2}\frac{m^2}{M^2}\left(\frac{1}{\varepsilon} + \log\frac{\bar{\mu}^2}{M^2} + 1\right)$$

$$I_s^{(1)} = \mu^{2\varepsilon}\int (d\ell)\frac{1}{-M^2(\ell^2-m^2)} = -\frac{i}{16\pi^2}\frac{m^2}{M^2}\left(\frac{1}{\varepsilon} + \log\frac{\bar{\mu}^2}{m^2} + 1\right)$$

$$\Rightarrow I = I_h^{(0)} + I_h^{(1)} + I_s^{(0)} = \frac{i}{16\pi^2}\left[\left(\frac{1}{\varepsilon} + \log\frac{\bar{\mu}^2}{M^2} + 1\right) + \frac{m^2}{M^2}\left(\log\frac{\bar{\mu}^2}{M^2} - \log\frac{\bar{\mu}^2}{m^2}\right)\right]$$

- Identify regions w/ DOF' Soft region $\Rightarrow \varphi$ soft mode

Heavy Quark Effective Theory

Eg $v = (1, \vec{0})$

$$\mathcal{L}_{QCD} = \bar{Q}(i\not{D} - m)Q$$

Let $p^\mu = m v^\mu + k^\mu$ w/ $k/m \ll 1$

Define

$$h_v(x) = e^{imv\cdot x}\frac{1+\not{v}}{2}Q(x) \qquad \text{(labels)}$$

$$H_v(x) = e^{imv\cdot x}\frac{1-\not{v}}{2}Q(x)$$

$$\Rightarrow Q(x) = e^{-imv\cdot x}\left[h_v(x) + H_v(x)\right]$$

$$\Rightarrow \mathcal{L} = \bar{h}_v(iv\cdot D)h_v - \bar{H}_v(iv\cdot D + 2m)H_v$$
$$+ \bar{H}_v\, i\not{D}_\perp h_v + \bar{h}_v\, i\not{D}_\perp H_v$$

w/ $D_\perp^\mu \equiv D^\mu - v^\mu(v\cdot D)$

$\not{v} h_v = h_v$

$\not{v} H_v = -H_v$

Int out $H_v \Rightarrow$

$$H_v = \frac{1}{iv\cdot D + 2m}\, i\not{D}_\perp h_v$$

$$\Rightarrow \mathcal{L} \supset \bar{h}_v\left(iv\cdot D + i\not{D}_\perp\frac{1}{iv\cdot D + 2m}i\not{D}_\perp\right)h_v$$

$$\Rightarrow \mathcal{L}_{EFT} \supset \bar{h}_v\left(iv\cdot D + i\not{D}_\perp\left(\frac{1}{2m} - \frac{1}{(2m)^2}(iv\cdot D) + \dots\right)\not{D}_\perp\right)$$

Bottom up: $k \sim \lambda$

Symmetry (Reparametrization Invariance)

$$k^\mu \to k^\mu + \delta k^\mu \qquad v^\mu \to v^\mu - \frac{\delta k^\mu}{m}$$

w/ $\quad v \cdot \delta k = \dfrac{\delta k^2}{2m}$

$\Rightarrow \quad p \to p$

Mixes orders in $1/m$ and restricts independent Wilson coeffs.

Decoupling

Diag couplings $h_v h_v G$ & $H_v H_v G$

$$\Rightarrow \quad I_{\text{diag}} \sim \int \frac{d^d q}{(2\pi)^d} \left(\frac{1}{v \cdot q + m \pm i\varepsilon}\right)\left(\frac{1}{v \cdot (q+p) + m \pm i\varepsilon}\right)$$

Poles only on one side $\Rightarrow$ deform contour to get zero

$$I_{\text{off-diag}} \sim \int \frac{d^d q}{(2\pi)^d} \left(\frac{1}{v \cdot q + m + i\varepsilon}\right)\left(\frac{1}{v \cdot (q+p) + 2m - i\varepsilon}\right)$$

Either contour $\Rightarrow$ contribution

Functional Matching (tree-level)

(For 1-loop, see e.g. 2011.02484)

Integrate out field using equations of motion

"Semiclassical expansion": evaluate action on a solution to EOM

$$S_{\text{eff}}[\varphi] = S[\varphi, \Phi_{cl}] + \mathcal{O}(\hbar) \quad \text{w/} \quad \frac{\delta S[\varphi, \Phi]}{\delta \Phi}\bigg|_{\Phi = \Phi_{cl}} = 0$$

EOM

$$\Rightarrow \Box \Phi + m_\Phi^2 \Phi + \frac{g}{2}\varphi^2 + \frac{g'}{2}\Phi^2 + \frac{\lambda'}{2}\Phi\varphi^2 + \frac{\lambda''}{6}\Phi^3 = 0$$

Assume $g \sim g' \sim m_\Phi \sim M$ and $\lambda \sim \lambda' \sim \mathcal{O}(1)$

Solve itteratively: $\Phi_{cl}^{(1)} = \frac{-g}{2m_\Phi^2}\varphi^2 \sim \mathcal{O}\left(\frac{1}{M}\right)$

$$\Rightarrow \Phi_{cl} = \underbrace{\frac{-g}{2m_\Phi^2}\varphi^2}_{\mathcal{O}\left(\frac{1}{M}\right)} - \underbrace{\frac{1}{m_\Phi^2}\Box\Phi_{cl}}_{\mathcal{O}(1/M^3)} - \underbrace{\frac{\lambda'}{2m_\Phi^2}\Phi_{cl}\varphi^2}_{\mathcal{O}(1/M^3)} - \underbrace{\frac{g'}{2m_\Phi^2}\Phi_{cl}^2}_{\mathcal{O}(1/M^3)} - \underbrace{\frac{\lambda''}{6m_\Phi^2}\Phi_{cl}^3}_{\mathcal{O}(1/M^5)}$$

$\Rightarrow$ To go to $\mathcal{O}(1/M^3)$ sub $\Phi_{cl}^{(1)}$ into EOM

$$\Rightarrow \Phi_{cl}^{(3)} = \frac{-g}{2m_\Phi^2}\varphi^2 + \frac{g}{2m_\Phi^4}\Box\varphi^2 + \left(\frac{g\lambda'}{4m_\Phi^4} + \frac{g^2 g'}{4m_\Phi^6}\right)\varphi^4 + \mathcal{O}\left(\frac{1}{M^5}\right)$$

Sub into $\mathcal{L}_{UV} \Rightarrow \mathcal{L}_{\text{eff}} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m_\varphi^2\varphi^2 - \frac{1}{4!}\left(\lambda - \frac{3g^2}{m_\Phi^2}\right)\varphi^4$

$$- \frac{1}{6!}\left(\frac{45\lambda' g^2}{m_\Phi^4} - \frac{15 g' g^3}{m_\Phi^6}\right)\varphi^6 + \frac{g^2}{8m_\Phi^4}(\partial_\mu\varphi^2)(\partial^\mu\varphi^2) + \mathcal{O}(1/M^4)$$

Field Redefinition Invariance of S-Matrix

We can extract S-matrix elements from connected correlation functions using the LSZ reduction formula, which schematically takes the form

$$A_n \sim \int \left[\prod d^4x_i \, e^{ip_i \cdot x_i}\right] D^{-1}_{x_1 y_1} \cdots D^{-1}_{x_n y_n} \langle \varphi(y_1) \ldots \varphi(y_n) \rangle_{conn} \Big|_{J=0}$$

where $D^{-1}_{x_1 y_1}$ are the inverse propagators from $x_1 \to y_1$.

We compute the correlation functions from the path integral:

$$Z[J] = \int \mathcal{D}\varphi \, \exp\left(i S'[\varphi] + i \int d^4x \, \varphi(x) J(x)\right)$$

Writing $Z[J] = \exp i W[J]$ we have

$$\langle \varphi(x_1) \ldots \varphi(x_n) \rangle_{connected} = (-i)^n \frac{\delta^n (iW)}{\delta J(x_1) \ldots \delta J(x_n)}$$

Now, lets do a field redef:

$$\varphi(x) = F(\varphi'(x))$$

The new Lagrangian is

$$\mathcal{L}(\varphi) = \mathcal{L}(F(\varphi')) = \mathcal{L}'(\varphi')$$

The path integral becomes

relabeled integration variable

$$Z'[J] = \int \mathcal{D}\varphi' \, e^{iS(\mathcal{L}'(\varphi') + J\varphi')} = \int \mathcal{D}\varphi \, e^{iS(\mathcal{L}'(\varphi) + J\varphi)}$$

We can compare this with the original path int after making the field redef:

$$Z[J] = \int \mathcal{D}\varphi' \left| \frac{\delta F}{\delta \varphi'} \right| e^{iS(\mathcal{L}'(\varphi') + JF(\varphi'))}$$

The Jacobian $\left| \frac{\delta F}{\delta \varphi'} \right| = 1$ in dim reg (scaleless int)

Relabeling $\varphi' \to \varphi$, we have

$$Z[J] = \int \mathcal{D}\varphi \, e^{iS(\mathcal{L}'(\varphi) + JF(\varphi))}$$

The only difference between Z and Z' is the coupling to the source: $JF(\varphi)$ vs. $J\varphi$

Clearly the connected correlators will not be the same. However, all the S-matrix depends on are the poles in the correlation functions. This is what the factors of D^{-1} extract.

All that is required to determine the propagator is the interpolating field formula: $\langle p | \varphi(x) | 0 \rangle \neq 0$.

As long as we ensure that this condition holds for φ and F, then we have a good interpolating field in both cases.

Setting the wave function renormalization to 1, we know $\langle p|\varphi(x)|0\rangle = e^{ip\cdot x} \neq 0$

So we must restrict ourselves to field redefs of the form $\phi = F(\varphi') = \varphi' + f(\varphi')$

Then $\langle p|F(\varphi')|0\rangle = \langle p|\phi'|0\rangle + \cancel{\langle p|f(\phi')|0\rangle}^{\,0}$

$$= e^{ip\cdot x} \neq 0$$

For this class of field redefs, the S-matrix is unchanged. Specifically, $f(\phi')$ is a local analytic expansion in fields and derivatives.

Finally, note that field redefinitions can mix terms at different order in power counting.

Ex: Let $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2$. Define $F(\phi') = \phi' + \frac{\phi'^3}{\Lambda^2}$

Show that one still gets a free theory for $\varphi'\varphi' \to \varphi'\varphi'$ scattering at tree-level. (Do this)

Simplifying $\mathcal{L}_{eff}$

Two stradigies : (1) integration by parts
(2) field redefinitions

Ex: Classify all possible terms of the form $\partial^2 \varphi^n$

Using $\partial_\mu \varphi^r = r\varphi^{r-1}\partial_\mu \varphi$ rewrite operator so each derivative acts on single field.

$\Rightarrow$ Most general operator is linear combo of

$$\varphi^{n-1}\Box\varphi \quad \text{and} \quad \varphi^{n-2}\partial^\mu\varphi\,\partial_\mu\varphi$$

Then
$$\varphi^{n-2}\partial^\mu\varphi\,\partial_\mu\varphi = \frac{1}{n-1}\partial^\mu\varphi^{n-1}\,\partial_\mu\varphi$$
$$= -\frac{1}{n-1}\varphi^{n-1}\Box\varphi + \text{total der}$$

$\Rightarrow$ Only single independent operator for each n.

Ex: Field redefinitions (aka "using the equations of motion")

Let $\varphi \to \varphi + f(\varphi)$ and expand in powers of $f(\varphi)$:

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - V \to \frac{1}{2}(\partial\varphi)^2 - V - f(\varphi)\underbrace{\left[\Box\varphi + V'\right]}_{EOM} + \mathcal{O}(f^2)$$

Ex: Let's simplify our previous example

$$\varphi \to \varphi + c\frac{g^2}{m_\Phi^4}\varphi^3$$

$$\Rightarrow \mathcal{L}_{eff} \to \mathcal{L}_{eff} + \frac{cg^2}{m_\Phi^4}\varphi^3\left[\Box\varphi + m_\varphi^2\varphi + \frac{\lambda}{3!}\varphi^3\right] + \mathcal{O}(\Lambda^{-4})$$

Taking $c = \frac{1}{2} \Rightarrow$

$$\mathcal{L}_{eff} \to \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m_\varphi^2\varphi^2 - \frac{1}{4!}\left(\lambda - \overbrace{\frac{3g^2}{m_\Phi^2}}^{\mathcal{O}(1)} - \overbrace{\frac{6g^2 m_\varphi^2}{m_\Phi^4}}^{\mathcal{O}(1/\Lambda^2)}\right)\varphi^4$$

$$+ \frac{1}{6!}\left[\frac{g^2(45\lambda' - 60\lambda)}{m_\Phi^4} - \frac{15g'g^3}{m_\Phi^6}\right]\varphi^6 + \mathcal{O}(1/\Lambda^4)$$

We have eliminated the $\partial^2\varphi^4$ term!

$\Rightarrow$ All indirect effects from η can be modeled by modified φ^4 and φ^6 terms up to $\mathcal{O}(E^2/\Lambda^2)$

This justifies using the classical EOMs to rewrite the $\mathcal{L}$ into a more convenient form.

SMEFT, HEFT, and EFT Geometry

For simplicity, we can assume custodial symmetry

$\Rightarrow SU(2) \times U(1) \to SU(2)_L \times SU(2)_R \cong O(4)$

Custodial sym is only approximate in SM

Explicitly broken by gauging $U(1)_Y \subset SU(2)_R$

and due to fermion mass splittings

SMEFT

Focus on scalar sector. Let $\vec{\varphi}$ be fundamental of $O(4)$:

$$\vec{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} \qquad \text{w/} \quad \vec{\varphi} \to O\,\vec{\varphi} \quad \text{under } O(4)$$

(O is 4×4 orthogonal matrix.)

Identify
$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_4 + i\varphi_3 \end{pmatrix}$$

$$\mathcal{L}_{SMEFT} = A(|H|^2)\,|\partial H|^2 + \frac{1}{2} B(|H|^2)\left[\partial(|H|^2)\right]^2 - \tilde{V}(|H|^2) + \mathcal{O}(\partial^4)$$

w/ $A, B, \tilde{V}$ are real analytic at origin $|H| = 0$.

Geometrically, φ_i are Cartesian coordinates.

Goldstones of $O(4)/O(3)$ $\vec{\pi}$ ← transform non-linearly

Singlet scalar field h

Define $\vec{n} = \begin{pmatrix} n_1 = \pi_1/v \\ n_2 = \pi_2/v \\ n_3 = \pi_3/v \\ n_4 = \sqrt{1-n_1^2-n_2^2-n_3^2} \end{pmatrix}$

Under $O(4)$ $h \to h$ and $\vec{n} \to O\vec{n}$

$\vec{n}(\vec{\pi}) \in S^3$ is 4-component unit vector w/ $\vec{n}\cdot\vec{n} = 1$.

The constrained vector $\vec{n}$ transforms linearly.

The rotations in the 12, 13, and 23 planes act linearly on (n_1, n_2, n_3) and leave n_4 invariant. However, if one does eg a 14 rotation (infinitesimal)

$\delta n_1 = \theta n_4$, $\delta n_2 = 0$, $\delta n_3 = 0$, $\delta n_4 = -\theta n_4$

Then the transformation of the unconstrained π fields is $\delta \pi_1 = \theta\sqrt{v^2 - \vec{\pi}\cdot\vec{\pi}}$, $\delta\pi_{2,3} = 0$

$\Rightarrow$ non-linear.

$$\mathcal{L}_{HEFT} = \frac{1}{2}\left[\mathbb{K}(h)\right]^2(\partial h)^2 + \frac{1}{2}\left[v F(h)\right]^2(\partial \vec{n})^2 - V(h) + \mathcal{O}(\partial^4)$$

w/ $\mathbb{K}, F, V$ are real analytic about the physical vacuum $h=0$.

Geometrically, HEFT is like polar coordinates.

* Ultimately, HEFT is description used to do physical calculations, since need to work in physical vacuum.

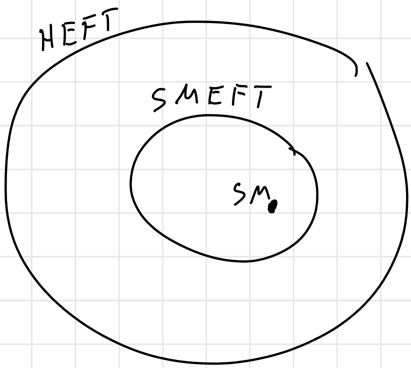
Remember EFT requires truncation of power counting expansion.

Compare $\tilde{V}(H)$ up to dim 6 and $V(h)$ up to 6 fields

$$\tilde{V}(H) = -\mu^2 |H|^2 + \lambda |H|^4 + \frac{1}{\Lambda^2} |H|^6$$

$$V(h) = m^2 h^2 + c_3 h^3 + c_4 h^4 + c_5 h^5 + c_6 h^6$$

Clearly HEFT has larger parameter space then SMEFT.



If we parametrize BSM searches w/ SMEFT, are we potentially missing anything?

Motivates understanding the relationship between HEFT and SMEFT.

Note: preference is to work w/ SMEFT since that is already hard enough.

Also much more natural from model building perspective.

Assume no obstruction to mapping between SMEFT and HEFT:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_4 + i\varphi_3 \end{pmatrix} \quad \text{and} \quad \vec{\varphi} = (v_0 + h)\vec{n}$$

How to determine v_0? ← Revisit

(Note v sets gauge boson masses, etc)

Let's write some $O(4)$ symmetric objects setting $v = v_0$ for simplicity:

$$|H|^2 = \frac{1}{2}\vec{\varphi}\cdot\vec{\varphi} = \frac{1}{2}(v+h)^2$$

$$|\partial H|^2 = \frac{1}{2}(\partial\vec{\varphi})^2 = \frac{1}{2}(\partial h)^2 + \frac{1}{2}(v+h)^2(\partial\vec{n})^2$$

$$(\partial|H|^2)^2 = (\vec{\varphi}\cdot\partial\vec{\varphi})^2 = (v+h)^2(\partial h)^2$$

The using This, we can write (Exercise)

$$\mathcal{L}_{HEFT} = \frac{1}{2}\left[K(h)\right]^2(\partial h)^2 + \frac{1}{2}\left[v\,F(h)\right]^2(\partial\vec{n})^2 - V(h) + \ldots$$

$$= \frac{v^2F^2}{2|H|^2}|\partial H|^2 + \frac{1}{2}(\partial|H|^2)^2\frac{1}{2|H|^2}\left(K^2 - \frac{v^2F^2}{2|H|^2}\right) - \tilde{V}(|H|^2) + \ldots$$

(Notice non-analiticity.)

Field Redefinitions and EFT

An EFT Lagrangian is a local expansion in terms of fields and derivatives.

We can only make field redefinitions of the form $\tilde{\varphi}^i = \varphi^j F^{ij}(\varphi)$ where F is a real analytic function of the fields (F has a convergent Taylor expansion about $\varphi = 0$.) and $F^{ij}(0) = \delta^{ij}$.

This implies that we are working with a real analytic manifold.

However, polar coordinates obscure the analyticity of the origin as we now explain:

Consider a 2d manifold $\mathbb{R}^2$. Define polar coordinates that map all points except the origin to (r, θ) w/ line element $ds^2 = dr^2 + r^2 d\theta^2$.

Now consider two Cartesian-like charts $C1$ w/ (x_1, y_1) and C_2 w/ (x_2, y_2)

Away from the origin, we have invertable and analytic relations $x_1 = r\cos\theta,\ y_1 = r\cos\theta$.

and $x_2 = (r+r^2)\cos\theta,\ y_2 = (r+r^2)\cos\theta$

So we can relate them to each other:

$$x_2 = x_1\left(1+\sqrt{x_1^2+y_1^2}\right)$$

$$y_2 = y_1\left(1+\sqrt{x_1^2+y_1^2}\right)$$

These are not real analytic at the origin.

This non-analyticity manifests when computing the components of the metric:

$$ds^2 = dr^2 + r^2 d\theta^2 = dx_1^2 + dy_1^2$$

$$= \frac{1}{x_1^2+x_2^2}\left[\frac{(x_2\,dx_2 + y_2\,dy_2)^2}{1+4\sqrt{x_2^2+y_2^2}} + \frac{(x_2\,dy_2 - y_2\,dx_2)^2}{\left(1+\sqrt{1+4\sqrt{x_1^2+x_2^2}}\right)^2}\right]$$

This is in exact analogy with what can go wrong when mapping from HEFT $\to$ SMEFT.

Want to distinguish this "unphysical" non-analyticity from "physical" ones. A tool to do this is to look for physical singularities on the manifold using curvature invariants.

For example, take $ds^2 = dr^2 + T(r)\, d\theta^2$

Then we can compute the Ricci scalar

$$R(r) = \frac{(T')^2}{2T^2} - \frac{T''}{T}$$

For the flat space case, $T = r^2$, and we have

$R = 0 \Rightarrow$ no physical singularities

If we had e.g. $T = r \Rightarrow R(r) = 1/2r^2 \Rightarrow R(0) \to \infty$,

so this case has a physical obstruction

at the origin.

Applying these ideas to our EFT, take the following example

Claim (Exercise)

$$\mathcal{L}_H = \frac{1}{2}\left(1 + \frac{\sqrt{2|H|^2}}{v} + \frac{|H|^2}{2v^2}\right)|\partial H|^2 + \frac{1}{4v^2}\left(\frac{v}{\sqrt{2|H|^2}} + \frac{3}{4}\right)\frac{1}{2}\left(\partial |H|^2\right)^2 - \tilde{V}_H$$

but sending $h \to h_1 = h + \frac{1}{4v} h^2$

$$\Rightarrow \mathcal{L}_{H_1} = |\partial H_1|^2 - \tilde{V}_{H_1}$$

Field redefs of h can completely obscure

the analytic properties in terms of H.

Electroweak Symmetry Restoration

How do we identify the point on the field space manifold where EW symmetry is restored? AJM showed that this corresponds to identifying an "O(4) invariant fixed point" on the manifold. This is a point $\vec{\varphi}_0$ where $O\vec{\varphi}_0 = \text{zero}$ where O is an $O(4)$ transformation. Clearly if a linear rep exists s.t. $\vec{\varphi} \to O\vec{\varphi}$ Then $\vec{\varphi} = \text{zero}$ is such a fixed point.

To show the converse is true, assume That a set of coordinates exist That transform under $O(4)$ That contains an $O(4)$ invariant fixed point. Then The Coleman, Wess, Zumino "Linearization Lemma" tells us That a set of coordinates exist in the neighborhood of the fixed point That transform linearly under $O(4)$.

So now we know that the existance of an $O(4)$ invariant point on the manifold implies that we can write the coordinates in a linear representation (a necissary condition to have SMEFT). How do we identify such a point from

$$\mathcal{L}_{HEFT} = \frac{1}{2}(\partial h)^2 + \frac{1}{2}v^2 F(h)^2 (\partial \vec{n})^2 + O(4) \text{ sym terms}$$

Note that $\partial \vec{n}$ is only invariant under $O(3)$ transformations. So we are looking for a point h_* such that $F(h_*) = 0$.

Then we can identify $h_* = -v$.

This allows us to find a possible SMEFT point within the HEFT framework.

Field Space Geometry

Start with a set of coordinates (fields) φ^i

Under a coordinate change (aka a field redefinition without derivatives)

$\varphi^i = \varphi^i(\tilde{\varphi})$ then $\partial_\mu \varphi^i$ transforms as

$$\partial_\mu \tilde{\varphi}^i = \left(\frac{\partial \tilde{\varphi}^i}{\partial \varphi^j}\right) \partial_\mu \varphi^j$$ due to the chain rule.

Check this (e.g. for polynomial field redefs)

Similarly, a tensor transforms according to its index structure. E.g.

$$\tilde{T}^{ij} = \left(\frac{\partial \tilde{\varphi}^i}{\partial \varphi^k}\right)\left(\frac{\partial \tilde{\varphi}^j}{\partial \varphi^l}\right) T^{kl} \quad \text{and} \quad \tilde{T}_{ij} = \left(\frac{\partial \varphi^k}{\partial \tilde{\varphi}^i}\right)\left(\frac{\partial \varphi^l}{\partial \tilde{\varphi}^j}\right) T_{kl}$$

Infinitesmal line element in field space is

$$ds^2 = G_{ij}(\varphi)\, d\varphi^i d\varphi^j$$

We call $G_{ij}(\varphi)$ the field space metric

The metric has the following properties:

- Transforms as a 2-tensor
- Symmetric: $G_{ij} = G_{ji}$
- Non-singular: No row or column can be the zero vector.

Write a generic 2 derivative scalar field Lagrangian in the form

$$\mathcal{L}_{kin} = \frac{1}{2} G_{ij}(\varphi)\, \partial_\mu \varphi^i \partial^\mu \varphi^j$$

G_{ij} is symmetric ✓

G_{ij} is non-singular since there must be a non-zero kinetic term for each scalar field ✓

G_{ij} transforms as a 2-tensor?

Under a transformation $\varphi \to \tilde{\varphi}(\varphi)$

Then $$\mathcal{L}_{kin} = \frac{1}{2} \underbrace{G_{ij}(\varphi) \frac{\partial \varphi^k}{\partial \tilde{\varphi}^i} \frac{\partial \varphi^l}{\partial \tilde{\varphi}^j}}_{\equiv\, \tilde{G}_{kl}(\tilde{\varphi})} \partial_\mu \tilde{\varphi}^k \partial^\mu \tilde{\varphi}^l$$

So it transforms as a tensor ✓

and $$\tilde{\mathcal{L}}_{kin} = \mathcal{L}_{kin}\Big|_{\substack{\varphi \to \tilde{\varphi} \\ G \to \tilde{G}}}$$

What about derivatives of scalar functions (e.g V)

$$V(\varphi) \to V(\tilde{\varphi})$$

$$\frac{\partial V}{\partial \varphi^i} = \frac{\partial \tilde{\varphi}^j}{\partial \varphi^i} \frac{\partial V}{\partial \tilde{\varphi}^j}$$ ← tensor

not tensor

$$\frac{\partial^2 V}{\partial \varphi^i \partial \varphi^j} = \frac{\partial \tilde{\varphi}^k}{\partial \varphi^i} \frac{\partial \tilde{\varphi}^l}{\partial \varphi^j} \frac{\partial^2 V}{\partial \tilde{\varphi}^k \partial \tilde{\varphi}^l} + \frac{\partial^2 \tilde{\varphi}^k}{\partial \varphi^i \partial \varphi^j} \frac{\partial V}{\partial \tilde{\varphi}^k}$$

$\Rightarrow$ Need to covariantize

Introduce Christoffel symbols

$$\Gamma^i_{jk} = \frac{1}{2} G^{il} \left(\partial_j G_{kl} + \partial_k G_{lj} - \partial_l G_{jk} \right)$$

Then covariant derivative ∇_i acts on V as

$$\nabla_i V = \frac{\partial}{\partial \varphi^i} V \,, \quad \nabla_i \nabla_j V = \frac{\partial^2 V}{\partial \varphi^i \partial \varphi^j} - \Gamma^k_{ij} \frac{\partial V}{\partial \varphi^k}$$

Geometry of HEFT

To identify the metric in HEFT, write

$$\mathcal{L}_{HEFT} = \frac{1}{2}(\partial h)^2 + \frac{1}{2}\left[v F(h)\right]^2 \left(\delta^{ij} + \frac{n^i n^j}{1-n^2} \right) (\partial^\mu n^i)(\partial_\mu n^j)$$

w/ $\vec{n} = \left(n_1, n_2, n_3, \sqrt{1 - \sum_i n_i^2}\right)$

Then we identify the components of the metric:

$$g_{hh} = K^2 \quad , \quad g_{h\pi_i} = 0 \qquad \Bigg| \qquad g^{hh} = 1/K^2$$

$$g_{\pi_i \pi_j} = F^2 \left(\delta_{ij} - \frac{\pi_i \pi_j}{v^2 - \vec{\pi}\cdot\vec{\pi}} \right) \qquad \Bigg| \qquad g^{ij} = \frac{1}{F^2}\left(\delta_{ij} - \frac{\pi_i \pi_j}{v^2} \right)$$

Turn the GR crank... (show this)

$\Rightarrow$ Ricci scalar curvature

$$R = 6(\mathcal{K}_h + \mathcal{K}_\pi) \quad \text{w/ } \mathcal{K} \text{ are sectional curvatures}$$

$$R_{\pi_i h h \pi_j} = -g_{hh}\, g_{\pi_i \pi_j}\, \mathcal{K}_h \quad , \quad R_{\pi_i \pi_k \pi_l \pi_j} = (g_{il}\, g_{kj} - g_{ij}\, g_{kl})\, \mathcal{K}_\pi$$

$$\mathcal{K}_h = -\frac{1}{K^2}\left[\frac{F''}{F} - \frac{K'}{K}\frac{F'}{F} \right] , \quad \mathcal{K}_\pi = \frac{1}{(vF)^2}\left[1 - \frac{(vF')^2}{K^2} \right]$$

$$\nabla^2 V = \left(\frac{1}{K}\partial_h \right)^2 V + 3\frac{F'}{FK}\left(\frac{1}{K}\partial_h \right) V$$

Geometrizing Amplitudes

Notation

$$T_{\alpha_1\ldots\alpha_m,\beta_1\ldots\beta_n} \equiv \frac{\partial}{\partial\varphi^{\beta_n}}\cdots\frac{\partial}{\partial\varphi^{\beta_1}}T_{\alpha_1\ldots\alpha_m}$$

$$T_{\alpha_1\ldots\alpha_m;\beta_1\ldots\beta_n} \equiv \frac{\nabla}{\nabla\varphi^{\beta_n}}\cdots\frac{\nabla}{\nabla\varphi^{\beta_1}}T_{\alpha_1\ldots\alpha_m}$$

E.g. $$T_{\alpha_1\ldots\alpha_m;\beta} = T_{\alpha_1\ldots\alpha_m,\beta} - \sum_{i=1}^{m} T_{\alpha_1\ldots\underbrace{\hat{\alpha}_i\rho}_{\text{replace } \alpha_i \text{ with } \rho}\ldots\alpha_m}\Gamma^{\rho}_{\alpha_i\beta}$$

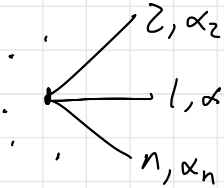
Taylor expand metric and potential about vacuum

$$\mathcal{L} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n!}\bar{g}_{\alpha\beta,\gamma_1\ldots\gamma_n}(\partial_\mu\varphi^\alpha)(\partial^\mu\varphi^\beta)\varphi^{\gamma_1}\cdots\varphi^{\gamma_n}$$
$$-\sum_{n=0}^{\infty}\frac{1}{n!}\bar{V}_{,\gamma_1\ldots\gamma_n}\varphi^{\gamma_1}\cdots\varphi^{\gamma_n}$$

Note $\bar{V}_{,\alpha\beta} = \bar{g}_{\alpha\beta}m_\alpha^2$ (bar denotes evaluated at vacuum)

Feynman rules

$$\alpha \;\text{———}\; \beta = \frac{i\bar{g}^{\alpha\beta}}{p^2 - m_\alpha^2}$$



omit

$$= -i\bar{V}_{,\alpha_1\ldots\alpha_n} - i\sum_{1\le i<j\le n} p_i\cdot p_j\,\bar{g}_{\alpha_i\alpha_j,\alpha_1\ldots\hat{\alpha}_i\ldots\hat{\alpha}_j\ldots\alpha_n}$$

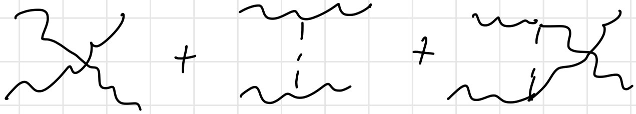
High energy Goldstone / Higgs scattering

$\pi\pi \to hh$

$$\Rightarrow \mathcal{M} = -s\,\bar{\alpha}_h + \mathcal{O}(g^4, t/s)$$

(compute these)

$\pi\pi \to \pi\pi$



$$\Rightarrow \mathcal{M} = s\,\bar{\alpha}_\varphi + \mathcal{O}(g^4, t/s)$$

Manifestly invariant under field redefs!
A useful trick to compute amplitudes is to do a field redefinition to normal coordinates at the vacuum. This implies that the metric is flat at this point on the manifold, so $\nabla \to \partial$. Additionally, the derivative of the metric vanishes

$\Rightarrow$ 3-pt vertex with derivative couplings vanishes.

Criteria for SMEFT

How can we develop a criterion for SMEFT that is robust against field redefinition ambiguities? Clearly we should try to frame the question in terms of field space geometry.

We already have one necissary condition, which is there must exist an $O(4)$ fixed point on the field space manifold h_*. Here we want to understand if a SMEFT expansion exists at that point (analytic expression in "Cartesian coordinates")

The logic of the argument is as follows:

1) Write the most general SMEFT (up to 2-derivative order)
2) Canonically normalize the h kinetic term. This fixes the choice of field basis.

3) Rephrase the basis specific criteria in terms of curvature invariants, so that the criteria can be applied in any basis.

1) First, notice that when we map from SMEFT to HEFT:

$$\mathcal{L}_{SMEFT} = A|\partial H|^2 + \frac{1}{2}B\left(\partial |H|^2\right)^2 - \tilde{V}$$
$$= \frac{1}{2}\left(A + (v+h)^2 B\right)(\partial h)^2 + \frac{1}{2}(v+h)^2 A (\partial \vec{n})^2 - V$$

$$\Rightarrow K = \sqrt{A + (v+h)^2 B} \;\; \& \;\; vF = (v+h)\sqrt{A}$$

where $A, B, \tilde{V}$ are real analytic functions of $|H|^2 = (v+h)^2/2$, $A(0) = 0$ and $V'(v^2/2) = 0$

This Lagrangian has the following properties

1) $F(h_* = -v) = 0$, $V'(h=0) = 0$
2) $K(h_*)$, $F(h_*)$, $V(h_*)$ are real analytic functions of h
3) Expanding about $h = h_*$, K & V are even and F is odd in $(h - h_*) = (v+h)$

Also, $A(0) = 1 \Rightarrow K(h_*) = vF'(h_*) = 1$.

Note that condition (3) is not field redefinition invariant. So this set of criterion are necissary but not sufficient to guarantee that SMEFT exists.

2) We want to fully fix the HEFT basis.

A natural choice is to canonically normalize the kinetic term. Let

$$v_1 + h_1 = Q(v+h) = \int_0^{v+h} dt\, \bar{K}(t) \Rightarrow dh_1 = \bar{K}\, dh_2$$

Claim that this fully fixes the freedom to redefine the fields h & $\vec{n}$, see "Is SMEFT Enough" Sec 4.1.2 for argument.

The HEFT Lagrangian in this basis is

$$\mathcal{L}_{HEFT} = \frac{1}{2}(\partial h)^2 + \frac{1}{2}(v F(h))^2 (\partial \vec{n})^2 - V(h) + \dots$$

3) To geometrize these basis dependent statements, we use the map

$$\left.\begin{array}{l} F^{(2k)}(h_*) = 0 \quad \forall k \in \mathbb{N} \\ v' F(h_*) = 1 \\ V^{(2k+1)}(h_*) = 0 \quad \forall k \in \mathbb{N} \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} F(h_*) = 0 \\ \nabla^{\mu_1}\nabla_{\mu_1} \cdots \nabla^{\mu_n}\nabla_{\mu_n} R\big|_{h_*} < \infty \quad \forall n \in \mathbb{N} \\ \nabla^{\mu_1}\nabla_{\mu_1} \cdots \nabla^{\mu_n}\nabla_{\mu_n} V\big|_{h_*} < \infty \quad \forall n \in \mathbb{N} \end{array}\right.$$

See Appendix B of "Is SMEFT Enough?" for derivation.

This motivates the approximate "Leading Order Criteria" for the existence of SMEFT:

a) $F(h)$ has a zero at some h_*

b) $\underline{K}$, F, and V have convergent Taylor expansions about h_*

c) The scalar curvature R is finite at h_*

When is HEFT Required?

Let's explore what can cause the criteria

$R|_{h_*} < \infty$ to fail.

Ex: Integrate out scalar singlet at tree-level

The UV model is

$$\mathcal{L}_{UV} = |\partial H|^2 + \frac{1}{2}(\partial S')^2 - V$$

w/ $V = -\mu_H^2 |H|^2 + \lambda_H |H|^4 + \frac{1}{2}\left(m^2 + \kappa |H|^2\right) S'^2 + \frac{1}{4}\lambda_S S'^4$

Integrate out S' using its EOMs

The Effective $\mathcal{L}$ is then (Check this)

$$\mathcal{L}_{Eff} = |\partial H|^2 - \frac{\kappa^2}{8\lambda_S(m^2 + \kappa |H|^2)}\left(\partial |H|^2\right)^2$$

$$+ \mu_H^2 |H|^2 - \lambda_H |H|^4 + \frac{1}{4\lambda_S}\left(m^2 + \kappa |H|^2\right)^2$$

Note this is <u>not</u> an EFT since it contains all orders in H. EFT requires truncation.

Next, express this in terms of h and $\vec{n}$ to find

$$K(h) = \sqrt{1 - \frac{\kappa^2 (v+h)^2}{2\lambda_S(2m^2 + \kappa(v+h)^2)}}, \quad vF = v + h$$

$$V = -\frac{1}{2}\mu_H^2 (v+h)^2 + \frac{1}{4}\lambda_H (v+h)^4 - \frac{1}{16\lambda_S}\left(2m^2 + \kappa(v+h)^2\right)^2$$

From here we can derive

$$R = \frac{6}{(v+h)\bar{K}^3}(\partial_h \bar{K}) + \frac{6}{(v+h)^2}\left(1 - \frac{1}{\bar{K}^2}\right) \quad \leftarrow \begin{array}{l} \partial_h F = 1/v \\ \partial_h^2 F = 0 \end{array}$$

$$= 3\lambda_d^2 \frac{2\lambda_d(v+h)^2(\lambda_d - 2\lambda_s) - 16m^2\lambda_s}{\left(\lambda_d(v+h)^2(\lambda_d - 2\lambda_s) - 4m^2\lambda_s\right)^2}$$

Let's check all the criteria:

Clearly $F(h_* - v) = 0$, and there are no obstructions to Taylor expanding any terms about h_*.

Then we can evaluate $R(h_*) = \frac{-3\lambda_d^2}{m^2\lambda_s}$ is finite.

So we can expand $\mathcal{L}_{Eff}$ in H to find

$$\mathcal{L}_{SMEFT} = |\partial H|^2 + \mu_H^2 |H|^2 - \lambda_H |H|^4 + \frac{1}{4\lambda_s}(m^2 + \lambda_d |H|^2)^2 - \frac{1}{8}\frac{\lambda_d^2}{\lambda_s m^2}(\partial |H|^2)^2 + \text{dim } 8$$

What about the UV theory with $m^2 = 0$?

We have $R\Big|_{m^2 \to 0} = \frac{6\lambda_d}{(\lambda_d - 2\lambda_s)(v+h)^2} \xrightarrow{h \to v} \infty$

So the criteria fails and HEFT is required! See "Is SMEFT Enough?" for more examples (loop level, fermions, ...).

Physical Interpretation of HEFT

We have learned that there are essentially two ways HEFT can fail:

1) The field space manifold does not contain an $O(4)$ invariant fixed point. This has the physical interpretation that there is an additional source of EW symmetry breaking that does not go to zero when the Higgs vev vanishes.

2) The fixed point exists but the curvature (or its covariant derivatives) diverges at the fixed point. This has the interpretation that we have integrated out a particle that gets all of its mass from the Higgs. Therefore, there is a BSM massless state at the $O(4)$ fixed point and SMEFT does not have all the necissary dofs.

The lesson is an intuitive one: SMEFT fails if the BSM physics is "non-decoupling". Then it is not possible to match onto SMEFT, and one must match onto HEFT.

In fact, we showed that these arguments can be used to show that HEFT violates perturbative unitarity at a scale $\mathcal{O}(4\pi v)$ when the EFT is modeling a BSM state that gets all (or most) of its mass from v.
(see "Unitarity violation and the Geo of HEFT")

Practicle Criterion for HEFT

One should match onto HEFT when integrating out a state whose mass is near or below the weak scale.

The point is that while SMEFT may exist, the expansion might converge so slowley that it is not practically useful.

See Sec 8 of "Is SMEFT Enough?"

Outlook

How should we organize indirect searches for BSM physics?

- If we want to use EFT, we have to decide if we want to assume that the new physics is of the "decoupling" type or not, i.e., should we use SMEFT? Then it is natural to ask if there are any "non-decoupling" UV completions that are consistent with the data. We studied this (w/ Ian Banta) and named these types of new particles "loryons." Some param space is still viable!
- We can also try to organize searches in a more bottom up approach. This is the idea of "primaries".
- My personal view is that the only thing that matters is to not miss the signs of new physics!

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