Light scalar fields as dark matter

What do we know about dark matter?

Abundance $\Omega_{DM} h^2 \approx 0.12$ (Planck CMB data)

where $\Omega_{DM} \equiv \frac{\rho_{DM}}{\rho_c}$, $\rho_c = 3 \underbrace{H_0^2}_{(\dot{a}/a)\ \text{today}} M_{Pl}^2$

and $h \equiv \frac{H_0}{100\, \text{km/s/Mpc}}$.

$H_0 \approx 67$ km/s/Mpc so $\Omega_{DM} \approx 0.27$

Compare ordinary matter, "baryons" $\Omega_b h^2 \approx 0.02$ so $\approx 4.5\%$.

[Rest of energy mostly "dark energy", at least approx. cosmological constant]

Perturbations The CMB has small temperature perturbations $\delta T/T \sim 10^{-5}$.

The DM perturbations must be correlated: the data is consistent w/ the so-called "adiabatic mode"

$$\frac{\delta\rho_i}{\dot{\rho}_i} = \frac{\delta\rho_j}{\dot{\rho}_j}$$ for diff species (photons, baryons, DM)

Specifically $$\frac{\delta\rho_{DM}}{\rho_{DM}} = \frac{3}{4}\frac{\delta\rho_\gamma}{\rho_\gamma}$$ for the modes visible in the CMB.

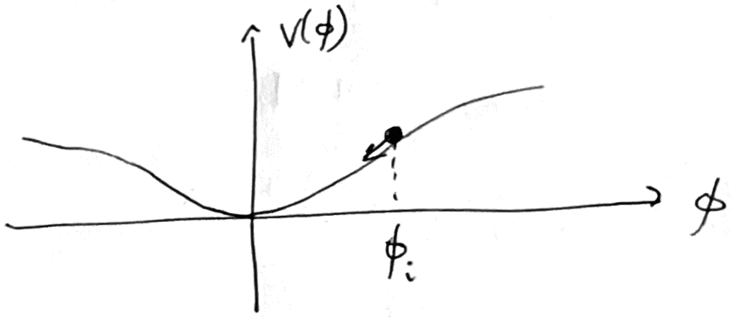
ratio $1 + \underbrace{w_{DM}}_{\text{eq of state, } p = w\rho} = 1$ to $1 + w_\gamma = 1 + 1/3 = 4/3$.

This relationship is automatic in thermal equilibrium, less obvious why it would be true for non-thermal DM candidates.

Main goal: explain abundance and perturbations for axion or ALP DM.

Misalignment production for scalar fields:

Assume a light scalar field in the early universe that interacts weakly with the SM.



The homogeneous part of the field begins at some initial value ϕ_i displaced from the minimum of the potential.
(Maybe random, maybe during inflation there was a large mass around ϕ_i).

How does $\phi(t)$ evolve?

Metric $ds^2 = -dt^2 + a(t)^2 dx^i dx^j \delta_{ij}$

$\Rightarrow$ equation of motion

$$\partial_t \left[a(t)^3 \partial_t \phi \right] + a(t)^3 \frac{\partial V}{\partial \phi} = 0,$$

ie,
$$\ddot{\phi}(t) + \underbrace{3H\dot{\phi}(t)} + \frac{\partial V}{\partial \phi} = 0.$$

"Hubble friction": like a damping term for a harmonic oscillator.

What does $\phi(t)$ do? If the potential is shallow, it sits still for a long time.

ex 1 If $V = \frac{1}{2} m^2 \phi^2$, $H = \frac{\gamma}{t}$ (e.g. $\gamma = 1/2$ in rad dom, $2/3$ in mat dom),
solution $\sim t^{\frac{1}{2}(1-3\gamma)} J_{\pm\frac{1}{2}(1-3\gamma)}(mt)$, ϕ essentially at rest until
$mt \sim 1 \Rightarrow H \sim m$.

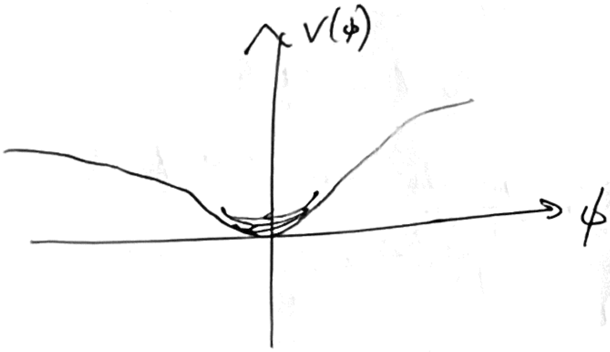
eg 2 $\quad V(\phi) = \Lambda^4 \underbrace{g(\phi/f)}_{\text{some } \mathcal{O}(1) \text{ fn, reasonably smooth}}$, f a characteristic field range so $\phi_i \sim f$

Then $$\phi(t) - \phi_i \approx -\frac{1}{2(1+3\gamma)} \frac{\Lambda^4 g'(\phi_i/f)}{f} t^2 = \mathcal{O}(f)$$

when $t \sim \frac{f}{\Lambda^2}$, which again is $\sim \frac{1}{m}$

So, as a rule, we expect scalar fields to stay put until $H \sim m$.

At this point, the scalar field rolls down its potential and begins to oscillate around the minimum, gradually redshifting.



This is again described by the Bessel fn. solution

$$\phi(t) \propto \frac{1}{t^{(3\gamma-1)/2}} J_{(3\gamma-1)/2}\big(m(t-t_0)\big)$$

For large arguments $\quad J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$

so the amplitude redshifts like $\quad |\phi(t)| \sim \frac{1}{t^{3\gamma/2}} \sim \frac{1}{a(t)^{3/2}}$.

The stored energy density then redshifts as $\frac{1}{a(t)^3}$.

This is just how the energy density of a collection of massive particles redshifts! It dilutes w/ volume.

Indeed, a <u>coherently oscillating scalar field</u> essentially <u>is</u> <u>a collection of massive particles at rest</u>.

This "misalignment mechanism" thus produces <u>intrinsically non-relativistic</u>, <u>non-thermal</u> particles that could account for dark matter.

How do we calculate the abundance of dark matter produced in this way?

Let's do the calculation first for a scalar with a simple potential $\frac{1}{2}m^2\phi^2$, and then for a QCD axion which has a temperature-dependent potential $V(\phi, T)$.

The main fact you need for this is that for the Standard Model plasma, the entropy density $s = \frac{2\pi^2}{45}\underbrace{g_{*s}(T)}_{\text{a count of degrees of freedom}}T^3$ redshifts like $1/a(t)^3$, just the same way as non-interacting dark matter! We know

$$\frac{\rho_{DM}}{s} = \frac{m_{DM}n_{DM}}{s} \quad \text{at the time of the CMB, from data,}$$

so we just need to compare to ρ_{DM}/s at the time that ϕ oscillations start. (Turns out $\rho_{DM}/s\big|_{t_{CMB}} \approx 4\times10^{-10}$ GeV).

Oscillations begin at $3H \approx m$, and in a radiation-dominated phase $3H^2M_{Pl}^2 = \frac{\pi^2}{30}g_*(T_{osc})T_{osc}^4$. So: solve for T_{osc} in terms of m, plug in to get $s(T_{osc})$, compare to $\rho_{DM} = \frac{1}{2}m^2\phi(T_{osc})^2$.

(If ϕ begins high enough up that potential isn't quadratic, need to correct this).

Putting everything together: (homework!)

$$\Omega_{ALP}h^2 \approx 0.12\left(\frac{\phi_i}{4.7\times10^{16}\,\text{GeV}}\right)^2\left(\frac{m_\phi}{10^{-21}\,\text{eV}}\right)^{1/2}.$$

We see that fields with a tiny mass and a large (but still sub-Planckian) field range can naturally be dark matter.

The regime $m_\phi \sim 10^{-21}$ eV where the de Broglie wavelength of DM is of order the size of a small galaxy is referred to as "Fuzzy Dark Matter" (Hu, Barkana, Gruzinov 2000)

What changes for the QCD axion?

- We know the full shape of $V(\theta)$, which is not quadratic (and not quite a cosine). This matters only if $\theta_0 \gtrsim 2$. ("Anharmonicity corrections")
- $V(\theta)$ has a steep temperature dependence for $T \gtrsim 1\,\text{GeV}$. This is important. Suppose, e.g., $m_\phi \sim 1\,\mu\text{eV}$. Naively, oscillation begin at $3H \simeq 1\,\mu\text{eV} \Rightarrow T_{osc} \approx \left(\frac{10}{\pi^2}\right)^{1/4} \frac{\sqrt{m_\phi M_{pl}}}{g_*(T_{osc})^{1/4}}$
($g_* \approx 61.75$ between QCD + EWK scales)
$\approx 17\,\text{GeV}$.

But $17\,\text{GeV} \gg \Lambda_{QCD}$, and the axion potential is much smaller above the QCD phase transition.

Implication: the QCD axion starts to oscillate later than an ALP would, so its initial energy is less diluted and it gives rise to a larger DM abundance.

So we repeat the calc., but oscillations start at $H \sim m_a(T)$, where a decent first approximation is
$$m_a(T)^2 \sim \begin{cases} m_a^2, & T < T_{QCD} \\ m_a^2\,(T/T_{QCD})^{-b}, & T > T_{QCD} \end{cases}$$

and $b \simeq 8.16$ (lattice result: Borsanyi et al., 1606.07494, exponent in approx. agreement with dilute instanton gas approximation).
Omitting the details (a good exercise!) one finds $\Omega_a h^2 \propto \theta_i^2\, m_a^{-\left(\frac{b+6}{b+4}\right)}$,
and quantitatively
$$\Omega_a h^2 \approx 0.12\ \theta_i^2 \left(\frac{f_a}{9.0\times 10^{11}\,\text{GeV}}\right)^{1.165}$$

For $\theta_i \sim 1$, the right DM abundance is achieved for $f_a \sim 10^{12}\,\text{GeV}$ or, equivalently, $m_a \sim 6\,\mu\text{eV}$. Heavier axions tend to be too little dark matter and lighter axions tend to be too much dark matter. Why? All store $\sim m_\pi^2 f_\pi^2$ of energy density initially, but $T_{osc} \sim m_a^{\frac{2}{b+4}}$ so lighter axions start oscillating later and thus dilute less.

At larger f_a, can get correct DM abundance for QCD axion by tuning θ_i small or having dilution: a particle Φ dominating energy density & decaying + T_{RH} between T_{BBN} and T_{osc} will dilute DM abundance by $\sim T_{RH}/m_\Phi$.
(e.g.: moduli field, mass ~ 10s of TeV and $\frac{1}{M_{pl}}$ couplings → $T_{RH} \sim T_{BBN}$)

This story applies most directly to the "pre-inflation" axion (or ALP), which was an independent field during inflation: no PQ symmetry restored later, no thermalization.

For a pre-inflation axion (or ALP) why are the density perturbations adiabatic?
At the time of the CMB, need $\delta\rho_{DM}/\rho_{DM} = \frac{3}{4}\,\delta\rho_\gamma/\rho_\gamma$. But axion never equilibrated w/ photons. Why are perturbations related?

This is explained by Weinberg's theorem regarding adiabatic perturbations.
There is a gauge invariant density perturbation $\mathcal{R}_k$ that is conserved on super horizon scales ($k/a \ll H$). There is an adiabatic solution in conformal Newtonian gauge where $ds^2 = -(1+2\Phi)dt^2 + a^2(t)(1-2\Phi)\,dx^i dx^j \delta_{ij}$:
define the quantity $\lambda(t) \equiv \frac{1}{a(t)} \int_{\mathcal{I}}^{t} a(t')\,dt'$.

Then
$$\Phi_k(t) = -\mathcal{R}_k \dot{\lambda}(t)$$
$$\delta\varphi_k(t) = -\mathcal{R}_k \lambda(t) \dot{\bar{\varphi}}(t)$$
for any scalar quantity φ (where $\bar{\varphi} = \varphi_0$ is background value)

is a solution on the scales $k/a \ll H$.

In particular, for a decoupled scalar axion w/ $\dot{\theta} = 0$ (as in misalignment at $t \ll m^{-1}$), the adiabatic solution is $\underline{\delta\theta_k = 0}$: no perturbations at all.
At late times, this remains adiabatic: it has $\delta\rho_\theta/\rho_\theta = \frac{3}{4}\,\delta\rho_\gamma/\rho_\gamma$.
Weinberg's theorem guarantees the initial adiabatic mode w/ no perturbations

become the later mode w/ perturbations prop. + photon perturbations,
once $\bar{\theta}$ starts + evolve in time

So the question is: why $\delta\theta_k = 0$ initially?

We don't expect zero, we expect dark matter isocurvature perturbations

$\delta\theta \sim \frac{H_I}{2\pi f_a}$, from 2-pt. fn. of θ in de Sitter space.

The observed adiabatic perturbations thus constrain H_I.

Planck: $\beta_{iso} = \frac{\text{DM isocurvature power}}{\text{total power}} < 0.038$. Denominator: $A_s \simeq 2\times10^{-9}$.

$$\Rightarrow \quad H_I < \left(\frac{f_a}{10^{12}\text{ GeV}}\right)^{0.4175} 2.6\times10^{7}\text{ GeV},$$

in scenario where θ_i tuned to get DM abundance.

This is a much stronger bound than we get from BICEP-Keck,

$H_I \lesssim 4.5\times10^{13}$ GeV.

So, pre-inflation axion $\Rightarrow$ relatively low H_I

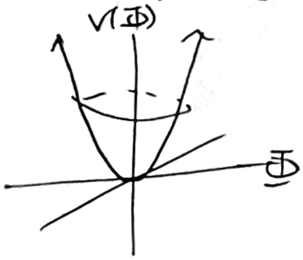
or m_a much bigger ($> \frac{3}{2}H_I$) during inflation

or f_a decreased after inflation.

All interesting options.

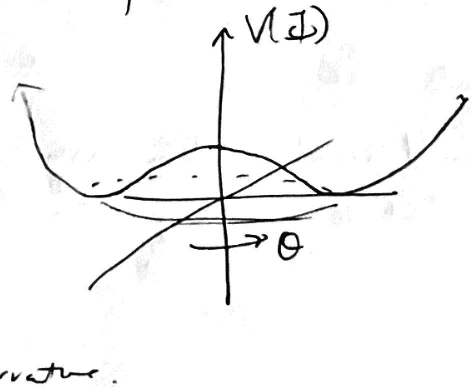
Post-inflation axion:

PQ symmetry restored after inflation. θ = phase of complex field Φ.



high-T behav.

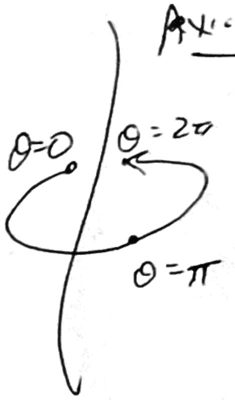
$\rightarrow$



low-T behavior

Thermal equil.: don't worry about isocurvature.

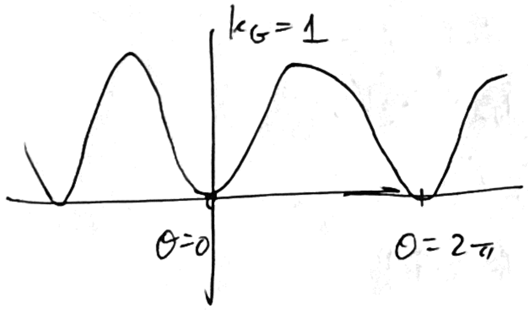
Do worry about topological defects.



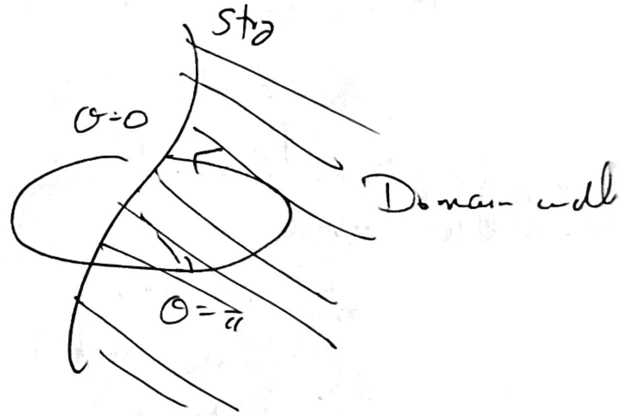
Axion string forms at time of PQ phase transition.

The string network radiates axions — these contribute significantly to axion dark matter (more than misalignment).

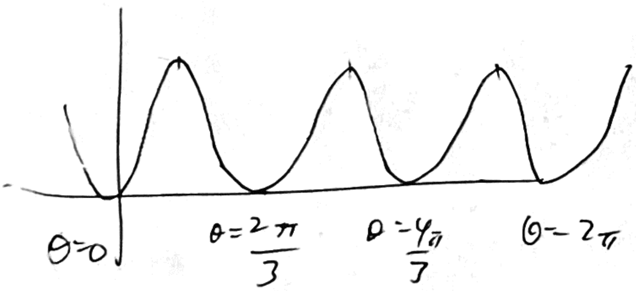
At the time of the QCD phase transition, the axion potential turns on, forming **domain walls** attached to axion strings.

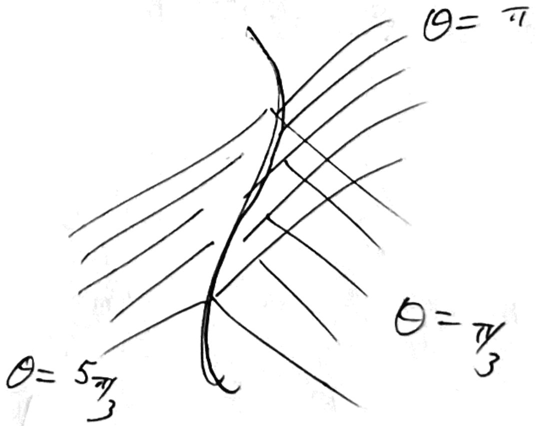


→ local max @ $\theta = \pi$:



$k_G > 1$:



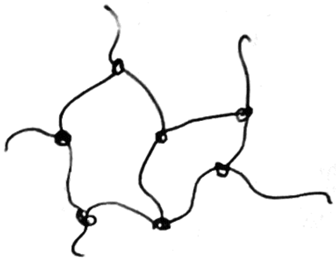


$|k_G|$ counts the "domain wall number": how many DWs attach to a string.

$|k_G| = 1$: DW/string network efficiently destroys itself

→ annihilate away. Often touted as predictive — calculate which m_a gives correct Ω_{DM}.

$|k_G| > 1$: DW/string network frustrated



→ cannot fully collapse.

DWs dominate energy density, cannot restore a conventional cosmology.

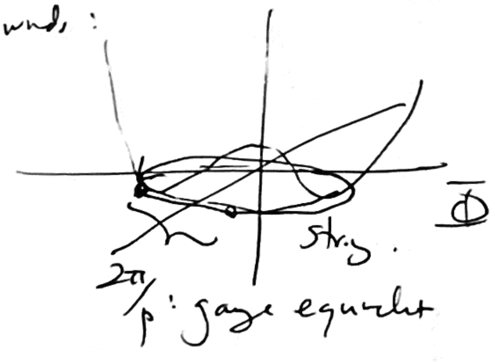
For a general ALP, could have symmetry-violating terms making the minima slightly non-degenerate.

For a QCD axion, this spoils the quality problem.

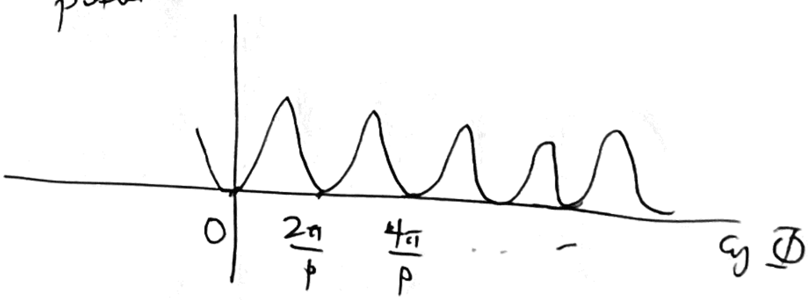
Even for $|k_G| = 1$, there is often a tension w/ the quality problem.

ex: try to forbid Φ^k w/ a $\mathbb{Z}_p$ symmetry $\Phi \mapsto e^{2\pi i/p}\Phi$.

Then the Kibble-Zurek mechanism forms strings when the phase of Φ winds:



potential:



so have DW number a multiple of p.

In principle,

string exists, but it isn't clear how to populate it cosmologically.

(Q. Lu, MR, Z. Sun '23)

A1

Backup: all of the relevant numbers, for the ALP relic abundance calculation.

ρ_{DM}'s today?

$$\rho_{DM} \approx 0.27 \cdot 3 H_0^2 M_{Pl}^2$$

$$M_{Pl} = 2.4 \times 10^{18}\ \mathrm{GeV}$$

$$H_0 \approx \frac{67 \frac{km}{s} \cdot \frac{1}{Mpc}}{3\times10^5\ km/s} \times \frac{1\ Mpc}{3.09\times10^{22}\ m} \times 0.197\times10^{-15}\ \mathrm{GeV}\cdot m$$

$$\approx 1.4\times10^{-42}\ \mathrm{GeV}.$$

$$\Rightarrow \rho_{DM} \approx 9.3\times10^{-48}\ \mathrm{GeV}^4.$$

Entropy density? Slightly tricky: photons, neutrinos, (some?) neutrinos non-relativistic

Back up to CMB:

$$\rho_{DM}(t_{CMB}) = \rho_{DM}(\text{today}) \cdot \frac{a(\text{today})^3}{a(t_{CMB})^3} = \rho_{DM}(\text{today}) \cdot (1+z_{CMB})^3$$

$z_{CMB} \approx 1100.$ so $\rho_{DM}(t_{CMB}) \approx 1.2\times10^{-38}\ \mathrm{GeV}^4.$

Photon temperature today:

$$T_{CMB} \approx 2.7K \approx 2.7K \times \frac{8.6\times10^{-5}\ eV}{K} \approx 2.3\times10^{-4}\ eV.$$

at the time of the CMB,

$$T_\gamma(t_{CMB}) = T_{CMB}(1+z_{CMB}) \approx 0.258\ eV \approx 2.6\times10^{-10}\ \mathrm{GeV}.$$

entropy contained in photons:

$$s_\gamma = \frac{2\pi^2}{45} \times \underset{\uparrow\ g_* \text{ for photons}}{2} \times T_\gamma(t_{CMB})^3 \approx 1.5\times10^{-29}\ \mathrm{GeV}^3.$$

> Recall: $e^+e^- \to \gamma\gamma$ after neutrinos decoupling reduces g_* from $2 + \underset{\text{Dirac!}}{4} \times \frac{7}{8} = \frac{11}{2}$ to $g_* = 2$, heating the photons by a factor $(11/4)^{1/3}$ by entropy conservation.

But then we should account for neutrinos at temperature

$$T_\nu(t_{CMB}) = T_\gamma(t_{CMB}) \times \left(\frac{4}{11}\right)^{1/3} \approx 1.9\times10^{-10}\ \mathrm{GeV}$$

and entropy

$$s_\nu(t_{CMB}) = \frac{2\pi^2}{45} \times (\underset{\text{d.o.f. (flavor)}}{3} \times \underset{\text{d.o.f. (spin)}}{2} \times \underset{\text{fermions}}{\tfrac{7}{8}})\, T_\nu(t_{CMB})^3 \approx 1.5\times10^{-29}\ \mathrm{GeV}^3$$

so $s(t_{CMB}) = s_\gamma(t_{CMB}) + s_\nu(t_{CMB}) \approx 3\times10^{-29}\ \mathrm{GeV}^3$

and the

$$\left.\frac{\rho_{DM}}{s}\right|_{t_{CMB}} \approx 4.0\times 10^{-10}\,\mathrm{GeV}.$$

Now we compare this to oscillations, for very light ALPs still at a time with photons and neutrinos, so

$$s(t_{osc}) \approx \frac{2\pi^2}{45}\Big(\underbrace{2+3\times 2\times\tfrac{7}{8}\times\left(\tfrac{4}{11}\right)}_{3.9}\Big)T_{osc}^3$$

and

$$\rho_{osc} \equiv 3H^2 M_{pl}^2 = \frac{1}{3}m_\phi^2 M_{pl}^2 = \frac{\pi^2}{30}\times\Big[\underbrace{2+3\times 2\times\tfrac{7}{8}\times\left(\tfrac{4}{11}\right)^{4/3}}_{3.36}\Big]T_{osc}^4$$

with

$$\rho_{DM}^{(t_{osc})} = \frac{1}{2}m_\phi^2\phi_i^2 = \frac{3}{2}\cdot\rho_{osc}\left(\frac{\phi_i}{M_{pl}}\right)^2 = 1.66\,T_{osc}^4\left(\frac{\phi_i}{M_{pl}}\right)^2,$$

$$\to\quad T_{osc} \approx 0.74\sqrt{m_\phi M_{pl}}$$

$$\frac{\rho_{DM}(t_{osc})}{s(t_{osc})} \approx 0.97\,T_{osc}\left(\frac{\phi_i}{M_{pl}}\right)^2 \simeq 0.72\sqrt{m_\phi M_{pl}}\left(\frac{\phi_i}{M_{pl}}\right)^2$$

$$\simeq 4.3\times 10^{-10}\,\mathrm{GeV}\sqrt{\frac{m_\phi}{10^{-21}\,\mathrm{eV}}}\left(\frac{\phi_i}{4.7\times 10^{16}\,\mathrm{GeV}}\right)^2$$