References : 1. PhD thesis of Maria Christodoulou : AdS / CFT : Dictionary and Applications

2. AdS / CFT lecture note by Christopher Herzog

3. AdS / CFT & related topics lecture note by Dionysios Anninos

I. Motivation

1. holographic principle ⇒ AdS / CFT

The holographic principle states that the physics governing quantum gravity in a $d+1$ dimensional volume of spacetime is encoded in a quantum field theory without gravity defined on the d dimensional boundary. Each degree of freedom of the gravitational theory can be holographic projected to a degree of freedom on the boundary in such a way that the two theories are in fact describing the same physics.

The holographic principle is a deep and fundamental property of quantum gravity that emerged from observations about black holes dating back to the early 70s. In 1972, Bekenstein introduced the notion of black hole entropy [3] as a measure of inaccessibility of information about the interior of a black hole, in analogy to thermodynamic entropy which is a measure of our ignorance about the microscopic configurations of a system, when our knowledge is restricted to its macroscopic properties. Since the entropy of any system must be non–decreasing, he asserted that the black hole entropy is proportional to the area of its event horizon which had already been shown to be non–decreasing by Christodoulou and Hawking [4, 5, 6].

$$S_{BH} = \frac{c^3 A_{hor}}{4 \underset{\substack{\uparrow \\ \text{gravitational Newton constant}}}{G_N} \hbar} = \frac{A_{hor}}{4 L_{pl}^2}$$

About twenty years after Bekenstein's area law for black hole entropy, 't Hooft proposed a radical interpretation for it. Combining black hole thermodynamics with ideas from quantum mechanics he postulated that at Planckian length scales where quantum gravity takes over, the world is not 3+1 dimensional but instead the observable degrees of freedom live on a 2 dimensional surface that evolves in time [7]. Said differently, given a closed surface in spacetime enclosing a quantum gravitational system, all information contained in the interior of the surface can be holographically projected onto the surface. Moreover, the theory of quantum gravity governing the interior or bulk physics can be described by a gauge field theory on the boundary surface. This was not the first time a gauge field theory description was proposed for a theory of quantum gravity and vice versa. Klebanov and Susskind [8], and Thorn [9] discovered that string theory can be described by a 2+1 dimensional gauge theory. However, 't Hooft's result is much stronger as it states that *any* theory of quantum gravity must be holographic.

$\Rightarrow$ 1997 Maldacena

string / M theory on AdS × compact manifold (Bulk theory) $\Leftrightarrow$ lower dimensional gauge field theory (Boundary theory)

In its strongest form, the AdS/CFT correspondence can be stated as follows:

The operator content and Hilbert space of an ultraviolet complete theory of quantum gravity in a $(d+1)$-dimensional asymptotically anti-de Sitter spacetime is equivalent to that of a d-dimensional, unitary and local conformal field theory.

Example: type IIB string theory on $AdS_5 \times S^5$ $\Leftrightarrow$ 4d $\mathcal{N}=4$ SYM with gauge group $SU(N)$

$\left(\frac{L}{l_{Pl}}\right)^4 = N$ (AdS radius in Planck units) $\quad g_{string} = g_{YM}^2 \quad \left(\frac{L}{l_s}\right)^4 = g_{YM}^2 N = \lambda \leftarrow$ 't Hooft parameter (AdS radius in string units)

· strong / weak duality

For example, in the large N limit of gauge theory at large 't Hooft coupling

$$\left(\frac{R}{l_{\rm Pl}}\right)^4 = N \qquad g_{\rm string} = g_{\rm YM}^2 \qquad \left(\frac{R}{l_{\rm s}}\right)^4 = g_{\rm YM}^2 N = \lambda$$

large, small, large

supergravity (point particle) approximation is good for AdS description. This is interesting since it gives insights into strongly coupled gauge theories using supergravity methods.

The **tensionless regime** arises in another corner of parameter space where the gauge theory is weakly coupled

$$\left(\frac{R}{l_{\rm Pl}}\right)^4 = N \qquad g_{\rm string} = g_{\rm YM}^2 \qquad \left(\frac{R}{l_{\rm s}}\right)^4 = g_{\rm YM}^2 N = \lambda$$

large, small

$l_s \to \infty$ `tensionless strings' [Sundborg '01] [Witten '01] [Sezgin,Sundell '01]

· scale / radius correspondence

Scale/Radius Correspondence

The identification of the diffeomorphisms of AdS with the conformal symmetry of the boundary suggests that the extra dimension of the bulk, namely the holographic or radial direction, is related to the energy scale of the field theory. In particular, studying the radial evolution of the bulk field equations tells us something about the renormalisation group (RG) flow of the dual operators in the field theory. To illustrate the relation between the bulk holographic direction and the field theory energy scale we consider AdS_{d+1} in Poincaré coordinates in which the metric takes the form

$$\mathrm{d}s^2 = \frac{\ell^2}{z^2}\left(\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}z^2\right). \tag{1.2.1}$$

z is the holographic direction and ℓ is the AdS radius. Constant z hypersurfaces are parametrised by x^μ, $\mu = 0, \dots d$, and their topology is $\mathbb{R}^{1,d-1}$. According to the AdS/CFT dictionary, the field theory metric is given by the asymptotic limit of the bulk metric, up to conformal transformations. Hence, in this case, it can be taken to be the Minkowski metric parametrised by the same coordinates x^μ.

The field theory is invariant under rigid scale transformations $x^\mu \to \alpha x^\mu$ which rescale the energies of particles according to $E \to E/\alpha$. In the bulk, this transformation corresponds to the diffeomorphism $x^\mu \to \alpha x^\mu$, $z \to \alpha z$. This leads to the identifications of the extra bulk dimension with the inverse energy scale of the gauge theory, $z \sim 1/E$, giving rise to a scale/radius or UV/IR duality. High energies or equivalently short distances on the field theory side translate in the bulk to large radii, that is, to moving closer to the boundary. Another way of understanding this duality is to say that, as we move a bulk excitation closer to the boundary of AdS, it localises in the field theory, i.e. the wavelength of the field theory excitation becomes smaller and its energy larger. Conversely, moving the excitation towards the interior of AdS smears the boundary excitation over a larger area.

2. Generalization ⇒ gauge / gravity duality

In the prototypical example of the correspondence the field theory is highly symmetric, making it unrealistic for real world applications. This, however, is not an issue as the duality can be extended to less symmetric more realistic setups such as field theories with some or all supersymmetries and/or the conformal symmetry broken. For example, perturbing the field theory Lagrangian by a relevant operator can cause the theory flow to less symmetric theories. Moreover, the bulk spacetime need not be AdS but only asymptotically (locally) AdS (AAdS or AlAdS). Such generalisations of AdS/CFT are referred to as gauge/gravity dualities, although the term AdS/CFT is also used to refer to them, and they allow for a wide range of applications of the duality.

3. Two routes to obtain the dual theories

· Top down

and it involves starting from string theory and M–theory and studying the low energy dynamics of brane configurations, in analogy to what Maldacena did. This method is quite involved and in principle it provides the dual field theory but there is no control over what this theory is. In other words, this method will provide the field theory Lagrangian which is fixed by the string theory configuration one considers.

· Bottom up

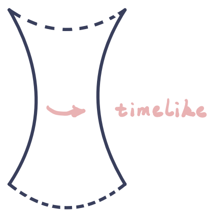
alternative, known as the bottom up approach, bypasses the high energy physics and it involves postulating a gravitational theory on an asymptotically AdS spacetime which contains supergravity fields dual to a desire set of field theory operators. The choice of "desired" operators depends on field theory system being modelled which could be for example a strongly coupled condensed matter system. The field theory is not known in this case and one only knows of the elements they placed in the theory by hand. This approach makes use of the *AdS/CFT dictionary*, the map that relates objects and features of the bulk theory to objects and features of the boundary field theory. For example, the bulk theory necessarily contains the gravitational field which, according to the AdS/CFT dictionary, sources the field theory stress energy tensor. In addition, one may want to have symmetry currents and operators of various dimensions in the field theory which requires turning on gauge fields and matter fields in the bulk. Moreover, one may want to study the field theory at finite temperature. In the bulk this translates to considering black hole solutions in AdS. Once the building blocks of the field theory under consideration have been placed in the bulk, one can compute its observables and study its properties by performing the corresponding bulk computations. In fact, in the bottom up approach there is no specific Lagrangian for the field theory and the only way to study it is through the bulk. The work presented in this thesis is an application of the

II. The AdS / CFT dictionary

1. Global symmetries

· Definition of AdS_d :

the AdS of signature of $(1, d-1)$ can be isometrically embedded in the space $\mathbb{R}^{d-1,2}$ with coordinates $(x_1, \cdots, x_{d-1}, t_1, t_2)$ and the metric



$$ds^2 = \sum_{i=1}^{d-1} dx_i^2 - \sum_{j=1}^{2} dt_j^2$$

with $\sum_{i=1}^{d-1} x_i^2 - \sum_{j=1}^{2} t_j^2 = -L^2$ the radius

solution of Einstein field eqns with negative cosmological constant $\Lambda = \frac{-(d-1)(d-2)}{2L^2}$

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

· global coordinates (cover all AdS)

$$t_1 = L \cosh\rho \cos\tau$$

$$t_2 = L \cosh\rho \sin\tau \qquad \tau \in [0, 2\pi) \qquad \rho \in \mathbb{R}^+$$

$$x_i = L \sinh\rho \, \hat{x}_i \qquad \underline{\sum_i \hat{x}_i^2 = 1} \text{ parametrizes a } S^{d-1} \text{ sphere}$$

$$\Rightarrow ds^2 = L^2 (-\cosh^2\rho \, d\tau^2 + d\rho^2 + \sinh^2\rho \, d\Omega^2_{d-2}) \qquad \text{boundary } \rho \to \infty$$

· Poincaré coordinates (don't cover all AdS)

$$t_1 = \frac{L^2}{2r} \left(1 + \frac{r^2}{L^4} (L^2 + \vec{x}^2 - \hat{t}^2) \right)$$

$$t_2 = \frac{r}{L} \hat{t}$$

$$x_i = \frac{r}{L} \hat{x}_i \qquad i = 1, \cdots, d-2$$

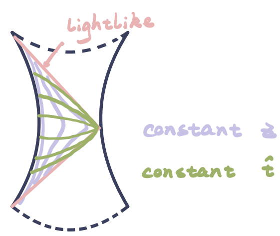
$$x_{d-1} = \frac{L^2}{2r} \left(1 - \frac{r^2}{L^4} (L^2 - \vec{x}^2 + \hat{t}^2) \right)$$

$$ds^2 = -\frac{r^2}{L^2} d\hat{t}^2 + \frac{L^2}{r^2} dr^2 + \frac{r^2}{L^2} d\vec{x}^2$$

$$r > 0$$

$$\Rightarrow u = \frac{r}{L^2} \qquad ds^2 = L^2 \left(\frac{du^2}{u^2} + u^2 dx_\mu dx^\mu \right) \qquad x^\mu = (\hat{t}, \vec{x})$$

$$\Rightarrow z = \frac{1}{u} \qquad ds^2 = \frac{L^2}{z^2} (dz^2 + dx_\mu dx^\mu)$$



boundary $z \to 0$

We begin by considering the global symmetries of the two theories. According to the AdS/CFT dictionary, gauge symmetries of the bulk theory are mapped to global symmetries of the boundary theory. In particular the isometries of the bulk are mapped to the global symmetry group of the boundary theory. For example, in the example of $\mathrm{AdS}_5 \times S^5$ the isometry group of AdS_5 is SO(4,2) and of the S^5 SO(6). In the $\mathcal{N} = 4$ SYM we encounter the same symmetry groups. The SO(4,2) is the conformal group in four dimensions and the SO(6)$\simeq$SU(4) is the group associated to the R–symmetry of the theory. Moreover, the two theories have the same number of supersymmetry generators.

2. Bulk Fields $\Leftrightarrow$ Boundary operators

In general, for every field $\Phi(x, z)$ that propagates in the bulk, there is a local, gauge invariant operator $\mathcal{O}(x)$. The boundary operator couples to the restriction of the bulk field on the boundary $\phi_{(0)}(x)$ via a term of the form $\int_{\mathcal{B}_d} \phi_{(0)}\mathcal{O}$ where $\mathcal{B}_d$ is the boundary manifold. Subleading terms in the asymptotic expansion of the bulk field are related to the expectation value of the field theory operator. Accordingly, the bulk field and operator must have the same Lorentz structure and quantum numbers.

$\Phi(x,z)$ in the bulk $\xleftrightarrow{\int_{B_d} \phi_{(0)} \mathcal{O}}$ $\mathcal{O}(x)$ in the boundary

- bulk scalar field $\Leftrightarrow$ scalar boundary field
- bulk dynamical gauge field $\Leftrightarrow$ conserved current associated with a boundary global sym.
- bulk metric $\Leftrightarrow$ boundary stress energy tensor
- bulk p-form $\Leftrightarrow$ boundary p-form
- bulk fermions $\Leftrightarrow$ boundary fermions

...

3. Observables

Explicit examples

- scalar 2-pt functions
- gauge fields in the bulk $\Leftrightarrow$ current in the boundary

3 How AdS/CFT works

The equivalence of $\mathcal{N} = 4$ SYM and type IIB string theory in $\mathrm{AdS}_5 \times S^5$ should imply an equality between their respective path integrals. On the the gauge theory side, we ought to be able to include gauge invariant sources in the path integral. On the gravity side, AdS_5 is a space with boundary (at $z = 0$ in the parametrization (25)). Thus to be well defined, we need to include boundary conditions. Consider probing the stack of D3-branes with wave packets sent in from the asymptotically flat part of the D3-brane geometry. From the near horizon point of view, these wave packets look alternately like sourcing gauge invariant operators on the D-branes or like setting boundary conditions for the AdS_5 space-time. This line of reasoning leads to the central postulate of the AdS/CFT correspondence, a result [2, 3] whose importance can not be over emphasized:

$$e^{W_{\mathrm{CFT}}[\phi_0]} \equiv \left\langle \exp \int \mathrm{d}^4x \, \phi_0(x) \mathcal{O}(x) \right\rangle_{\mathrm{CFT}} = Z_{\mathrm{string}} \left[\phi(x,z)|_{z\to 0} = \phi_0(x) \right] . \tag{33}$$

In this expression $\phi(x,z)$ is a field on the string side of the story. Its boundary value $\phi_0(x)$ can alternately be interpreted as a source for a gauge invariant operator $\mathcal{O}(x)$ in the conformal field theory. The CFT quantity W_{CFT} is then a generating functional for connected correlation functions of $\mathcal{O}(x)$ in the CFT.

While the correspondence (33) is expected to hold true in general, we will be interested in it primarily in the limit g_s and $\ell_s/L \to 0$. In field theory terms, this double limit is $g_{\mathrm{YM}}^2 N$ and $N \to \infty$. Given that we are working in AdS_5 with a scale set by the radius of curvature L, we can first replace Z_{string} by the corresponding supergravity partition function Z_{SUGRA}. Then the effective gravitational coupling constant in the supergravity action (8) we can identify as

$$\frac{(2\pi)^7 g_s^2 \ell_s^8}{L^8} = \frac{8\pi^2}{N^2} \, \mathrm{Vol}(S^5) . \tag{34}$$

Because N is large, a saddle point approximation of $Z_{\mathrm{SUGRA}} \sim e^{-S_{\mathrm{os}}}$ becomes accurate. (We work in Euclidean signature here.) In other words, the on-shell gravitational action $S_{\mathrm{os}}[\phi_0]$ (i.e. the action evaluated using the equations of motion) is a good approximation to the generating functional, $W_{\mathrm{CFT}}[\phi_0] \approx -S_{\mathrm{os}}[\phi_0]$. We can therefore use classical gravity to compute connected correlation functions in the CFT in the limit $N \to \infty$.

We would like to explore the consequences of the postulate (33) for a free scalar field. Consider then the action for a real scalar in the Poincaré patch of (Euclidean) AdS_{d+1}:

$$S = \frac{1}{2} \int \mathrm{d}^{d+1}x \, \sqrt{-g} \left[(\partial\phi)^2 + m^2 \phi^2 \right] . \tag{35}$$

We focus here just on the AdS_{d+1} geometry and use the line element

$$\mathrm{d}s^2 = L^2 \left[\frac{\delta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}z^2}{z^2} \right] . \tag{36}$$

To produce a generating function for the CFT correlation functions, we need to evaluate this action on-shell with a prescribed boundary condition for ϕ at $z = 0$. To that end, let us start with the equation of motion for ϕ:

$$\left(z^{d+1}\partial_z z^{-d+1}\partial_z + z^2\eta^{\mu\nu}\partial_\mu\partial_\nu - m^2L^2\right)\phi = 0 \ . \tag{37}$$

Typically boundary conditions for second order differential equations are either Dirichlet or Neumann. Here, however, $z=0$ is a singular point, and the boundary behavior is described instead by two characteristic exponents which satisfy the following indicial equation:

$$\Delta(\Delta - d) = m^2L^2 \ , \tag{38}$$

as can be seen by plugging $\phi \sim z^\Delta$ into the equation of motion (37) and expanding the result near $z=0$. Generically, one finds the following behavior for ϕ near $z=0$:

$$\phi = az^{d-\Delta}(1+O(z^2)) + bz^\Delta(1+O(z^2)) \ . \tag{39}$$

(Interesting issues arise when Δ is an integer and the series overlap. Extra logarithmic terms appear which we shall ignore.) If we assume $\Delta > d/2$, then a describes the leading small z behavior and we can tentatively identify $a = \phi_0$ with the source term in the CFT. The singular behavior at $z=0$ means we should really work with a $z=\epsilon$ cutoff and modify the basic statement (33) to include an ϵ dependence, $\phi|_{z=\epsilon} = \phi_0\epsilon^{d-\Delta}$, taking the $\epsilon \to 0$ limit only at the end.

Given that the boundary $z=0$ is a singular point and we cannot use typical Dirichlet or Neumann boundary conditions, it is not obvious that the action (35) has a well defined variational principle. In varying the action, we are left with the following boundary term

$$\begin{aligned} \delta S &= -\int_{z=\epsilon} \mathrm{d}^dx \left(\frac{L}{z}\right)^{d-1} \delta\phi(x,z)\partial_z\phi(x,z) \\ &= -L^{d-1}\int_{z=\epsilon}\mathrm{d}^dx\,\frac{1}{z^d}(\delta a\, z^{d-\Delta} + \delta b\, z^\Delta + \ldots)(a(d-\Delta)\, z^{d-\Delta} + b\Delta\, z^\Delta + \ldots) \\ &= -L^{d-1}\int_{z=\epsilon}\mathrm{d}^dx\left[(d-\Delta)a\,\delta a\, z^{d-2\Delta} + (\Delta\,\delta a\, b + (d-\Delta)\,\delta b\, a)\right. \\ &\qquad\qquad \left. +\Delta\, b\,\delta b\, z^{2\Delta-d} + \ldots\right] \ . \end{aligned} \tag{40}$$

There are really three potentially overlapping power series in the last line. The boundary variation (40) includes only the leading term in each power series; the ellipses denote the subleading terms. In the context of the variational principle, we fix the boundary behavior $a = \phi_0$. Thus, we insist that $\delta a = 0$. There remains a term proportional to $\delta b\, a$ which we need to cancel through the addition of a boundary term. (The $b\,\delta b$ term will vanish given our assumption that $2\Delta > d$.) We add

$$S_{\text{bry}} = \frac{c}{L}\int_{z=\epsilon}\mathrm{d}^dx\sqrt{-\gamma}\,\phi^2(x,z) \tag{41}$$

where $\gamma_{\mu\nu}$ is the induced metric on the $z=\epsilon$ slice of the geometry, and c is a constant to be determined. The choice of counter-terms is guided by the requirements that S_{bry} be local, Lorentz invariant, and depend only intrinsically on the geometry of the boundary. One could imagine also terms of the form $\phi\Box\phi$ and $\phi\Box^2\phi$ where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ or even, in the case of a curved boundary,

$R\phi^2$ where R is the Ricci scalar curvature of the boundary. By dimensional analysis, these higher derivative terms must come with additional powers of z and cannot cancel the leading $a\,\delta b$ term.

Given the boundary term (41), the variation is then

$$\delta S_{\text{bry}} = 2cL^{d-1} \int_{z=\epsilon} \mathrm{d}^d x \, \left[a\,\delta a\, z^{d-2\Delta} + (a\,\delta b + b\,\delta a) + b\,\delta b\, z^{2\Delta-d} + \ldots \right] \; , \tag{42}$$

To cancel the $a\,\delta b$ term in (40), we should set the constant $c = (d-\Delta)/2$.

Having ensured that the on-shell value of the action is indeed an extremum, and thus that the saddle-point approximation is sensible, we can ask what the response of the system is to small changes δa in the source term. The calculation is essentially already done. The leading $a\,\delta a$ term cancels and one finds

$$\delta S_{\text{tot}} = \delta S + \delta S_{\text{bry}} = L^{d-1} \int_{z=\epsilon} \mathrm{d}^d x \, (d-2\Delta) b\, \delta a \; . \tag{43}$$

The expectation value of the operator dual to ϕ then follows from the basic postulate (33):

$$\langle \mathcal{O} \rangle = -\frac{\delta S_{\text{tot}}}{\delta \phi_0} = -\frac{\delta S_{\text{tot}}}{\delta a} = L^{d-1} (2\Delta - d)\, b \; . \tag{44}$$

We have come to a second omission in the discussion. The ellipses in the variations (40) and (42) contain subleading terms in the $a\,\delta a$ series which may be dominant compared to the $b\,\delta a$ term considered in (43). In general, we require further counter-terms to cancel these subleading $a\,\delta a$ pieces and to prevent $\langle \mathcal{O} \rangle$ from being UV divergent. As an example, one may consider the subleading term in the $a\,\delta a$ series, proportional to $z^{d-2\Delta+2} \delta a \Box a$. Assuming $2\Delta > d+2$, this term is dominant compared to $b\,\delta a$, but it can be canceled by adding a $\phi \Box \phi$ boundary term to the action. That these counterterms can be identified in general and that $\langle \mathcal{O} \rangle$ can be renormalized is discussed in more detail in for example ref. [8]. The procedures described above for scalar fields can be generalized for higher spin fields. These techniques usually go by the name of "holographic renormalization".

Note that the characteristic exponent Δ is also the scaling dimension of the operator $\mathcal{O}$. The transformation rule $x \to \Lambda x$ and $z \to \Lambda z$ is a symmetry of the line element (36) and of the geometry of AdS_{d+1}. The restriction of the scaling symmetry to the boundary $z=0$ corresponds to scale transformations of the CFT. Under this scale transformation, the field ϕ transforms as $\phi'(z,x) = \phi(\Lambda z, \Lambda x)$. Thus we find that

$$\langle \mathcal{O}' \rangle = \Lambda^{\Delta} \langle \mathcal{O} \rangle \; . \tag{45}$$

Primary scalar operators in CFT satisfy a unitarity bound [9], $\Delta > (d-2)/2$, saturated by the free field case. The assumption $\Delta > d/2$ thus leaves out a set of operators with scaling dimension in the range $(d-2)/2 < \Delta < d/2$. To close this gap, let us now assume that $\Delta < d/2$ and repeat the exercise we went through above. We still freeze the value of a and thus set $\delta a = 0$. Now, in addition to canceling the $a\,\delta b$ term in the variation (40), we also need to cancel the $b\,\delta b$ term, which no longer vanishes in the limit $z \to 0$. Breaking from our rule that counterterms should depend only on the intrinsic geometry of the boundary, we add a Gibbons-Hawking like term that depends on a

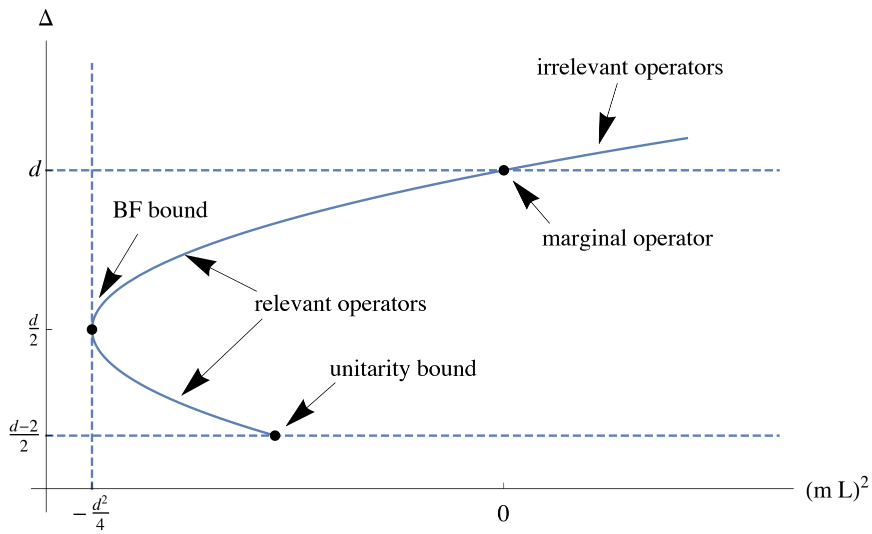


Figure 1: A plot of the scaling dimension Δ of $\mathcal{O}$ versus the mass m of the AdS_{d+1} scalar ϕ.

normal derivative

$$S_{\mathrm{bry}} = \int_{z=\epsilon} \mathrm{d}^d x \sqrt{-\gamma} \left(\frac{c}{L} \phi^2 + c' n^\mu \phi \partial_\mu \phi \right) . \tag{46}$$

where c and c' are constants and $n^\mu = (0, z/L)$ is a unit normal to the boundary. We leave it as an exercise to show that $c' = 1$ and $c = -\Delta/2$ for a good variational principle. Just as we did earlier, we can then consider the response of the system to a small δa. We find that $\langle \mathcal{O} \rangle = L^{d-1}(2\Delta - d)\, b$, just as before. In the window $(d-2)/2 < \Delta < d/2$, there are no subleading divergences in the $b\,\delta b$ series, and no further counter-terms are needed.

The set of scalar fields considered in this lecture is summarized pictorially in figure 1. The point $\Delta = d/2$ where the curve turns around is known as the Breitenlohner-Freedman (BF) bound. It is the smallest mass-squared for a scalar field in AdS_{d+1} that allows for a sensible stress-energy tensor [10, 11].

While for simplicity, we have focused on the simplest case of the Poincaré patch, the techniques here generalize to situations where the space is only asymptotically, in the limit $z \to 0$, of AdS type. From a CFT point of view, this restriction on the asymptotics means keeping the UV behavior of the field theory the same. One could imagine, for example, providing a nonzero source $\phi_0 \neq 0$ for a relevant operator $\Delta < d$, in which case the large z (i.e. low energy) geometry will generally be modified. The small z asymptotics remain the same, and now we may calculate correlation functions in the presence of the source. On the other hand, if we add a source for an irrelevant operator $\Delta > d$, the small z (i.e. high energy) geometry will be modified and the preceding results can no longer be applied.

3.1 Scalar Two-Point Functions in Pure AdS$_{d+1}$

Above, in expanding the field $\phi(x, z)$ near the boundary

$$\phi(x, z) = z^{d-\Delta} a(1 + \ldots) + z^{\Delta} b(1 + \ldots)$$

and positing $a = \phi_0$, we found that the one-point function $\langle O \rangle \sim b$ was determined by the coefficient of the second series. Here, we will use a second boundary condition to find a relation between b and the source a. Given that relation, we can then compute a two-point correlation function $\langle OO \rangle$ by varying $\langle O \rangle$ with respect to $a = \phi_0$.

In pure AdS$_{d+1}$, we can find an explicit solution of the equations of motion (37) for the scalar field. We first make a plane wave ansatz, $\phi \sim e^{k \cdot x} \phi(z)$. The equation of motion simplifies to an ordinary differential equation

$$z^{d+1}(z^{1-d}\phi')' - (z^2 k^2 + m^2 L^2)\phi = 0 \ , \tag{47}$$

where $'$ denotes ∂_z. Next, we make the substitution $\phi(z) = z^{d/2} H(z)$,

$$z^2 H'' + z H' - \left(k^2 z^2 + m^2 L^2 + \frac{d^2}{4} \right) H = 0 \ , \tag{48}$$

and recognize a second order differential equation of Bessel type. In the Euclidean or space-like case where $k^2 > 0$, we find a solution in terms of Hankel functions:

$$H = c_1 H_\nu^{(1)}(ikz) + c_2 H_\nu^{(2)}(ikz) \ , \tag{49}$$

where we have defined $\nu \equiv \sqrt{m^2 L^2 + d^2/4}$. To fix the second boundary condition, consider the large z behavior where $H_\nu^{(1)}(ikz) \sim e^{-kz}$ and $H_\nu^{(2)}(ikz) \sim e^{kz}$, allowing us to set $c_2 = 0$ and throw out the second, exponentially growing solution.

To extract the two-point function, consider the small z expansion of the solution, assuming $\Delta > d/2$ and that ν is not an integer,

$$\phi = c_1 \left[z^{d-\Delta} \left(-\frac{i}{\pi} \left(\frac{2}{ik} \right)^\nu \Gamma(\nu) + \ldots \right) + z^{\Delta} \left(\left(\frac{ik}{2} \right)^\nu \frac{1 + i \cot(\pi\nu)}{\Gamma(1+\nu)} + \ldots \right) \right] \ . \tag{50}$$

From the leading and subleading coefficients of the series expansion, we can read off the values of ϕ_0 and $\langle O \rangle$:

$$\phi_0 = c_1 \left(-\frac{i}{\pi} \right) \left(\frac{2}{ik} \right)^\nu \Gamma(\nu) \ , \tag{51}$$

$$\langle O \rangle = (2\Delta - d) L^{d-1} c_1 \left(\frac{ik}{2} \right)^\nu \frac{1 + i \cot(\pi\nu)}{\Gamma(1+\nu)} \ . \tag{52}$$

The (Fourier transform of the) two-point function can then be extracted by varying the one-point function:

$$G^{\mathcal{O}\mathcal{O}}(k) = \frac{\delta \langle \mathcal{O} \rangle}{\delta \phi_0} = \frac{\langle \mathcal{O} \rangle}{\phi_0} = (-2\nu) \left(\frac{ik}{2} \right)^{2\nu} (i\pi) \frac{1 + i \cot(\pi\nu)}{\Gamma(\nu)\Gamma(1+\nu)} L^{d-1} \tag{53}$$

We need now to Fourier transform back to position space. Focusing on the $k^{2\nu} = k^{2\Delta-d}$ behavior, note that by translational symmetry and dimensional analysis, the only possible result is that

$$\langle \mathcal{O}(x_2)\mathcal{O}(x_1)\rangle = \int \frac{\mathrm{d}^d k}{(2\pi)^d} G^{\mathcal{O}\mathcal{O}}(k) e^{ik\cdot(x_2-x_1)} \sim \frac{1}{|x_2-x_1|^{2\Delta}} \ . \tag{54}$$

Two-point functions in CFT are indeed constrained to have precisely this form.

3.2 Gauge fields in the bulk, global symmetries in the boundary

Having gained some experience with scalar fields, we move on to gauge fields in AdS_{d+1}, which in the context of the holographic renormalization are actually somewhat simpler, requiring fewer counter-terms. Consider the following abelian gauge field in the bulk:

$$S = -\frac{1}{4e^2}\int \mathrm{d}^{d+1}x\sqrt{-g}F_{AB}F^{AB} \ . \tag{55}$$

The equations of motion are simply $\partial_A\sqrt{-g}F^{AB} = 0$. To keep the discussion simple, we pick a radial gauge $A_z = 0$. The equations of motion $\partial_A\sqrt{-g}F^{A\mu}$ expand, using the line element (36), to give

$$\partial_z z^{3-d}\partial_z A_\mu + z^{3-d}\partial_\lambda \eta^{\lambda\nu}F_{\nu\mu} = 0 \ . \tag{56}$$

In analogy to the scalar discussion, we consider a small z expansion of the gauge field, $A_\mu \sim z^\Delta$. The corresponding indicial equation

$$\Delta(\Delta + 2 - d) = 0 \ , \tag{57}$$

has the two roots $\Delta = 0$ and $\Delta = d-2$, leading to the following small z series solution

$$A_\mu = a_\mu(1+\ldots) + b_\mu z^{d-2}(1+\ldots) \ . \tag{58}$$

We should also consider the remaining equation of motion $\partial_A\sqrt{-g}F^{Az} = 0$ which expands to give

$$\partial_\mu z^{3-d}\partial_z \eta^{\mu\nu}A_\nu = 0 \ . \tag{59}$$

Inserting the small z series solution into this equation of motion produces the constraint $\partial_\mu\eta^{\mu\nu}b_\nu = 0$. In other words, $\eta^{\mu\nu}b_\nu$ satisfies a current conservation condition.

In determining the equations of motion, we produced a boundary term which we now consider more carefully:

$$\begin{aligned}
\delta S &= \frac{L^{d-3}}{e^2}\int_{z=\epsilon} \mathrm{d}^d x\, z^{3-d}\eta^{\mu\nu}\delta A_\mu \partial_z A_\nu & (60)\\
&= \frac{L^{d-3}}{e^2}\int_{z=\epsilon} \mathrm{d}^d x\, z^{3-d}\eta^{\mu\nu}(\delta a_\mu + \delta b_\mu\, z^{d-2})((d-2)b_\nu z^{d-3} + \ldots) & (61)\\
&= \frac{L^{d-3}}{e^2}\int_{z=\epsilon} \mathrm{d}^d x\,(d-2)\eta^{\mu\nu}\delta a_\mu\, b_\nu \ . & (62)
\end{aligned}$$

To get a good variational principle, where we set $\delta a_\mu = 0$, we need no further counter-terms. To extract the one-point function however, we may find that even though the leading $a\,\delta a$ term cancels because of the ∂_z derivative, there could be subleading divergences that are nonetheless dominant compared to the $\delta a\, b$ term. In fact the situation here is further complicated by the fact that $d-2$ is integer and the two series may overlap, generating logarithms. There is a $z \to -z$ symmetry of the equations of motion which implies that the series expansion is in even powers of z. Thus, the series only overlap when d is an even integer. While in $d=3$, we may take the variation (62) at face value, in $d=4$ a logarithmic singularity appears which requires more careful treatment. In $d>4$, there can be further complications. Ignoring these gritty details, we take the variation (62) at face value and compute the one-point function:

$$\langle J^\mu \rangle = \frac{\delta S}{\delta a_\mu} = \frac{(d-2)L^{d-3}}{e^2} \eta^{\mu\nu} b_\nu \ . \tag{63}$$

We are now in a position to identify the operator J^μ. From the point of view of the CFT, it is sourced by an external gauge field a_μ and satisfies a current conservation condition $\partial_\mu J^\mu = 0$. Thus it must be a conserved current. Note that a_μ is not dynamical both from the gravity and CFT point of view.

3.3 The stress tensor

The stress-tensor operator in the CFT is one of the more difficult fields to study through AdS/CFT but also one of the most useful and interesting. It naturally couples to the boundary value of the metric. To analyze this case, let us first set some notation. The bulk metric shall be G_{AB}. We will pick a gauge where the line-element is

$$\mathrm{d}s^2 = \frac{L^2}{z^2}\mathrm{d}z^2 + \gamma_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \ , \tag{64}$$

where $\gamma_{\mu\nu}$ is the boundary metric. We further define

$$g_{\mu\nu} \equiv \frac{z^2}{L^2}\gamma_{\mu\nu} \ . \tag{65}$$

In general, $g_{\mu\nu}$ will have a nontrivial z dependence which we can write for small z as

$$g_{\mu\nu} = \begin{cases} g^{(0)}_{\mu\nu} + z^2 g^{(2)}_{\mu\nu} + \ldots + z^d g^{(d)}_{\mu\nu} + z^{d+2} g^{(d+2)}_{\mu\nu} + \ldots \ , & \text{odd } d \\ g^{(0)}_{\mu\nu} + z^2 g^{(2)}_{\mu\nu} + \ldots + z^d g^{(d)}_{\mu\nu} + z^d \log z \, h^{(d)}_{\mu\nu} + \ldots \ , & \text{even } d \end{cases} \tag{66}$$

Note that the CFT metric is not $g_{\mu\nu}$ but the boundary value $g^{(0)}_{\mu\nu}$. The full tensor structure $g_{\mu\nu}$ contains more information, as we will see.

Given the earlier discussion of scalars and gauge fields, we can anticipate that the action will contain a bulk contribution, a boundary contribution to have a good variational principle, and further counter-terms to render the correlation functions finite:

$$S = S_{\mathrm{EH}} + S_{\mathrm{GH}} + S_{\mathrm{ctr}} \ . \tag{67}$$

The bulk term is Einstein-Hilbert plus a negative cosmological constant, required so that AdS_{d+1} is a solution of the equations of motion:

$$S_{\mathrm{EH}} = \frac{1}{2\kappa^2}\int_M \mathrm{d}^{d+1}x\sqrt{-G}\left(R + \frac{d(d-1)}{L^2}\right) \ . \tag{68}$$

However, anti-de Sitter space has a boundary and second derivatives $R \sim \partial^2 G_{AB}$ in the action will generate boundary terms of the form $\partial_A(\delta g_{BC})$ which need to be canceled. The standard procedure is to add a Gibbons-Hawking term

$$S_{\rm GH} = \frac{1}{\kappa^2}\int_{\partial M} {\rm d}^d x \sqrt{-\gamma} K \ , \tag{69}$$

where $K = G^{AB}\nabla_A n_B$ is the trace of the extrinsic curvature and n_B is an outward pointing unit normal vector. Such a boundary term will cancel normal derivatives of the metric variation $n^A\partial_A(\delta g_{BC})$.

The variation of the Einstein-Hilbert term gives

$$\delta S_{\rm EH} = \frac{1}{2\kappa^2}\int_M {\rm d}^{d+1}x \left[\sqrt{-G}(\delta R_{AB})G^{AB} + \sqrt{-G}R_{AB}\delta G^{AB} + \left(R + \frac{d(d-1)}{L^2}\right)\delta(\sqrt{-G})\right] \ . \tag{70}$$

The variation of the second two terms produces Einstein's equation, which vanish on-shell. The variation of the Ricci tensor is a covariant derivative

$$\delta R_{AB} = -(\delta\Gamma^C_{AC})_{;B} + (\delta\Gamma^C_{AB})_{;C} \ , \tag{71}$$

a result sometimes known as the Palatini identity. Inside the action, this variation becomes a total derivative

$$\sqrt{-G}G^{AB}\delta R_{AB} = -(\sqrt{-G}G^{AB}\delta\Gamma^C_{AC})_{,B} + (\sqrt{-G}G^{AB}\delta\Gamma^C_{AB})_{,C} \tag{72}$$

Skipping some steps which we will flesh out in the next section, this total derivative reduces to the boundary term

$$\delta S_{\rm EH} = -\frac{1}{2\kappa^2}\int_{\partial M} {\rm d}^d x\sqrt{-\gamma}\left(n^A\gamma^{CD}\delta G_{CD;A} - K n^A n^B \delta G_{AB} + K^{AB}\delta G_{AB}\right) \ . \tag{73}$$

Meanwhile, varying the Gibbons-Hawking term leads to

$$\delta S_{\rm GH} = \frac{1}{\kappa^2}\int_{\partial M} {\rm d}^d x \left[\sqrt{-\gamma}\,\delta K - K\,\delta(\sqrt{-\gamma})\right] \ . \tag{74}$$

Again skipping some steps, the variation of the extrinsic trace produces

$$\delta K = \frac{1}{2}\gamma^{CD}\delta G_{CD;A}n^A - \frac{K}{2}n^A n^B \delta G_{AB} \ .$$

Assembling the pieces, the boundary variation is then

$$\delta S_{\rm EH} + \delta S_{\rm GH} = -\frac{1}{2\kappa^2}\int_{\partial M} {\rm d}^d x\sqrt{-\gamma}(K^{\mu\nu} - K\gamma^{\mu\nu})\delta\gamma_{\mu\nu} \tag{75}$$

where $K_{AB} = \nabla_{(A}n_{B)}$. Thus the "bare" stress tensor will be[5]

$$(T_{\rm bare})^{\mu\nu}\frac{\sqrt{-g^{(0)}}}{2} = \frac{\delta S}{\delta g^{(0)}_{\mu\nu}} = -\frac{L^{d+2}}{2\kappa^2}\sqrt{-g}\frac{1}{z^{d+2}}(K^{\mu\nu} - K\gamma^{\mu\nu}) \ . \tag{76}$$

This stress tensor appears in the early AdS/CFT paper [12]. The factor of z^{-d-2} in this expression suggests that the bare stress tensor may be divergent. Indeed, combined with an inverse metric

[5] In Lorentzian signature, conventionally the variation of the action is proportional to the stress tensor. In Euclidean, there should be a relative minus sign. We are implicitly working in Lorentzian signature here.

factor $\gamma^{\mu\nu}$, there will in general be divergent terms starting at order z^{-d}. These terms need to be regulated. The form of the counter terms in $d \leq 6$ is

$$S_{\rm ctr} = \frac{1}{\kappa^2} \int_{\partial M} \mathrm{d}^d x \sqrt{-\gamma} \left[\frac{d-1}{L} + \frac{L}{2(d-2)} \mathcal{R} + \frac{L^3}{(d-4)(d-2)^2} \left(\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} - \frac{d}{4(d-1)} \mathcal{R}^2 \right) + \ldots \right] . \quad (77)$$

The Ricci tensor $\mathcal{R}_{\mu\nu}$ is computed with the boundary metric $\gamma_{\mu\nu}$. We include as many of these counter-terms as are necessary to cancel the divergences. A term of the form $\sqrt{-\gamma}\mathcal{R}^n$ can cancel a divergence of order z^{-d+2n}. As a result, we need to include counter terms up to but not including $O(\mathcal{R}^{d/2})$ to cancel potential divergences. (In even d, there is an ambiguity in the definition of the stress tensor that comes from including terms of precisely $O(\mathcal{R}^{d/2})$. This ambiguity parallels a similar ambiguity on the the field theory side. In $d = 4$, for example, there is an analogous ambiguity in the coefficient of the $\Box R$ term in the trace anomaly.) In AdS_3, only the first term is needed. For AdS_4 and AdS_5, the first and second are needed. The second term proportional to $\mathcal{R}$ can be thought of as an analog of the $\phi\Box\phi$ counter term we needed for the scalar field. For AdS_6 and AdS_7, all three are needed, and higher order terms we have not written down would need to be constructed to regulate the divergences in $d > 6$.

Deriving the Boundary Stress Tensor

Similar discussions to the following can be found in textbooks on general relativity, for example appendix E.1 of Wald's book. However, in most of the general relativity literature, the variation of the metric on the boundary is set to zero, $\delta G_{AB}|_{z=0} = 0$. Like in the the case of the scalar we studied before, we would like to discover the response of the system to small variations in the boundary value of δG_{AB}. Thus we need to redo the classic textbook calculations, keeping a nonzero value for the metric fluctuations on the boundary.

We begin by studying the term proportional to δR_{AB} in the variation of the Einstein-Hilbert action (70). Using that δR_{AB} becomes a total derivative (72) inside the integral, the variation (70) becomes

$$\begin{aligned} \delta S_{\rm EH} &= -\frac{1}{2\kappa^2} \int_{\partial M} \mathrm{d}^d x \sqrt{-\gamma} \left[G^{AB} \delta\Gamma^C_{AC} n_B - G^{AB} \delta\Gamma^C_{AB} n_C \right] & (78) \\ &= -\frac{1}{2\kappa^2} \int_{\partial M} \mathrm{d}^d x \sqrt{-\gamma}\, G^{AB} G^{CD} \left(\delta G_{CD;A} n_B - \delta G_{AD;B} n_C \right) & (79) \\ &= -\frac{1}{2\kappa^2} \int_{\partial M} \mathrm{d}^d x \sqrt{-\gamma}\, n^A G^{CD} \left(\delta G_{CD;A} - \delta G_{CA;D} \right) . & (80) \end{aligned}$$

We can write the boundary metric as an operator $\gamma^{AB} = G^{AB} - n^A n^B$ that projects onto the subspace orthogonal to n^A. In the variation, we can replace G^{CD} with γ^{CD} as the terms proportional to $n^A n^C n^D$ will drop out of the difference:

$$\delta S_{\rm EH} = -\frac{1}{2\kappa^2} \int_{\partial M} \mathrm{d}^d x \sqrt{-\gamma}\, n^A \gamma^{CD} \left(\delta G_{CD;A} - \delta G_{CA;D} \right) . \quad (81)$$

But now $\gamma^{CD}\delta G_{CA;D}$ becomes almost a total tangential derivative which we can integrate by parts. In more detail, we have the identity

$$\gamma^{ED} (\gamma^C_D n^A \delta G_{AC})_{;E} = -K n^A n^C \delta G_{AC} + K^{AC} \delta G_{AC} + \gamma^{CD} n^A \delta G_{AC;D} , \quad (82)$$

where now the quantity on the left really is a total boundary derivative because the covariant derivative acts on a quantity with projected indices. In this identity we have replaced the covariant derivative of the unit normal with the extrinsic curvature, $n^{A;C} = K^{AC}$. This identity combined with the intermediate result (81) leads to

$$\delta S_{\rm EH} = -\frac{1}{2\kappa^2} \int_{\partial M} {\rm d}^d x \sqrt{-\gamma} \left(n^A \gamma^{CD} \delta G_{CD;A} - K n^A n^C \delta G_{AC} + K^{AC} \delta G_{CA} \right) \ . \tag{83}$$

Next we consider the variation of the Gibbons-Hawking term:

$$\delta S_{\rm GH} = \frac{1}{\kappa^2} \int_{\partial M} {\rm d}^d x \left(\sqrt{-\gamma}\, \delta K + K \delta(\sqrt{-\gamma}) \right) \ . \tag{84}$$

Rewriting δK in terms of the connection leads to

$$\delta K \ = \ (\delta \nabla_A) n^A + \nabla_A \delta n^A \ .$$

The first term in this variation can be simplified straightforwardly:

$$\begin{aligned} (\delta \nabla_A) n^A &= (\delta \nabla_A) n^A \\ &= (\delta \Gamma^A_{AC}) n^C \\ &= \frac{1}{2} G^{AD} (\delta G_{AD;C} + \delta G_{CD;A} - \delta G_{AC;D}) n^C \\ &= \frac{1}{2} G^{AD} \delta G_{AD;C} n^C \ . \end{aligned}$$

The constraint $n^A n_A = 1$ implies that the variation of the unit normal must take the form

$$\delta n_A = \left(\frac{1}{2} n_A n^B n^C + c \gamma_A^B n^C \right) \delta G_{BC} \ , \tag{85}$$

where c is an as yet undetermined constant. To fix $c = 0$, we know that the tangent vectors $\partial X^A / \partial x^\mu$ do not depend on the metric and must be orthogonal to δn_A. But to vary K, we need $\delta n^A = \delta(g^{AB} n_B)$ which must then be

$$\delta n^A = -\left(\frac{1}{2} n^A n^B n^C + \gamma^{AB} n^C \right) \delta G_{BC} \ . \tag{86}$$

The variation of the trace of the extrinsic curvature is thus

$$\delta K = \frac{1}{2} \gamma^{AD} \delta G_{AD;C} n^C - \frac{K}{2} n^B n^C \delta G_{BC} - \nabla_A (\gamma^{AB} n^C \delta G_{BC}) \ . \tag{87}$$

The variation of the Gibbons-Hawking term then becomes

$$\delta S_{\rm GH} = \frac{1}{2\kappa^2} \int_{\partial M} {\rm d}^d x \sqrt{-\gamma} \left(n^A \gamma^{BC} \delta G_{BC;A} - K n^A n^B \delta G_{AB} + K \gamma^{AB} \delta G_{AB} \right) \ , \tag{88}$$

where we have discarded a total boundary derivative. As is well known, the normal derivatives in $\delta S_{\rm EH}$ and $\delta S_{\rm GH}$ cancel. As is less well known, the terms proportional to $K n^A n^B \delta G_{AB}$ cancel as well, leaving the boundary stress tensor (75).