Instanton number

The quantities $\int F \wedge F$ in $U(1)$ gauge theory, or $\int \mathrm{tr}(F \wedge F)$ in $SU(N)$ gauge theory, are special – they are **quantized** (integer multiples of a base unit) when integrated for any gauge bundle on any spacetime manifold. Because we're integrating over space and time, makes the most sense to focus on the Euclidean QFT where space + time are treated on equal footing.

U(1) example take spacetime geometry to be a torus. Coords $x_1, \ldots x_4$, each periodic,

$$x_i \cong x_i + 2\pi r_i.$$

Consider the config. $$F = \frac{n}{2\pi r_1 r_2} dx_1 \wedge dx_2 + \frac{m}{2\pi r_3 r_4} dx_3 \wedge dx_4.$$

$n, m \in \mathbb{Z}$ so that $\frac{1}{2\pi}\int_{S^1_1 \times S^1_2} F = n$, $\frac{1}{2\pi}\int_{S^1_3 \times S^1_4} F = m$

are allowed (quantized) fluxes.

Then $$F \wedge F = \frac{nm}{2\pi^2 r_1 r_2 r_3 r_4} \underbrace{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}_{\text{volume element on torus}}.$$

Then $$\int_{T^4} F\wedge F = \underbrace{(2\pi)^4 r_1 r_2 r_3 r_4}_{\text{volume}} \cdot \frac{nm}{2\pi^2 r_1 r_2 r_3 r_4}$$

$$= 8\pi^2 nm.$$

This is an integer multiple of $8\pi^2$.

Turns out this is <u>always</u> true: $\frac{1}{8\pi^2}\int_{M^4} F\wedge F \in \mathbb{Z}$, for any 4-manifold M^4 on which we can define spinor fields (a <u>spin</u> manifold).

Can show that $F\wedge F \propto \vec{E}\cdot\vec{B}$

Thus, also obtain nontrivial $\int F\wedge F$ in presence of magnetic monopoles + electric charges. (Witten effect)

Can add $\frac{i\theta}{8\pi^2}\int F\wedge F$ to Euclidean action or

just $\frac{\theta}{8\pi^2}\int F\wedge F$ to Minkowski action. This is

a <u>periodic</u> term, i.e., $\theta \to \theta + 2\pi$ leaves physics unchanged because $\exp(iS_E) \mapsto \exp(iS_E)\exp\left(\frac{2\pi i}{8\pi^2}\int F\wedge F\right)$

$= \exp(iS_E)\cdot\exp(2\pi i n) = \exp(iS_E)$ for any field config.

SU(N) gauge theory and instantons

Here $A = \underbrace{A_\mu{}^a T^a}_{\equiv A_\mu,\ \text{a matrix}} dx^\mu$, $F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu = dA - i A\wedge A$.

fundamental rep $\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$.

Kinetic term $\int \left(-\frac{1}{g^2}\mathrm{tr}(F\wedge \star F)\right)$.

The function $g: x^\mu \mapsto \dfrac{x^0 + i\vec{x}\cdot\vec{\sigma}}{|x|}$, σ the Pauli matrices,

maps rays in $\mathbb{R}^4$ to elements of $SU(2)$, bijectively.

The configuration
$$A = \frac{|x|^2}{|x|^2+\rho^2}\left(-i g^{-1} d g\right)$$

is self-dual: $F^a_{\mu\nu} = \tilde{F}^a_{\mu\nu}$. This means its action can be related to the $\int \mathrm{tr}(F\wedge F)$ by the Bogomolnyi trick:

$$\int \frac{1}{g^2}\mathrm{tr}(F\wedge\star F) = \int \frac{1}{g^2}\left\{ \frac{1}{2}\mathrm{tr}\left[(F\mp\star F)\wedge(F\mp\star F)\right] \pm \underbrace{\mathrm{tr}(F\wedge F)}_{\text{topological invariant ("instanton number")}} \right\}$$

Perfect square, ≥ 0, equals zero for self-dual $F = \star F$.

You can check that $\int \mathrm{tr}(F\wedge F) = 8\pi^2$ for this configuration.
The Bogomolnyi trick shows it has action $8\pi^2/g^2$ and solves the Euclidean equation of motion.

This is the BPST instanton solution (Belavin, Polyakov, Schwarz, Tyupkin).

Fact: for $SU(N)$ gauge theory, $\int_M \frac{1}{8\pi^2}\mathrm{tr}(F\wedge F) \in \mathbb{Z}$ for any field configuration, M a closed 4-manifold.

The quantization of instanton number is important because it allows us to couple to a periodic scalar field, $\theta(x) \cong \theta(x) + 2\pi$.

$$S = \int \frac{k_G \, \theta(x)}{8\pi^2} \operatorname{tr}(F \wedge F).$$

$\theta(x) \mapsto \theta(x) + 2\pi$:

$$S \mapsto S + \int \frac{2\pi k_G}{8\pi^2} \operatorname{tr}(F \wedge F) = S + 2\pi k_G \, n_{inst}.$$

This leaves $\exp(iS)$ invariant iff $k_G \in \mathbb{Z}$, since $n_{inst} \in \mathbb{Z}$.

Generalization to Standard Model:

$$G = [SU(3) \times SU(2) \times U(1)] / \mathbb{Z}_6$$

allows

$$\int \left[\frac{k_G \, \theta(x) \operatorname{tr}(G \wedge G)}{8\pi^2} + \frac{k_W \, \theta(x) \operatorname{tr}(W \wedge W)}{8\pi^2} + \frac{k_B \, \theta(x) \, B \wedge B}{8\pi^2} \right]$$

conventionally normalized hypercharge

if $k_G \in \mathbb{Z}$, $k_W \in \mathbb{Z}$, $\frac{2}{3} k_G + \frac{1}{2} k_W + k_B \in \mathbb{Z}$.

Chiral anomaly (in 4d)

There is a fascinating interplay between instantons, instanton number, and chiral fermions, which we will only scratch the surface of. This is the source of a deep connection between QFT and mathematics, via the Atiyah-Singer index theorem.

ABJ (Adler-Bell-Jackiw) anomaly: a chiral rotation of a massless charged fermion is not a symmetry of the quantum theory.

Specifically, consider QED w/ massless charged Weyl fermions ψ (charge $+1$) and $\bar\psi$ (charge -1):

$$\frac{1}{\sqrt{g}}\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}^2 + \psi^\dagger i \not{D}\psi + \bar\psi^\dagger i\not{D}\bar\psi,$$

where $\not{D}\psi = \bar\sigma^\mu(\partial_\mu - iA_\mu)\psi$

$$\not{D}\bar\psi = \bar\sigma^\mu(\partial_\mu + iA_\mu)\bar\psi.$$

This theory w/out A_μ has 2 global symmetries,

$$\psi \mapsto e^{i\alpha}\psi \text{ and } \bar\psi \mapsto e^{i\beta}\bar\psi \text{ (independent).}$$

A_μ gauges the combination $\beta = -\alpha$

(the "vector" comb.: LH ψ and RH $\bar\psi^\dagger$ w/ equal charge)

Naively, the "axial" comb. is still a global symmetry. In fact, it is not!

chiral anomaly

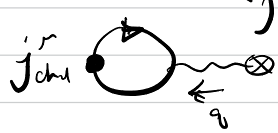
Consider the field redefinition $\psi \mapsto e^{i\alpha}\psi$ (leaving $\bar\psi$ untouched). Leaves the classical action unchanged. However, it does change the action of the quantum theory!

Multiple ways to see this, all somehow tied to needing to regulate the theory to define it (though the answer is regulator-independent).

This is all described in great detail in sections 19.1 – 19.4 of the QFT textbook by Peskin and Schroeder, which I <u>highly recommend</u> you work through carefully. (It begins w/ the 2d case, which is easier & in some ways more intuitive.)

Idea: compute the <u>expectation value</u> of the chiral current in a background electric field.

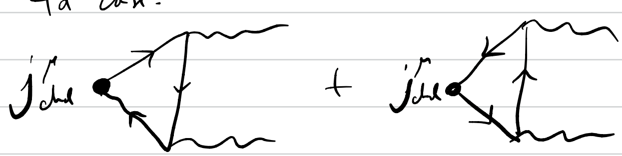
2d case: $\langle j^\mu_{\text{chiral}}(q)\rangle = \frac{1}{\pi}\epsilon^{\mu\nu}\left(A_\nu(q) - \frac{q_\nu q^\lambda}{q^2}A_\lambda(q)\right)$



→ doesn't vanish when dotting into q!

$$\partial_\mu j^\mu_{\text{chiral}} = \frac{1}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}.$$

4d case:



"triangle diagrams"

$$j^\mu_{\text{chiral}} = i\psi^\dagger\bar\sigma^\mu\psi.$$

chiral anomaly

Naively these 2 diagrams cancel. However, must be careful: they are divergent. Regulating them properly, one finds

$$\partial_\mu j^\mu_{\text{chiral}} = -\frac{1}{32\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}.$$

ONE-LOOP EXACT: INTEGRAL OF EACH SIDE $\in \mathbb{Z}$ (!)

<u>Fujikawa approach</u>: we can vary the fields in the path integral, $\psi(x) \mapsto e^{i\alpha(x)}\psi(x)$ (can take α constant at the end).

Problem: the path integral is $\int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{-S_E[\cdots]}$.

The classical action S_E is invariant under $\psi \mapsto e^{i\alpha}\psi$.

But the <u>measure</u> $\mathcal{D}\psi$ is actually <u>not</u>!

Expand in orthonormal modes ϕ_m, $i\not{D}\phi_m = \lambda_m\phi_m$,

$\psi(x) = \sum_m a_m \phi_m(x)$; then $\mathcal{D}\psi \to \prod_m da_m$.

But it turns out that (again carefully regulating) $\prod_m da_m$ has a nontrivial Jacobian:

$$\mathcal{D}\psi \mapsto \mathcal{D}\psi \cdot \exp\left[-\frac{i\alpha}{8\pi^2}\int F\wedge F\right].$$

CHIRAL TRANSFORMATION $\Rightarrow$ THETA TERM.

We just discussed an ABJ anomaly:

$$\partial_\mu j^\mu_{chiral} \neq 0$$

so this is not a symmetry.

But RHS involved gauge field.
w/o gauge fields, had

$$\frac{1}{\sqrt{g}} \mathcal{L} = \psi^\dagger i \partial \psi + \bar{\psi}^\dagger i \partial \bar{\psi} .$$

2 symmetries: $\psi \mapsto e^{i\alpha} \psi$
$\bar{\psi} \mapsto e^{i\beta} \bar{\psi}$

Both perfectly good symmetries! $U(1)_\psi \times U(1)_{\bar{\psi}}$.

But they have an 't Hooft anomaly, which is an obstruction to gauging: as the ABJ anomaly showed we can't gauge both.

't Hooft anomalies are a powerful nonperturbative tool to analyze QFT, but we will not discuss this in more detail in these lectures.

(Key idea: match anomalies in IR, UV)

QCD: quarks q (fundamental), $\bar{q}$ (antifundamental)

$$S_{QCD} = \int -\frac{1}{g^2}\, tr(F \wedge {*F}) + \int \left[i\, q^\dagger \not{D}\, q + i \bar{q}^\dagger \not{D}\, \bar{q} + m \bar{q} q + h.c. \right] \sqrt{g}\, d^4x$$

$$+ \int \frac{\theta}{8\pi^2}\, tr(F \wedge F)$$

Invariant physics under <u>field redefinitions</u>, like $q \mapsto e^{i\alpha} q$.

[Chiral anomaly: ψ a fermion w/ $U(1)$ charge q, $SU(N)$ rep R:

$$\psi \mapsto e^{i\alpha(x)} \psi(x) \text{ sends } S \mapsto S + \int \frac{\alpha(x)}{8\pi^2}\left[q^2\, F\wedge F + 2 I(R)\, tr(F \wedge F) \right]$$

$I(R)$ = Dynkin index (e.g., $\frac{1}{2}$ for fundamental rep, N for adjoint of $SU(N)$)

So for QCD, field redef $q \mapsto e^{i\alpha(x)} q$, $\bar{q} \mapsto e^{i\bar{\alpha}(x)} \bar{q}(x)$ mean S_{QCD} is equivalent to

$$S'_{QCD} = \int -\frac{1}{g^2}\, tr(F \wedge {*F}) + \int d^4x \sqrt{g} \left(i q^\dagger \not{D} q + i \bar{q}^\dagger \not{D} \bar{q} \right)$$

$$+ \underbrace{\int d^4x \sqrt{g} \left[m e^{i(\alpha+\bar{\alpha})} \bar{q} q + h.c. \right]}_{\substack{\text{Mass term changed, because} \\ \text{it violates chiral symmetry}}} + \underbrace{\int d^4x \sqrt{g} \left(\partial_\mu \alpha\, q^\dagger \bar{\sigma}^\mu q + \partial_\mu \bar{\alpha}\, \bar{q}^\dagger \bar{\sigma}^\mu \bar{q} \right)}_{\substack{\text{only appear if} \\ \alpha, \bar{\alpha} \text{ not constant} \\ \text{(relevant when we get} \\ \text{to axions)}}}$$

$$+ \underbrace{\int \frac{\theta + \alpha + \bar{\alpha}}{8\pi^2}\, tr(F \wedge F)}_{\theta \text{ term changed} \Rightarrow \theta \text{ not directly physical!}}$$

If no explicit classical breaking of chiral sym, θ <u>unphysical</u>.

But $\arg(m) \mapsto \arg(m) + \alpha + \bar{\alpha}$, so <u>$\bar{\theta} \equiv \theta - \arg(m)$ is a physical parameter.</u>

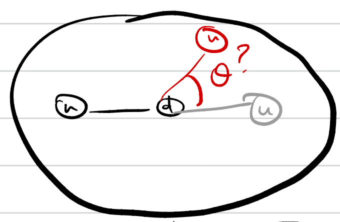
Strong CP Problem

A θ-term in QCD has physical effects.
It violates CP. Implications:

- θ-dependent vacuum energy, minimized at $\theta = 0$ (Vafa-Witten theorem)

- CP-violating pion-nucleon coupling, e.g., $\pi^0 \bar{N} N$ or $(\bar{N}\tau^i N)\pi^i$ w/ τ^i an SU(2) (global) isospin generator.

- Neutron electric dipole moment



A. Hook, TASI

$$-i\,\frac{d_n}{2}\,\bar{N}\sigma_{\mu\nu}\gamma^5 N F^{\mu\nu}.$$

exp: $|d_n| \lesssim 10^{-26}$ e·cm.

Expect: $d_n \sim 10^{-16}\,\theta_{QCD}$ e·cm.

$$\Rightarrow \quad |\theta_{QCD}| \lesssim 10^{-10}. \quad \text{Why so small?}$$

Only symmetry it violates is CP (equiv, T), but we know CP is ~maximally violated in the CKM matrix.

Basically 3 candidate explanations:

- massless up quark (renders θ_{QCD} unphysical)
  Ruled out ~~by~~ precision data + theory
- CP as spontaneously broken symmetry + careful model-building (Nelson-Barr; unfortunately no time...)
- AXIONS

Axion: core idea (Peccei–Quinn–Weinberg–Wilczek)

All the axion solution to Strong CP requires is a dynamical scalar field $\theta(x)$ that has an approx. continuous shift symmetry $\theta \mapsto \theta + \text{const.}$ (like a Nambu-Goldstone boson) only (to v. good approx.) broken by coupling to gluons:

$$S_{\text{axion}} = \int d^4x \, \tfrac{1}{2} f^2 (\partial\theta)^2 + \frac{N}{8\pi^2} \int \theta(x) \, \underbrace{\text{tr}(G \wedge G)}_{\text{QCD gauge fields}}$$

where we assume $\theta(x) \cong \theta(x) + 2\pi$ is a periodic scalar of period 2π. $\Rightarrow N \in \mathbb{Z}$.

This is a gauge symmetry, i.e., a redundancy in our description. $\partial_\mu \theta(x)$, $e^{in\theta(x)}$ $(n \in \mathbb{Z})$ are valid local operators. $\theta(x)$ is not.

Why does this solve strong CP?

Non-perturbative dynamics: θ acquires a potential.

- High energy, $E \gg \Lambda_{QCD}$: small instantons ($\rho \sim E^{-1}$) generate an effective potential $\sim E^4 e^{-\frac{8\pi^2}{g^2(E)}} \left[e^{iN\theta} + \text{c.c.} \right]$ (QCD scale is $e^{-8\pi^2/bg^2(E)}$ - suppressed w/ $b = \frac{11}{3} N_c - \frac{2}{3} N_f \gg 1$, so this is much smaller)
- Low-energy, $E \sim \Lambda_{QCD}$: need to match to the chiral Lagrangian.

axion: core idea

At first glance, not obvious how $\theta \, \mathrm{tr}(G \wedge G)$ affects interactions of pions, kaons, ... below the confinement scale. But there's a clever trick: we can use the inverse logic of chiral rotation to remove $\theta \, \mathrm{tr}(G \wedge G)$ and put it in a quark mass:

$$q_L \mapsto q_L e^{iN\theta(x)}$$

removes the $\frac{N}{8\pi^2} \theta(x) \, \mathrm{tr}(G \wedge \tilde{G})$ term,

produces a derivative coupling $\sim q_L^\dagger \bar{\sigma}^\mu q \, \partial_\mu \theta$ (not super important), and changes the quark mass

$$m \, q_L \bar{q}_R + \text{h.c.} \longrightarrow m e^{iN\theta} q_L \bar{q}_R + \text{h.c.}$$

We know how quark mass terms appear in the chiral Lagrangian:

$$\mu^3 \left[M U^\dagger + M^\dagger U \right].$$

Now M has $\theta(x)$ dependence in it! Result:

$$V(\theta) = -m_\pi^2 f_\pi^2 \sqrt{1 - \frac{4 m_u m_d}{(m_u + m_d)^2} \sin^2 \frac{N\theta}{2}}.$$

$\Rightarrow$ axion mass $\quad m_a = \frac{m_\pi f_\pi}{f/N} \sqrt{\frac{m_u m_d}{2(m_u + m_d)^2}} \approx 6\,\mu\mathrm{eV} \cdot \frac{10^{12}\,\mathrm{GeV}}{f/N}$

axion: core idea

$m_q \to 0$ limit: axion potential vanishes.

In this limit, field redef. on quark can remove $\theta \, \mathrm{tr}(G \wedge G)$ and produce only deriv. couplings of θ.

So constant part of θ has no physical effect when there is a massless quark.

Crewther, di Vecchia, Veneziano, Witten 1979

In fact this is important for the Strong CP problem!

$r_{neutron} \sim 10^{-13}$ cm, but EDM est $\sim \left(\frac{m_u}{\Lambda_{QCD}}\right) r_{neutr} \sim 10^{-16}$ cm.

See Srednicki's QFT textbook, section 94.

$$d_n = \frac{e\bar{\theta}\, g_A c_+ \tilde{m}}{8\pi^2 f_\pi^2} \ln \frac{\Lambda^2}{m_\pi^2} + (\text{not log enhanced})$$

where $\tilde{m} = \frac{m_u m_d}{m_u + m_d}$, g_A; c_+, coeff of diff pion-nucleon interaction terms.

$$-\frac{\theta c_+}{f_\pi} \tilde{m}\, \pi^a N_L T^a N_R^\dagger - i \frac{g_A m_N}{f_\pi} \pi^a N_L T^a N_R^\dagger$$