Third Post-Newtonian dynamics of inspiralling compact binaries in the Effective Field Theory approach

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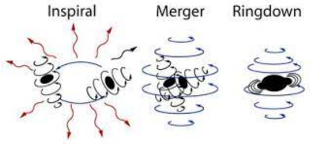
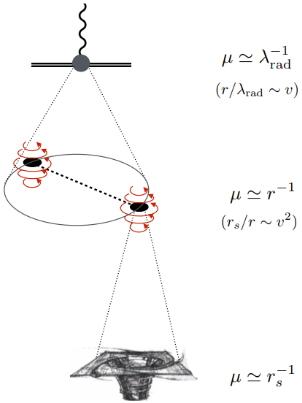
The Inspiral Phase of a Binary System



Picture from https://lisa.nasa.gov/ representing the three phases of a black hole collision. After an initial **inspiral**, the merger follows. A new asymmetric black hole is formed.



Picture from: [1601.04914]. During the so-called inspiral phase, the dynamics of the binary problem can be separated into different parts:

- The internal zone: This is the scale of finite size effects. For compact neutron stars or black holes we have $r_s \simeq 2\,Gm$.
- The near (or potential) zone: The intermediate region is the orbit scale, r, given by the typical separation between the constituents of the binary.
- The far (or radiation) zone: This is the scale of gravitational waves, emitted with typical wavelength $\lambda_{\text{rad}} \sim r/v$.

Effective Field Theory Setup

Slow inspiral phase ($v \ll 1$), hierarchical structure (Goldberger, W. D., and Rothstein, I. [0409.156]):

$$r_s \ll r \ll \lambda_{\mathsf{GW}} \,,$$

further constrained by the virial theorem $v^2 \sim GM/r$.

Relevant degrees of freedom:

- The gravitational field $g_{\alpha\beta}(x)$.
- The black hole's worldline coordinate $x^\alpha(\lambda)$, with λ an affine parameter.

Relevant symmetries of the problem:

- General coordinate and wordline reparametrization invariance: $x^\alpha \to \tilde{x}^\alpha(x)$ and $\lambda \to \tilde{\lambda}(\lambda)$.
- $\mathrm{SO}(3)$ invariance *i.e.* compact objects are perfectly spherical.

Expansion around Minkwoski + "background field" method:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \frac{h_{\mu\nu}(x)}{m_{Pl}} = \eta_{\mu\nu} + \frac{\bar{h}_{\mu\nu}(x)}{m_{Pl}} + \frac{H_{\mu\nu}(x)}{m_{Pl}} \,,$$

with the following scaling rules for potential and radiation modes:

$$\partial_0 H_{\mu\nu} \sim \left(\frac{v}{r}\right) \,, \quad \partial_i H_{\mu\nu} \sim \left(\frac{1}{r}\right) \,, \quad \partial_\alpha \bar{h}_{\mu\nu} \sim \left(\frac{v}{r}\right) \,.$$

$$\hbar = 1, \quad c = 1, \quad m_{Pl} = 1/\sqrt{32\pi G}, \quad \eta_{\mu\nu} = \mathrm{diag}(+,-,-,-).$$

Fully Diagrammatic Approach

"Integrating-out" the shortest scale r_s, the action reads:

$$S = S_{EH} + S_{GF} + S_{pp} \, ,$$

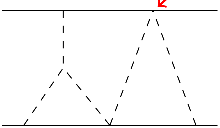
with $S_{EH} = -2m_{Pl}^2 \int \mathrm{d}^4x \sqrt{-g}\, g_{\mu\nu} R^{\mu\nu}$ and $S_{pp} = -\sum_{a=1,2} m_a \int \mathrm{d}\tau_a$.

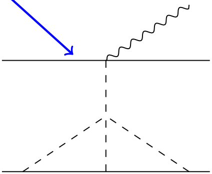
The NRGR (Non-Relativistic-General-Relativity) effective action yields:

$$e^{i\, S_{\mathsf{NRGR}}[x_a, \bar{h}]} = \int DH_{\mu\nu}\, e^{i\, S_{EH}[\bar{h}+H] + i\, S_{GF}[\bar{h}+H] + i\, S_{pp}[x_a, \bar{h}+H]} \, .$$

Expanding in the radiation field:

$$S_{\mathsf{NRGR}}\left[x_a, \bar{h}\right] = S_0\left[x_a\right] + S_1\left[x_a, \bar{h}\right] + \mathcal{O}\left(\bar{h}^2\right) \, .$$





Extracting Feynman Rules

We are now able to compute to some (**Post-Newtonian**) order in the $v \ll 1$ expansion the quantities

- from $S_0 \to$ mechanical energy of binary
- from $S_1 \to$ power emitted in gravitational waves (via optical theorem)

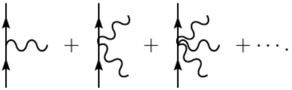
Solving the energy conservation equation $dE/dt = -P$

- we obtain a differential equation for $v(t)$ (or, equivalently, for $\omega(t)$ and thus $\phi(t)$).
- the phase is needed by current GW-detectors, since the typical signal is decomposed as $h(t) = A(t)\cos(\phi(t))$

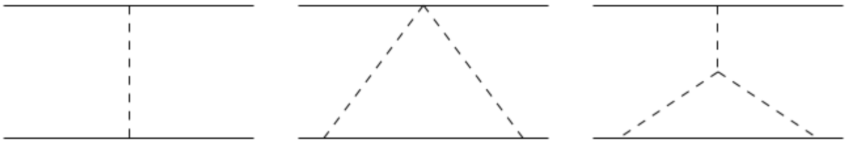
From the Einstein-Hilbert term we can extract graviton self-interactions with Feynman vertices containing any number of graviton lines:

$$-2m_{Pl}^2 \int d^4x \sqrt{g} R(x) \quad \to \quad \int d^4x \left[(\partial h)^2 + \frac{h(\partial h)^2}{m_{Pl}} + \frac{h^2(\partial h)^2}{m_{Pl}^2} + \cdots \right],$$

From the point-particle term, we can extract the non-linear interactions between the gravitational field and the two "particles" (recalling $d\tau = \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$)



Example: Next-to Leading Order Lagrangian



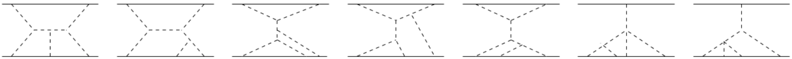
The leading order Lagrangian is derived from the single-graviton exchange diagram:

$$\mathcal{L}_{LO} = \frac{1}{2} m_a \sum_a v_a^2 + \frac{G_N m_1 m_2}{r} \, ,$$

The Next-to Leading Order Lagrangian (also called Einstein-Infeld-Hoffman Lagrangian), comes from the "seagull" and three-graviton vertex diagrams (among others):

$$\mathcal{L}_{NLO} = \frac{1}{8} \sum_a m_a v_a^4 + \frac{G_N m_1 m_2}{2\, r} \left\{ 3 \left(v_1^2 + v_2^2 \right) - 7 \left(\mathbf{v}_1 \cdot \mathbf{v}_2 \right) - \frac{\left(\mathbf{v}_1 \cdot \mathbf{r} \right) \left(\mathbf{v}_2 \cdot \mathbf{r} \right)}{r^2} \right\}$$
$$- \frac{G_N^2 m_1 m_2 (m_1 + m_2)}{2\, r^2} \, .$$

Conservative Sector @3PN



Applying the Legendre transformation to the effective Lagrangian (64 diagrams in total!):

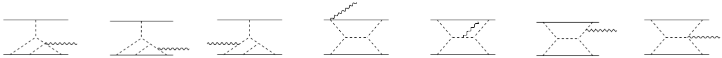
$$\mathcal{H} = \sum_{a=1,2} v_a^i \left(\frac{\partial \mathcal{L}}{\partial v_a^i}\right) + a_a^i \left(\frac{\partial \mathcal{L}}{\partial a_a^i}\right) + \dot{a}_a^i \left(\frac{\partial \mathcal{L}}{\partial \dot{a}_a^i}\right) - v_a^i \left(\partial_t \frac{\partial \mathcal{L}}{\partial a_a^i}\right) - a_a^i \left(\partial_t \frac{\partial \mathcal{L}}{\partial \dot{a}_a^i}\right) + v_a^i \left(\partial_t^2 \frac{\partial \mathcal{L}}{\partial \dot{a}_a^i}\right) - \mathcal{L}\left(\mathbf{r}, \mathbf{v}_1, \mathbf{a}_1, \dot{\mathbf{a}}_1, \mathbf{v}_2, \mathbf{a}_2, \dot{\mathbf{a}}_2\right) .$$

Binding energy for circular orbits ($x \equiv (GM\omega)^{2/3}$):

$$E_{N^3LO}(x) = -\frac{M\nu x}{2}\left\{1 + \left(-\frac{3}{4} - \frac{\nu}{12}\right)x + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24}\right)x^2 + \left[-\frac{675}{64} + \left(\frac{34445}{576} - \frac{205\pi^2}{96}\right)\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3\right]x^3\right\} .$$

Our result agrees with *e.g.* Blanchet L. Living Rev. Rel. 17 [1310.1528].

Radiative Sector @3PN



Gravitational Flux for circular orbits ($x \equiv (GM\omega)^{2/3}$)

$$
\begin{aligned}
\mathcal{P}_{N^3LO}(x) = & \frac{32\nu^2}{5G} x^5 \Bigg\{ 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu \right) + 4\pi x^{3/2} \\
& + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2 \right) x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu \right) \pi x^{5/2} \\
& + \Bigg[\frac{6643739519}{69854400} - \frac{1712}{105}\gamma_E + \frac{16\pi^2}{3} + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2 \right) \nu \\
& - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3 - \frac{856}{105} \,\mathrm{Log}\,(16\,x) \Bigg] x^3 \Bigg\} .
\end{aligned}
$$

Once again, our result agrees with *e.g.* `Blanchet L. Living Rev. Rel. 17 [1310.1528]`.
Within "dimensional regularization", extensively used in the EFT approach, we set $d \equiv 3 - 2\epsilon$ and employ *e.g.*

$$
\Gamma(d-3) \simeq -\frac{1}{2\epsilon} - \gamma_E - \left(\gamma_E^2 + \frac{\pi^2}{6} \right) \epsilon + \mathcal{O}(\epsilon^2) .
$$

Conclusions

Available reviews for those interested in the topic:

- Goldberger, Walter D. "Les Houches lectures on effective field theories and gravitational radiation." arXiv preprint hep-ph/0701129 (2007).
- Porto, Rafael A. "The effective field theorist's approach to gravitational dynamics." Physics Reports 633 (2016): 1-104.

Work done throughout my PhD:

- Amalberti, Loris, François Larrouturou, and Zixin Yang. "Multipole expansion at the level of the action in d-dimensions." Physical Review D 109.10 (2024): 104027.
- Amalberti, Loris, Zixin Yang, and Rafael A. Porto. "Gravitational radiation from inspiralling compact binaries to N ^{3} LO in the Effective Field Theory approach." arXiv preprint arXiv:2406.03457 (2024).

How to advance the EFT state of the art within the Post-Newtonian framework:

- 4PN radiative dynamics for non-spinning inspiralling binaries
- 3PN radiative dynamics for spinning inspiralling binaries

Thank you for your attention!